Dynamic Facility Location with Stochastic Demands

Martin Romauch and Richard F. Hartl

University of Vienna, Department of Management Science,
Brünner Straße 72, 1210 Vienna, Austria
{martin.romauch, richard.hartl}@univie.ac.at

Abstract. In this paper, a Stochastic Dynamic Facility Location Problem (SDFLP) is formulated. In the first part, an exact solution method based on stochastic dynamic programming is given. It is only usable for small instances. In the second part a Monte Carlo based method for solving larger instances is applied, which is derived from the Sample Average Approximation (SAA) method.

1 Introduction

The Uncapacitated Facility Location Problem (UFLP) has been enhanced into many directions. In [9] and [2] you can find numerous approaches that consider either dynamic or stochastic aspects of location problems. An exact solution method to an UFLP with stochastic demands is discussed in [6]. The problem considered there could be interpreted as a two stage stochastic programm. In [8] you can find dynamic (multi period) aspects as well as the multi commodity aspect. The approach in [10] could be seen as the integration of stochastics into the UFLP. In the work in hand a model will be presented, where the UFLP gets enriched by inventory and randomness in the demand. The UFLP and its generalizations are part of the class of NP-hard problems, where no exact efficient solution methods are known. First of all, the aim of this work is the preparation of tools to develop and investigate heuristics for this problem type. For this reason, an exact method for small instances was developed. This makes possible both, to carve out the range of exact solvability and to compare exact and heuristic solutions. A more detailed description of the problem is now following.

2 Stochastic Dynamic Warehouse Location Problem

Our aim is to find the optimal decisions for production, inventory and transportation, to serve the customers during a certain number of periods, $t \in \{1, ..., T\}$. Assume that the company runs a number for the production sites $i \in I = \{1, 2, ..., n\}$ that have limited storage capacities, $\Delta_i^{(t)}$. These production sites need not be used in all periods. When a production site i is operated at time t, this is denoted by the binary variable $\delta_i^{(t)} = 1$. In this case the fixed costs $o_i^{(t)}$ arise. If a location is active, then the exact production quantity $u_i^{(t)}$ must be

O.B. Lupanov et al. (Eds.): SAGA 2005, LNCS 3777, pp. 180–189, 2005.

fixed. For each period, the production decision is the first stage of the decision process. It has to be done before the demand of the customers is known. Only the current level of inventory $y_i^{(t-1)}$ as well as the demand forecasts are known in advance.

Demand occurs at various customer locations $j \in J = \{1, 2, ..., m\}$. At any given period t the demand d_j^t at customer j will occur with probability p_j^t, whereas customer j will not require any delivery with probability $1 - p_j^t$. Hence, demand can be described by a dichotomous random variable[1] $\mathcal{D}_j^{(\tau)}$ $(\tau \geq t)$. We also assume that the random variables $\mathcal{D}_j^{(t)}$ are stochastically independent.

$$\mathbf{P}(\mathcal{D}_j^{(t)} = d_j^{(t)}) = p_j^{(t)} \quad \mathbf{P}(\mathcal{D}_j^{(t)} = 0) = 1 - p_j^{(t)}$$

In the second stage, when the demand is known, we must decide upon the transportation of appropriate quantities $x_{ij}(t)$ from the production sites i to the customers j. We assume that the time needed for transportation can be neglected (i.e. the transportation lead time is less than one period). Stockouts (shortages) $f_j^{(t)}$ are permitted and are penalized by shortage costs $p_j^{(t)}$ per unit time and per unit of the product. We assume that backordering is not possible and that these potential sales are lost.

The periods are linked by the inventories $y_j^{(t)}$ at the production sites and the usual inventory balance equations (1) apply. Here $\eta_i^{(t)}$ denotes the surplus in period t at site i.

$$y_i^{(t)} + \eta_i^{(t)} = y_i^{(t-1)} + u_i^{(t)} - \sum_{j \in J} x_{ij}^{(t)} \tag{1}$$

In this paper we assume free disposal, therefore the variable $\eta_i^{(t)}$ can be eliminated by turning the equality (1) into the inequality (2).

$$y_i^{(t)} \leq y_i^{(t-1)} + u_i^{(t)} - \sum_{j \in J} x_{ij}^{(t)} \tag{2}$$

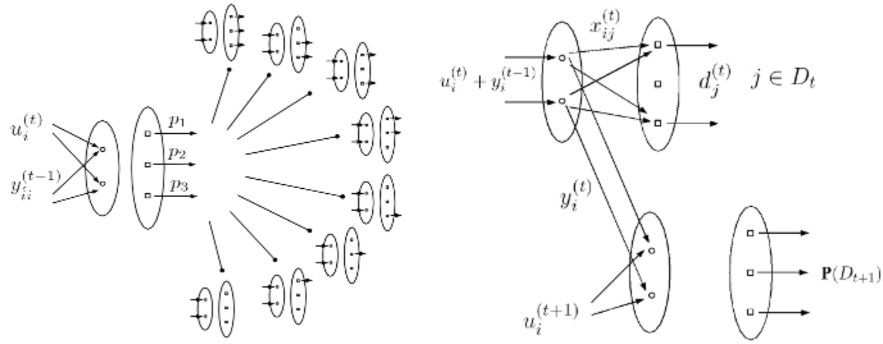
After the completion of the production and transportation decisions and after updating the inventories, the next period can be considered. We note here, that for all periods we have to pay attention to the capacity restrictions (3).

$$0 \leq u_i^t + y_i^{(t-1)} \leq \Delta_i^{(t)} \tag{3}$$

In order to have a convenient notation, we introduce the concept of scenarios. A scenario $D_t \subset J$ is a subset of customers where the demand gets realized. Since the demands of the different customers are independent, the corresponding probability of a scenario to occur is given in formula (4).

$$\mathbf{P}(D_t) = \prod_{j \in D_t} p_j^{(t)} \prod_{j \notin D_t} \left(1 - p_j^{(t)}\right) \tag{4}$$

[1] The embedding of stochastics is similar to the embedding of stochastics into the TSP, see [4].



(a) Scenarios ($|I| = 2, |J| = 3$)

(b) Transition

Fig. 1. Sequencing of Decisions

Solving the SDFLP means finding a strategy that minimizes the expected costs. Because of the sequencing of the decisions and the uncertain demand, the solutions could be understood as scenario dependent strategies, where the decisions are dependent on the forecasts and the level of inventory at hand. Figures 1(a) and 1(b) illustrate the dependency of operative planning[2] and the scenarios (realization of demand).

The left hand side of Figure 1(a) shows the production decisions $u_i^{(t)}$ and all of the possible subsequent scenarios (8 in number). One of the scenarios is magnified in the upper part of Figure 1(b). In each scenario the decisions for transportation, inventory and shortage are necessary.

In order to complete the model formulation, we summarize the decision variables and the corresponding costs in Table 1.

Table 1. Variables and Costs

variable	cost	description
$\delta_i^{(t)} \in \{0, 1\}$	$o_i^{(t)}$	operating decision and fixed costs
$x_{ij}^{(t)} \in \mathbf{Z}_+$	$c_{ij}^{(t)}$	transportation decision and unit transportation cost
$y_i^{(t)} \in \mathbf{Z}_+$	$s_i^{(t)}$	inventory level and unit holding cost
$u_i^{(t)} \in \mathbf{Z}_+$	$m_i^{(t)}$	production decision and variable production cost
$f_j^{(t)} \in \mathbf{Z}_+$	$p_j^{(t)}$	shortage (lost sales) and unit shortage cost

The decisions $\delta_i^{(t)}$ and $u_i^{(t)}$ are linked by formula (5)

$$\delta_i^{(t)} = \begin{cases} 1 & \text{if } u_i^{(t)} > 0 \\ 0 & \text{if } u_i^{(t)} = 0 \end{cases} \tag{5}$$

[2] To keep the figure as simple as possible shortages and disposal are not integrated.

while shortages are defined as

$$f_j^{(t)} = \mathcal{D}_j^{(t)} - \sum_{i \in I} x_{ij}^{(t)}.$$

The total cost F is the sum over all periods of fixed operating costs, variable production costs,

$$F = \mathbf{E}\left(\sum_{t=1}^{T} \sum_{i \in I} \left[o_i^{(t)} \delta_i^{(t)} + m_i^{(t)} u_i^{(t)} + s_i^{(t)} y_i^{(t)} \right] + \sum_{t=1}^{T} \sum_{i \in I} \sum_{j \in J} c_{ij}^{(t)} x_{ij}^{(t)} + \sum_{t=1}^{T} \sum_{j \in J} p_j^{(t)} f_j^{(t)} \right)$$

Since all relevant information about the past is contained in the inventory levels, this model is well suited to be solved by dynamic programming. This will be outlined in the next section.

3 Exact Solution Method

3.1 Stochastic Dynamic Programming

The principle of dynamic programming is the recursive estimation of the value function. This value function, henceforward denoted by F, contains the aggregate value of the optimal costs in all remaining periods. It can be derived recursively. It is convenient to first describe the method in general and to apply it to the problem afterwards. Let $z \in \mathbf{R}_+^m$ be the vector of state variables and $u \in \mathbf{R}_+^n$ be the vector of decisions. The set of feasible decisions in state z and period t is denoted by $U_t(z)$. The random influence in period t is represented by the random vector $r^{(t)}$ for which the corresponding distribution is known. It is important to note that the random variables $\{r^{(t)}\}$ have to be stochastically independent. The state transformation is described by

$$z_{t+1} = A(z_t, u_t, r_t)$$

and depends on the current state z_t, the random influence r_t at time t, and the chosen decision u_t. The single period costs in period t and state z when decision u is taken and random variable r is realized is denoted by $g_t(z, u, r)$.

The value function $F_t(z)$ gives the minimal expected remaining costs when starting in state z in period t. We now present a variant of the stochastic Bellman equation (compare Schneeweiß[11] S.151 (10.25) or Bertsekas [1] S.16).

$$F_T(z) = \min_{u \in U_T(z)} \Big\{ \mathbf{E}\big[g_T(z,u,r^{(T)})\big] \Big\}$$
$$F_t(z) = \min_{u \in U_t(z)} \Big\{ \mathbf{E}\big[g_t(z,u,r^{(t)}) + F_{t+1}(A_t(z,u,r^{(t)}))\big] \Big\} \quad t = T-1,\ldots,1 \tag{6}$$

Recursively solving the equation (6) we get an optimal strategy that balances the cost for implementing the decision u and the expected resulting remaining costs.

Applying Stochastic Dynamic Programming (SDP) to the SDFLP is almost straight forward. For solving the problem we have to iteratively calculate the functions F_t. We will show later that for the SDFLP it is sufficient to consider integer controls.

3.2 Application to the SDFLP

In order to apply the DP equation (6) to the SDFLP, we first introduce the notation $G\big(D, y_i^{start}, y_i^{end}, t\big)$ for the sum of inventory holding costs, shortage costs, and transportation costs in scenario D in period t when starting with initial inventory levels y_i^{start} and where the final inventories are required to be y_i^{end}. To every given inventory level y_i^{start} and scenario D, the best possible transportation plan has to be calculated. This can be done by solving a linear program:

$$G\big(D, y_i^{start}, y_i^{end}, t\big) = \sum_{i \in I} s_i^{(t)} y_i^{end} + \min_{x_{ij}, f_j} \sum_{j \in J} p_j^{(t)} f_j + \sum_{i \in I} \sum_{j \in J} c_{ij}^{(t)} x_{ij} \tag{7}$$

$$\begin{aligned} s.t. \quad & \textstyle\sum_{i \in I} x_{ij} + f_j = d_j^{(t)} \quad \forall j \in D \\ & \textstyle\sum_{j \in J} x_{ij} + y_i^{end} \le y_i^{start} \ \forall i \in I \\ & f_j, x_{ij} \ge 0. \end{aligned}$$

Now the value function F_T of the final period T can be computed. In G, the starting inventory is now given by $y_i^{start} = u_i^{(T)} + y_i^{(T-1)}$ while the terminal inventory must be zero, $y_i^{end} = 0$:

$$F_T(y_i^{(T-1)}) = \min_{\substack{u_i^{(T)} \ge 0 \\ 0 \le u_i^T + y_i^{(T-1)} \le \Delta_i^{(T)}}} \Big\{ \sum_{i \in I} \delta_i^{(T)} o_i^{(T)} + \sum_{i \in I} u_i^{(T)} m_i^{(T)} + + \sum_{D_T \subset J} \mathbf{P}(D_T) G\big(D_T, u_i^{(T)} + y_i^{(T-1)}, 0, T\big) \Big\} \tag{8}$$

Going back in time, we have to turn to the general case in period $t < T$. Now we have to take into account the remaining costs in periods $t+1, ..., T$ when making the decision in period t. The starting inventory is now given by

$y_i^{start} = u_i^{(t)} + y_i^{(t-1)}$ while the inventory at the end of period t is $y_i^{end} = y_i^{(t)}$. When determining $G(D_t, u_i^{(t)} + y_i^{(t-1)}, y_i^{(t)}, t)$ again a linear program has to be solved. The recursion for the value function becomes:

$$F_t(y_i^{(t-1)}) = \min_{\substack{u_i^{(t)} \geq 0 \\ 0 \leq u_i^{(t)} + y_i^{(t-1)} \leq \Delta_i^{(t)}}} \Big\{ \sum_{i \in I} \delta_i^{(t)} o_i^{(t)} + u_i^{(t)} m_i^{(t)} + \\ + \sum_{D_t \subset J} \mathbf{P}(D_t) \min_{0 \leq y_i^{(t)} \leq \Delta_i^{(t+1)}} \big\{ G_t\big(D_t, u_i^{(t)} + y_i^{(t-1)}, y_i^{(t)}, t\big) + F_{t+1}(y_i^{(t)}) \big\} \Big\} \tag{9}$$

In the SDFLP the data and the controls are assumed to be integer. In the problem (7) we therefore have to solve an integer linear program. It turns out to be a min cost flow problem and therefore it is totally unimodular, such that using the Simplex method for solving the relaxed linear program results in integer solutions for the transportation quantities x_{ij} and the shortages f_j.

The computational effort of this exact algorithm is increasing exponentially with the capacity at the locations and the number of customers. The additional effort that emerges from adding additional periods to the problem is linear.

This DP formulation is only applicable for small problem instances and for larger problem instances heuristic approaches are necessary. This is considered in the next section.

4 Heuristic Approach

A heuristic designed to solve stochastic combinatorial optimization problems is the Sample Average Approximation Method (SAA); see Kleywegt et al. [5]. Our model deals with a multi stage problem and that is the reason why this method is not directly applicable. In what follows we first present the classical SAA for solving static stochastic combinatorial optimization problems. Afterwards, we will explain how this method can be modified in order to be applicable to our problem.

Consider the following stochastic combinatorial optimization problem (10) in which W is a random vector with known distribution P, and S is the finite set of feasible solutions.

$$v^\star = \min_{x \in \mathcal{S}} g(x), \quad g(x) := \mathbf{E}_P G(x, W) \tag{10}$$

The main idea of the SAA method is to replace the expected value $\mathbf{E}_P G(x, W) = \int G(x, w) P(dw)$ (which is usually very time consuming) by the average of a sample. The following substitute problem (11) is an estimator of the original problem (10).

$$\min_{x \in \mathcal{S}} \hat{g}_N(x), \quad \hat{g}_N(x) := \frac{1}{N} \sum_{j=1}^{N} G(x, W^j) \tag{11}$$

The SAA method works in three steps

1. Generate a set of independent identically distributed samples: $\{W_i^1, \ldots, W_i^N\}_{i=1}^M$ of the random variable W.
2. Solve the corresponding optimization problems, i.e. optimize:

$$\hat{v}_i = \min_{x \in \mathcal{S}} \hat{g}_i(x), \quad \hat{g}_i(x) = \frac{1}{N}\sum_{j=1}^{N} G(x, W_i^j).$$

3. Estimate the solution quality. This is done by first computing mean and variance of the sample:

$$\hat{v} = \frac{1}{M}\sum_{i=1}^{M} \hat{v}_i, \qquad \hat{\sigma}^2 = \frac{1}{M(M-1)}\sum_{i=1}^{M}(\hat{v}_i - \hat{v})^2.$$

Then a solution $\tilde{x}$ is chosen (e.g. we can take the solution with the smallest $\hat{v}_i$) and its objective value is estimated more accurately by generating a larger sample $\{W^1, \ldots, W^{N'}\}$ $(N' >> N)$

$$\tilde{v} = \frac{1}{N'}\sum_{j=1}^{N'} G(\tilde{x}, W^j), \quad \tilde{\sigma}^2 = \frac{1}{N'(N'-1)}\sum_{j=1}^{N'}(G(\tilde{x}, W^j) - \tilde{v})^2.$$

Calculate the value *gap* and σ^2_{gap}:

$$gap = \tilde{v} - \hat{v}, \quad \sigma^2_{gap} = \tilde{\sigma}^2 + \hat{\sigma}^2$$

Since $\mathbf{E}(\hat{v}) \le v^\star \le \mathbf{E}(\tilde{v})$ the values $\tilde{v}$ and $\hat{v}$ can be interpreted as bounds on $v^\star$: let $x^\star \in \mathcal{S}$ denote an optimal solutions of (10) then the first inequality $\mathbf{E}(\hat{v}) \le v^\star$ comes from taking the expected value on the following inequality:

$$\hat{v}_i \le \hat{g}_i(x^\star)$$

which results in

$$\mathbf{E}(\hat{v}_i) \le \mathbf{E}(\hat{g}_i(x^\star)) = v^\star.$$

After completing Step 3, we have to inspect the values of *gap* and σ^2_{gap}. If these values are too large, one must repeat the procedure with increased values of N, M and N'. In [10] this method is applied for a Supply-Chain Management problem that includes location decisions.

Because of the multi-period structure of the SDFLP, the above SAA procedure has to be adapted. In particular, one must pay special attention to the way how the sampling is done. The sampling is done independently in every stage and every state of the SDP sipmly by modifying formulas (8) and (9), where for every period and inventory level we only sum over a small randomly chosen sets of scenarios. To be more specific, the expected value in formula (9) passes over into (12) where $\{D_i\}$ denotes the sample chosen in stage t and state $y_i^{(t)}$.

$$\frac{1}{N}\sum_{i=1}^{N} \min_{0 \le y_i^{(t)} \le \Delta_i^{(t+1)}} \left\{ G_t\big(D_i, u_i^{(t)} + y_i^{(t-1)}, y_i^{(t)}, t\big) + F_{t+1}(y_i^{(t)}) \right\} \tag{12}$$

5 Results

The implementation was done in C++ and an additional library from the GNU Linear Programming Kit (GLPK) [7] was used. For small examples where the product $\prod_{i\in I} \Delta_i^{(t)}$ is small enough (say ≤ 50) it is possible to find the exact solution using the dynamic programming approach described in Section 3. In Figure 2, we can see the cost distributions for the exact and the heuristc approach in a small example (3 periods, 3 customers, 2 facilities and capacity = 3). Here one can see that the shapes are quite different. The instance considered has very

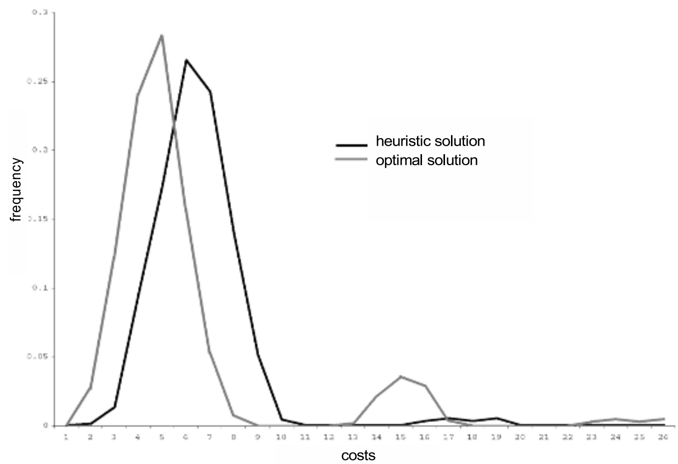


Fig. 2. Comparison of Different Solutions

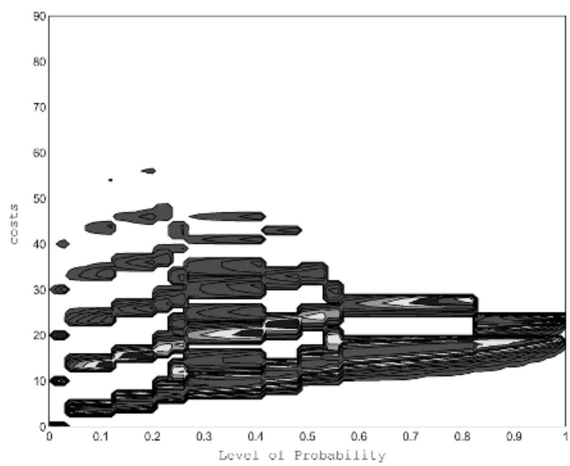


Fig. 3. Distributions of the Optimal Solution to Instances with Different Levels of Probability ($p_j^{(t)} = p \in \{0, 0.01, 0.02, \ldots, 1\}$)

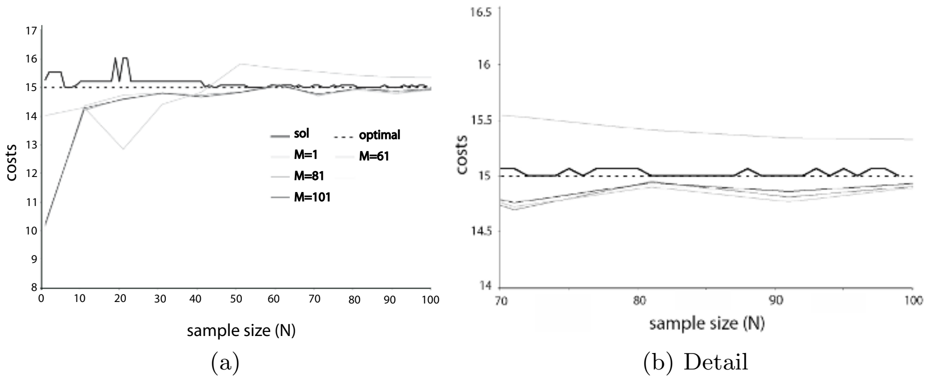


Fig. 4. Choice of sample size N and the number of samples M

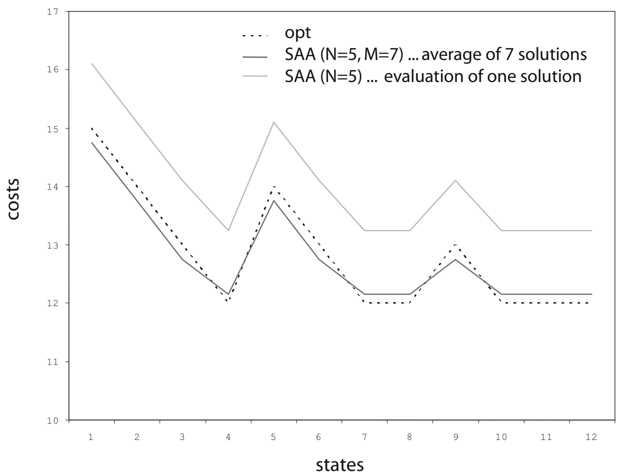


Fig. 5. Expected costs to different levels of inventory y ($1 : [0, 0]$; $2 : [0, 1]$; ...; $4 : [0, 3]$; $5 : [1, 0]$; ...; $12 : [2, 3]$)

uncertain demand (every customer has probability 50%). Hence the two peaks in the optimal solution are not very surprising. It is interesting to observe that the heuristic solution does not show these twin peaks.

This second peak diminishes if the probability is close to 0 or 1. An experiment was made where the probability for the demand varied from 0 to 1 for all customers, i.e.: ($p_j^{(t)} = p \in \{0, 0.01, 0.02, \ldots, 1\}$). The result is depicted in Figure 3 where grey areas represent positive probabilities that these cost values occur. Every vertical line (p fixed) corresponds to a distribution function. For instance, at $p = 0.1$ five peaks occur. When the probability p increases, the number of peaks in the distribution function decreases.

The key decision to make the heuristic work well is to choose the right sample size N and the right number of samples M. In Figure 4 the statistical lower bound $\hat{v}$ (calculated in step 3) is depicted for different values of N and M. Choosing a sample size N that is large enough seems to be more important than a large number of samples. In Figure 4(b) the region $N > 70$ of Figure 4) is magnified to see the effect of a choosing the number of samples more clearly. One also can see that the statistical gap stays positive if at least 11 samples of size 71 are chosen. It is also interesting to note that for small values of N and sufficient large M the corresponding bounds are quite good, although the corresponding individual solutions are quite bad. This situation is depicted in Figure 5.

6 Conclusion and Further Research

In this paper a stochastic dynamic facility location problem was proposed and exact and heuristic solution methods were presented. The examples that can be solved to optimality are quite small and therefore of minor practical interest. But the comparison of the SAA results and the exact solution method shows the applicability of the proposed method for larger instances of the SDFLP. To get more insight into this method, it will be necessary to make a transfer of the theoretical results known for the SAA method (see [5] for statistical bounds). For comparison purposes it would be interesting to adopt metaheuristic concepts: e.g. by using the variable-sample approach (for references see [3]). In our further research we also want to consider other exact solution techniques considering the SDFLP as a multistage stochastic program.

References

1. Bertsekas, D.P.: Dynamic Programming and Optimal Control, Athena Scientific, Belmont, MA (1995)
2. Hammer, P.L.: Annals of Operations Research: Recent Developments in the Theory and Applications of Location Models 1/2, Kluwer Academic Publishers (2002)
3. Homem-de-Mello, T.: Variable-Sample Methods for Stochastic Optimimization, ACM Trans. on Modeling and Computer Simulation (2003)
4. Jaillet, P.: Probabilistic Travelling Salesman Problems, Ph.D. thesis, MIT (1985)
5. Kleywegt, A.,Shapiro,A., Hohem-de-Mello, T.: The Sample Average Method for Stochastic Discrete Optimization, SIAM J. Optim. 12 (2001/02)
6. Laporte, G., Louveaux, F., Van Hamme, L.: Exact Solution to a Location Problem with Stochatic Demands, Transportation Science, Vol. 28,(1994)
7. Makhorin, A.:GNU Linear Programming Kit - Reference Manual - Version 4.1, Department for Applied Informatics, Moscow Aviation Institute, Moscow, Russia (2003)
8. Melo, M.T., Nickel, S., Saldanha da Gama, F.: Large-Scale Models for Dynamic Multi-Commodity Capacitated Facility Location, Berichtsreihe des Fraunhofer Inststituts für Techno- und Wirtschaftsmathematik (ITWM), ISSN 1434-997, Kaiserslautern (2003)
9. Hesse Owen, S., Daskin, M.S.: Strategic Facility Location: A Review, Eropean Journal of Operational Research, Vol. 111, (1998)
10. Santoso, T., Ahmed, S., Goetschalckx, M., Shapiro, J.: A Stochastic Programming Approach for Supply Chain Network Design under Uncertainty, The Stochastic Programming E-Print Series (SPEPS), (2003)
11. Schneeweiß, C.: Dynamisches Programmieren, Physica-Verlag, (1974)