ON CERTAIN UNITARY REPRESENTATIONS WHICH ARISE FROM A QUANTIZATION THEORY

by

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In this paper we are concerned with certain explicit constructions of unitary representations which arise from a general theory relating quantization and unitary representations. We shall not go into the general theory here but we can refer the reader to a forthcoming publication entitled "Quantization and Unitary Representations, Part I - Prequantization" which will appear as part of the series "Lectures in Modern Analysis and Applications" edited by C. T. Taam, in *Lecture Notes in Mathematics* published by Springer-Verlag. Those considerations here for solvable groups are part of a joint work of L. Auslander and myself.

1. THE REPRESENTATION $\mathrm{ind}_G(\eta_g, h)$

Let G be a Lie group, not necessarily connected, and let g be its Lie algebra.

Now let $g \in g'$ be a linear functional on g and let g_g be the Lie algebra of the isotropy subgroup $G_g \subseteq G$ with respect to the coadjoint representation of G on g'. Thus if B_g is the alternating bilinear form on g given by $B_g(x,y) = \langle g,[y,x]\rangle$ then

$$g_g = \{x \in g \mid B_g(x,y) = 0 \quad \text{for all} \quad y \in g\} \quad .$$

That is g_g is the radical of B_g.

We may regard g as a complex valued linear functional on $g_{\mathbb{C}} = g + ig$. A polarization at g is a complex subalgebra $h \subseteq g_{\mathbb{C}}$ such that

(1) $g_g \subseteq h$ and g_g is stable under $\mathrm{Ad}\, G_g$ (note that G_g is not necessarily connected even if G is connected)

(2) $\dim_{\mathbb{C}} g_{\mathbb{C}}/h = 1/2 \dim_{\mathbb{R}} g/g_g$ (recall $\dim_{\mathbb{R}} g/g_g$ is even since g_g is the radical of B_g)

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(3) $g|[h,h] = 0$, i.e., $g|h$ is a homomorphism

(4) $h + \bar{h}$ is a Lie algebra of $g_{\mathbb{C}}$.

Now let $d = h \cap g$ so that if $d_{\mathbb{C}} = d + id$ one has

$$d_{\mathbb{C}} = h \cap \bar{h} \quad .$$

Also let $e = (h + \bar{h}) \cap g$ so that if $e_{\mathbb{C}} = e + ie$ one has

$$e_{\mathbb{C}} = h + \bar{h} \quad .$$

Now clearly h is equal to its own orthogonal subspace relative to the extension of B_g to $g_{\mathbb{C}}$. It follows easily then that d is the orthogonal subspace to e relative to B_g and hence if $\hat{x} \in e/d$ denotes the image of $x \in e$ under the quotient map $e \to e/d$ one defines a non-singular alternating bilinear form $\hat{B}_g$ on e/d by the relation

$$(\hat{x},\hat{y}) = \langle g,[y,x]\rangle$$

for $x,y \in e$. Next note that we may identify $(e/d)_{\mathbb{C}}$ with $e_{\mathbb{C}}/d_{\mathbb{C}}$ so that

$$(e/d)_{\mathbb{C}} = h/d_{\mathbb{C}} \oplus \bar{h}/d_{\mathbb{C}}$$

is a linear direct sum. Since $\bar{h}/d_{\mathbb{C}} = (\overline{h/d_{\mathbb{C}}})$ relative to conjugation over the real form e/d of $(e/d)_{\mathbb{C}}$ one defines a non-singular operator $j \in \mathrm{End}\ e/d$ where $j^2 = -I$ and (upon complexification) $j = -i$ on $h/d_{\mathbb{C}}$ and $j = i$ on $\bar{h}/d_{\mathbb{C}}$.

Remark 1. Note that if $u \in e/d$ one has

$$u + iju \in h/d_{\mathbb{C}} \quad \text{and} \quad u - iju \in \bar{h}/d_{\mathbb{C}} \quad .$$

Let S_g be the bilinear form on e/d given by

$$\{u,v\} = (ju,v) \quad .$$

Proposition 1

S_g is a non-singular symmetric bilinear form on e/d. Moreover, j is orthogonal relative to both S_g and $\hat{B}_g$. That is, if $u,v \in e/d$ one has

$$\{ju,jv\} = \{u,v\} \quad \text{and} \quad (ju,jv) = (u,v) \quad .$$

Proof. It is clear that by definition $h/d_{\mathbb{C}}$ is orthogonal to itself relative to the extension of $\hat{B}_g$ to $(e/d)_{\mathbb{C}}$. Thus by Remark 1, one has for $u,v \in e/d$

$$0 = (u + iju, v + ijv) = [(u,v) - (ju,jv)] + i[(ju,v) + (u,jv)] \quad .$$

Since the imaginary part is zero this implies that

$$(ju,v) = -(u,jv) = (jv,u) \quad . \tag{1.1}$$

That is $\{u,v\} = \{v,u\}$ and hence S_g is symmetric. It is clearly non-singular since j is non-singular. The relation (1.1) together with $j^2 = -I$ clearly implies j is orthogonal relative to both S_g and $\hat{B}_g$.

We will say that the polarization h is positive in case S_g is a positive definite bilinear form. (This includes the case where $e/d = 0$, that is where $h = \bar{h}$.)

Remark 2. A simple criterion for the positivity of the polarization h without going to the quotient e/d is as follows: We assert that h is a positive polarization if and only if

$$-i(z,\bar{z}) \geq 0$$

for all $z \in h$. Indeed if $z \in h$ write $z = x + iy$ where $x,y \in e$. Thus $\hat{y} = j\hat{x}$ and hence $-i(z,\bar{z}) = -i(x + iy, x - iy) = 2(y,x) = 2(\hat{y},\hat{x}) = 2(j\hat{x},\hat{x}) = 2\{\hat{x},\hat{x}\}$. The relation then follows since the correspondence $z \mapsto x$ maps h onto e/d.

Now let $b = \{x \in d | \langle g,x \rangle\} = 0$. It follows that b has codimension 1 in d if and only if $g|d \neq 0$.

Remark 3. If g is nilpotent one knows that $g|g_g \neq 0$ and hence $g|d \neq 0$ if and only if $g \neq 0$.

Now let D_0 and E_0 be the connected Lie subgroups of G corresponding to $d = h \cap g$ and $e = (h + \bar{h}) \cap g$. Since h is stable under $\mathrm{Ad}\, G_g$ it follows that D_0 and E_0 are normalized by G_g and $D = G_g D_0$ and $E = G_g E_0$ are subgroups of G.

Proposition 2

The groups D *and* D_0 *are closed in* G. *Also* D_0 *is the identity component of* D *so that* d *is the Lie algebra of* D.

Proof. Since d and e are each other's orthogonal subspaces relative to B_g, one has that if $x \in g$ then $\langle x \cdot g, y \rangle = 0$ for all $y \in e$ if and only if $x \in d$. Thus

$$\langle a \cdot g - g, y \rangle = 0$$

for all $a \in D_0$ and hence for all $a \in \bar{D}_0$. But if x lies in the Lie algebra of $\bar{D}_0$ then clearly $\langle x \cdot g, y \rangle = 0$ for all $y \in e$ so that $x \in d$. Thus D_0 and $\bar{D}_0$ have the same Lie algebras and hence $D_0 = \bar{D}_0$.

Now let D_1 be the identity component of $\bar{D} = \overline{D_0 G_g}$. Then if $a \in D_1$ one has $\langle a \cdot g - g, y \rangle = 0$ for all $a \in D_1$, and $y \in e$. Then if d_1 is the Lie algebra of D_1 one has $d_1 \subseteq d$. But of course $d \subseteq d_1$ since $D_0 \subseteq D_1$. Thus

$d = d_1$ so that $D_0 = D_1$ is the identity component of $\overline{D}$. But $D_0 \subseteq D \subseteq \overline{D}$. Hence D is also closed and D_0 is the identity component of D.

QED

Now consider the D-orbit $D \cdot g \subseteq \mathcal{g}'$. For any subspace $a \subseteq \mathcal{g}$ let $\tilde{a}$ be its orthogonal subspace in $\mathcal{g}'$.

<u>Proposition 3</u>

$D \cdot g$ *is an open set of the affine plane* $g + \tilde{e}$ *in* $\mathcal{g}'$. *Also* $D \cdot g = D_0 \cdot g$.

<u>Proof</u>. We first observe that $g + \tilde{e}$ is stable under the action of D. Indeed since e is stable under Ad D clearly $\tilde{e}$ is stable under D. However, since $D = D_0 G_g$ one has $D \cdot g = D_0 \cdot g$ and hence if $b \in D$ and $f \in \tilde{e}$ one has $b \cdot (g + f) - g = a \cdot g - g + b \cdot f$ for some $a \in D_0$. But then $b \cdot (g + f) - g \in \tilde{e}$ (as above) so that $g + \tilde{e}$ is stable under D.

But now clearly $d \cdot g \subseteq \tilde{e}$. On the other hand one has a natural isomorphism $d \cdot g \cong d/\mathcal{g}_g$. But then $\dim d \cdot g = \dim d/\mathcal{g}_g = \dim \tilde{e}$. Hence $d \cdot g = \tilde{e}$. But $d \cdot g$ is the tangent space at g to the orbit $D_0 \cdot g \subseteq g + \tilde{e}$. Thus $D \cdot g$ is open in $g + \tilde{e}$. QED

We will say that the polarization h satisfies the Pukansky condition (see [4]) if $E \cdot g$ is closed; in which case E is closed and

$$D \cdot g = g + \tilde{e} \quad . \tag{1.2}$$

<u>Lemma 1</u>

If h *satisfies the Pukansky condition then* $D_0 \cap G_g = (G_g)_0$, *the identity component of* G_g. *Furthermore, if* D_1 *is the simply connected covering group to* D_0 *and* $\tau: D_1 \to D_0$ *is the covering map then* $\tau^{-1}((G_g)_0) = (G_g)_1$ is *connected.*

<u>Proof</u>. As a D_0 homogeneous space one has $D \cdot g = D_0 \cdot g \cong D_0/D_0 \cap G_g$. But since $(G_g)_0 \subseteq D_0$ one has that $(G_g)_0$ is the identity component of $D_0 \cap G_g$. However by (1.2) one has that $D_0 \cdot g$ is simply connected so that $D_0 \cap G_g$ is connected. Thus $D_0 \cap G_g = (G_g)_0$. But now also since $D_1/(G_g)_1 \cong D_0/(G_g)_0$ the simple connectivity of $D_0 \cdot g$ implies that $(G_g)_1 = \tau^{-1}((G_g)_0)$ is also connected.

QED

Now g vanishes on $[\mathcal{g}_g, \mathcal{g}]$ so that in particular g vanishes on $[\mathcal{g}_g, \mathcal{g}_g]$ or $g|\mathcal{g}_g$ is a homomorphism $\mathcal{g}_g \to \mathbb{R}$ of Lie algebras. We will say that g is integral if there exists a character $\eta_g: G_g \to \mathbb{T}$ whose differential is

$2\pi i g|\mathcal{g}_g$. That is if for all $x \in \mathcal{g}_g$

$$\frac{d}{dt}\,\eta_g(\exp\, tx)\Big|_{t=0} = 2\pi i\langle g,x\rangle\ .$$

When this is satisfied we will say that η_g corresponds to g.

Remark 5. If G is connected and simply connected one knows that the existence of η_g is equivalent to the integrality of the de Rham class of the canonical symplectic 2-form on the orbit $G \bullet g \subseteq \mathcal{g}'$ (see Kostant, Quantization and Unitary Representations, Part I).

Now since $\langle g,[d,e]\rangle = 0$ then $g|d$ also defines a Lie algebra homomorphism $d \to \mathbb{R}$.

Until otherwise stated we will assume g is integral and η_g is a character on G_g corresponding to g.

Proposition 4

If the Pukansky condition is satisfied then η_g *extends to a unique character*

$$\chi_g\colon D \to \mathbb{T}$$

whose differential is $2\pi i g|d$.

Proof. Now let the notation be as in Lemma 1 so that D_1 is the simply-connected covering group to D_0. Now since $\langle g,[d,d]\rangle = 0$ there exists a unique character $\chi_g^1\colon D_1 \to \mathbb{T}$ whose differential is $2\pi i g|d$. Now if the Pukansky condition is satisfied, then by Lemma 1 $(G_g)_1$ is connected and clearly $\chi_g^1|(G_g)_1 = \eta_g|(G_g)_0 \circ \tau$. But then if Z is the kernel of the covering map $\tau\colon D_1 \to D_0$ one has $Z \subseteq (G_g)_1 = \tau^{-1}((G_g)_0)$ and $\chi_g^1|Z$ is trivial. Hence there exists a unique character $\chi_g^0\colon D_0 \to \mathbb{T}$ such that $\chi_g^1 = \chi_g^0 \circ \tau$. Clearly $2\pi i g|d$ is the differential of χ_g^0.

Now G_g normalizes D_0 and hence G_g operates on the character group of D_0. However, χ_g^0 is invariant under this action since $G_g \bullet g = g$ and hence $G_g \bullet g|d = g|d$ (of course a character on a connected Lie group is determined by its differential). It follows then that if we form the semi-direct product $G_g \times D_0$ then (η_g,χ_g^0) defines a character on this group. However by Lemma 1 $G_g \cap D_0 = (G_g)_0$ and $\eta_g = \chi_g^0$ on $(G_g)_0$ so that (η_g,χ_g^0) is trivial on the kernel K of the surjection $\sigma\colon G_g \times D_0 \to D$ given by $(a,b) \to ab$. Thus (η_g,χ_g^0) is of the form $\chi_g \circ \gamma$ where χ_g is a character on D satisfying the conditions of the proposition. As such it is unique since $D = D_0G_g$ and χ_g is obviously uniquely determined on G_g and D_0. QED

Assume that h is a polarization satisfying the Pukansky condition.

Now let $X = E/D$. Since $E_0 D = E$ it is clear that X is connected. On the other hand since $\hat{B}_g$ is a non-singular alternating bilinear form on e/d which is invariant under the action of D it is clear that X has a measure μ_X invariant under the action of E.

Now consider the space $M(E,\chi_g)$ of all measurable functions ϕ on E such that $\phi(ab) = \chi_g(b)^{-1}\phi(a)$ for all $a \in E$, $b \in D$. Then $M(E,\chi_g)$ is an E-module where if $a \in E$, $\phi \in M(E,\chi_g)$ then $a \cdot \phi \in M(E,\chi_g)$ is given by $(a \cdot \phi)(b) = \phi(a^{-1}b)$. Then if $\mathcal{K}(E,\chi_g)$ is the space of equivalence classes (defined by sets of measure zero) of $\phi \in M(E,\chi_g)$ such that $\|\phi\|^2 = \int |\phi|^2 d\mu_X$ is finite then $\mathcal{K}(E,\chi_g)$ is the Hilbert space associated with the unitary representation $\mathrm{ind}_E\chi$. Since μ_X is an E-invariant measure one has $((\mathrm{ind}_E\chi)(a))\phi = a \cdot \phi$ for $a \in E$, $\phi \in \mathcal{K}(E,\chi_g)$ (conforming to the usual abuse of language).

Now recall $h \cap \bar{h} = d_{\mathbb{C}}$ and $h + \bar{h} = e_{\mathbb{C}}$.

If $C^\infty(E)$ is the space of all C^∞ functions on E we note that $C^\infty(E)$ is a right $e_{\mathbb{C}}$ module where if $z = x + iy \in e_{\mathbb{C}}$ with $x,y \in e$ then if $\phi \in C^\infty(E)$ one puts $\phi \cdot z = \phi \cdot x + i\phi \cdot y$ and if $a \in E$

$$(\phi \cdot x)(a) = \frac{d}{dt}\phi(a \exp - tx)\Big|_{t=0} \quad .$$

Clearly if $\phi \in C^\infty(E)$, $a \in E$, $z \in e_{\mathbb{C}}$ then

$$(a \cdot \phi) \cdot z = a \cdot (\phi \cdot z) \quad . \tag{1.3}$$

Now if $o \in X = E/D$ is the coset D then the tangent space $T_o(X)$ at o may be identified with e/d. Hence upon complexification

$$(T_o(X))_{\mathbb{C}} = e_{\mathbb{C}}/d_{\mathbb{C}} = h/d_{\mathbb{C}} \oplus \bar{h}/d_{\mathbb{C}} \quad .$$

<u>Proposition 5</u>

There is an E*-invariant complex structure on* X *such that* $h/d_{\mathbb{C}}$ *is the space of anti-holomorphic vectors at* o.

<u>Proof</u>. We define a complex distribution F on X such that for any $p \in X$ one has

$$(T_p(X))_{\mathbb{C}} = F_p \oplus \bar{F}_p$$

by putting $F_p = a_*(h/d_{\mathbb{C}})$ where $a \cdot o = p$, $a \in E$. This depends only on p and not on $a \in E$ since $h/d_{\mathbb{C}}$ is invariant under Ad D. Clearly F is E-invariant. By Nirenberg-Newlander, to prove that F_p is the space of anti-holomorphic tangent vectors at p, we have only to prove that F is involutory. That is, if ξ,η are two complex vector fields on X such that $\xi_p,\eta_p \in F_p$ for all X then $\zeta_p \in F_p$ for all $p \in X$ where $\zeta = [\xi,\eta]$. But this condition is purely local. If

$p \in X$ let $U \subseteq X$ be a neighborhood of p with the property that

$$\sigma\colon U \to E$$

is a smooth section of the projection $\pi\colon E \to E/D = X$. Then there exists an open neighborhood V of the identity on D such that the map

$$\tilde{\sigma}\colon U \times V \to W \in E$$

is a diffeomorphism onto an open set $W \subseteq E$ where $\tilde{\sigma}(a,b) = \sigma(a)b$. But let $\tilde{\xi},\tilde{\eta}$ be the complex vector fields on W defined by $\tilde{\xi} = (\tilde{\sigma})_*(\xi,0)$, $\tilde{\eta} = (\tilde{\sigma})_*(\eta,0)$. Clearly $\pi_*\tilde{\xi} = \xi, \pi_*\tilde{\eta} = \eta$. But then if F_h is the left invariant complex distribution on E defined by h, then F_h is involutory since h is a subalgebra (we are in the group case). However, $\tilde{\xi}_a,\tilde{\eta}_a \in (F_h)_a$ for any $a \in W$ since $h = \pi_*^{-1}(h/d_{\mathbb{C}})$. Then $[\tilde{\xi},\tilde{\eta}]_a \in (F_h)_a$ for any $a \in W$. However, $\zeta = \pi_*[\tilde{\xi},\tilde{\eta}]$ since $\tilde{\xi}$ is π-related to ξ, and $\tilde{\eta}$ is π-related to η. Thus $(\zeta)_p \in F_p$ for all $p \in U$. Hence F is involutory. QED

We can now speak of holomorphic functions on any open set $V \subseteq X = E/D$. In fact if

$$\pi\colon E \to X$$

is the quotient map then these are just the elements of $\phi \in C^\infty(V)$ such that, for all $z \in h$,

$$(\phi \circ \pi) \cdot z = 0 \tag{1.4}$$

in $\pi^{-1}(V)$.

Now let $C(E,\chi_g,h)$ be the set of all C^∞ functions ψ in $M(E,\chi_g)$ such that

$$\psi \cdot z = 2\pi i\langle g,z\rangle\psi$$

for all $z \in h$. By (1.3) it is clear that $C(E,\chi_g,h)$ is stable under the action of E and hence if

$$\mathcal{H}(E,\eta_g,h) = C(E,\chi_g,h) \cap \mathcal{H}(E,\chi_g)$$

(abuse of language) then $\mathcal{H}(E,\eta_g,h)$ is stable under $\mathrm{ind}_E\, \chi_g$.

Remark 6. Since χ_g is determined by η_g and h we use η_g in the notation rather than χ_g.

Proposition 6

$\mathcal{H}(E,\eta_g,h)$ is a closed subspace of the Hilbert space $\mathcal{H}(E,\chi_g)$.

Proof. We may assume $\mathcal{H}(E,\eta_g,h) \neq 0$. Let $a \in E$ and $p = \pi a \in X$. Since $\mathcal{H}(E,\eta_g,h) \neq 0$ there exists (by translation if necessary) an element $\psi \in \mathcal{H}(E,\eta_g,h)$ such that $\psi(a) \neq 0$. Let U be an open neighborhood of a with compact closure such that $A > |\psi| > \epsilon > 0$ in U. Let $V = \pi(U) \subseteq X$.

Now if $\beta \in M(E,\chi_g)$ then clearly one has that $\beta = (\phi \circ \pi)\psi$ in U where ϕ is a measurable function on V. Also $\phi \in C^\infty(V)$ if and only if $\beta|U \in C^\infty(U)$. But now $\beta \in \mathcal{H}(E,\eta_g,h)$ so that for $z \in h$ one has $2\pi i\langle g,z\rangle\beta = \beta \cdot z$ $= ((\phi \circ \pi) \cdot z)\psi + (\phi \circ \pi)(\psi \cdot z)$. But also $\psi \cdot z = 2\pi i\langle g,z\rangle\psi$ so that one has $((\phi \circ \pi) \cdot z)\psi = 0$ which implies $(\phi \cdot \pi) \cdot z = 0$. Thus by (1.4) one has ϕ is holomorphic and hence $\beta \mapsto \phi$ defines a map

$$\mathcal{H}(E,\eta_g,h) \to B_0(V)$$

where $(B_0(V))$ is the space of all bounded holomorphic functions in V.

On the other hand (taking U small enough) if $z^1,\cdots,z^m$ are the holomorphic coordinates in V then the measure $i^{m^2} dz_1\wedge\cdots\wedge dz_m\wedge d\bar{z}_1\wedge\cdots\wedge d\bar{z}_m$ is absolutely continuous with bounded (from above and below) Radon-Nikodyn derivative with respect to $\mu_X|V$. But now if β_n is Cauchy in $\mathcal{H}(E,\eta_g,h)$ and $\beta_n = (\phi_n \circ \pi)\psi$ in U where $\phi_n \in B_0(V)$ then clearly $\phi_n dz^1\wedge\cdots\wedge dz^m$ is Cauchy in $B(V)$ using the notation of (Weil, [5], p. 59). Since $B(V)$ is complete (see again Weil, p. 59) it follows that $\phi_n dz^1\wedge\cdots\wedge dz^m \to \rho dz^1\wedge\cdots\wedge dz^n$ in $B(V)$ where ρ is holomorphic in V. But ϕ_n converges to ρ uniformly on compact subsets of V by Proposition 5 in Weil. On the other hand if $\beta_n \to \beta$ in $\mathcal{H}(E,\chi_g)$ where $\beta = (\phi \circ \pi)\psi$ in U for ϕ a measurable function on V one has $\phi_n \to \phi$ almost everywhere. Thus $\phi = \rho$ almost everywhere. But clearly $((\rho \circ \pi)\psi) \cdot z = 2\pi i\langle g,z\rangle(\rho \circ \pi)\psi$ on U for $z \in h$. Thus the equivalence class of β contains an element in $\mathcal{H}(E,\eta_g,h)$ proving that $\mathcal{H}(E,\eta_g,h)$ is complete. QED

Now since $\mathcal{H}(E,\eta_g,h)$ is stable under $\mathrm{ind}_E\,\chi_g$ it defines a subrepresentation $\mathrm{ind}_E(\eta_g,h)$ of $\mathrm{ind}_E\,\chi_g$. But since

$$\mathrm{ind}_G(\mathrm{ind}_E\,\chi_g) = \mathrm{ind}_G\,\chi_g$$

it follows that if

$$\mathrm{ind}_G(\eta_g,h) = \mathrm{ind}_G\,\mathrm{ind}_E(\eta_g,h)$$

then $\mathrm{ind}_G(\eta_g,h)$ is a subrepresentation of $\mathrm{ind}_E\,\chi_g$. We denote the corresponding Hilbert space by $\mathcal{H}(G,\eta_g,h)$.

Remark 7. It is clear that if μ_Z is a G-quasi invariant measure on G/E then $\mathcal{H}(G,\eta_g,h)$ can be taken to be the set of all equivalence classes of measurable functions ϕ on G such that $\phi_a \in \mathcal{H}(E,\eta_g,h)$ for all $a \in G$, and such that

$$\int_Z \|\phi_a\|^2 d\mu_Z(\bar{a}) < \infty$$

where $\phi_a(b) = \phi(ab)$ for $b \in E$ and $\bar{a} \in Z$ is the image of a in Z.

Remark 8. We recall for emphasis that $\mathrm{ind}_G(\eta_g,h)$ is defined when (1) $g \in \mathfrak{g}'$ is integral and (2) h is a polarization satisfying the Pukansky condition. However it may reduce to the zero representation if $\mathcal{H}(E,\eta_g,h)$ reduces to zero. From the point of view of the general quantization theory $\mathrm{ind}_G(\eta_g,h)$ is a "zero cohomology" representation.

2. THE SOLVABLE CASE, EXISTENCE OF ADMISSIBLE POLARIZATIONS

Although one is forced into considering higher cohomology representations in the case where G is semi-simple, L. Auslander and I have shown that the representations of the form $\mathrm{ind}_G(\eta_g,h)$ for a solvable Lie group G of type I are sufficient to give $\hat{G}$, the set of equivalence classes of irreducible unitary representations of G.

More precisely assume G is a solvable simply connected Lie group. Then for one thing we have shown that G is of type I if and only if (1) $g \in \mathfrak{g}'$ are integrable and (2) all orbits $G \cdot g = O \subseteq \mathfrak{g}'$ are the intersections of a closed and open set. Furthermore in such a case we may explicitly give $\hat{G}$.

To do this consider first the maximal nilpotent ideal $\mathfrak{n} \subseteq \mathfrak{g}$. Let $g \in \mathfrak{g}$ and let $f = g|\mathfrak{n} \in \mathfrak{n}'$. Since $\mathfrak{n}$ is stable under Ad G one may consider contragrediently the representation of G on $\mathfrak{n}'$. Let G_f be the isotropy subgroup of G at f. Obviously $G_g \subseteq G_f$ and $\mathfrak{g}_g \subseteq \mathfrak{g}_f$ where $\mathfrak{g}_f$ is the Lie algebra of G_f.

A polarization h at g is called admissible in case (1) it is positive (i.e., the bilinear form S_g on e/d is positive definite) and (2) $h \cap \mathfrak{n}_{\mathbb{C}}$ is stable under G_f and is a polarization at f.

Then the following is proved in [1].

Theorem 1

For any $g \in \mathfrak{g}'$ *whether or not* G *is of type* I *there exists an admissible polarization at* g. *Moreover, any admissible polarization* h *satisfies the Pukansky condition so that if* g *is integrable,* $\mathrm{ind}_G(\eta_g,h)$ *is defined. Furthermore, assuming* g *is integrable then* $\mathrm{ind}_G(\eta_g,h)$ *is independent of the choice of polarizations* h *and if* G *is of type* I *then* $\mathrm{ind}_G(\eta_g,h)$ *is irreducible and every irreducible unitary representation is equivalent to a representation of this form. Finally if* G *is type* I *then* $\mathrm{ind}_G(\eta_g,h)$ *and* $\mathrm{ind}_G(\eta^1_{g_1},h_1)$ *are equivalent if and only if* $G \cdot g = G \cdot g_1$ *and* η_g *corresponds to* $\eta^1_{g_1}$ *under the action of an element* $a \in G$ *such that* $a \cdot g = g_1$.

We cannot go into the proof of this theorem here but we will prove two relevant facts which are needed in the proof. The first of these asserts the independence of the polarization in the nilpotent case. This generalizes a result of

Kirillov who proved a similar theorem for the case of real polarizations, i.e., where $h = \overline{h}$ or $e = d$. One is forced into non-real polarizations by the second fact to be proved. To begin with we need

Theorem 2

Assume that g *is nilpotent,* $0 \neq g \in g'$ *and the polarization* h *at* g *is positive. Let* $b = \mathrm{Ker}\,(g|d)$. *Then* b *is an ideal in* e *and* e/b *is a Heisenberg Lie algebra with* d/b *as the* 1-*dimensional center.*

In particular d *is an ideal in* e *and* e/d *is commutative.*

Proof. If $x \in d$ let $\pi(x) \in \mathrm{End}\ e/d$ be the operator on e/d induced by ad x. Since ad x is nilpotent so is $\pi(x)$. On the other hand the relation $\langle g,[d,e]\rangle = 0$ implies $\langle g,[d[e,e]]\rangle = 0$ since e is an algebra, it follows that $\pi(x)$ is skew-symmetric relative to $\hat{B}_g$. However, $\pi(x)$ obviously commutes with j so that it is skew-symmetric relative to S_g. Thus $\pi(x)$ is both nilpotent and skew-symmetric relative to a positive definite bilinear form. Hence $\pi(x) = 0$ so that d is an ideal in e.

But the relation $\langle g,[d,e]\rangle = 0$ then implies $[d,e] \subseteq b$ so that in particular $[b,e] \subseteq b$. Hence b is also an ideal in e. Furthermore d/b is obviously central in e/b. Also d/b is 1-dimensional since $g \neq 0$ (see Remark 3).

Now to prove that e/b is a Heisenberg Lie algebra with d/b as center, it suffices to show that e/d is abelian and d/b is the center of e/b. But for this it suffices only to show that e/d is abelian. Indeed if this were the case then for $x \in e-d$ one has $[x + b,y + b] \subseteq d/b$ for all $y \in e-d$. But from the non-singularity of $\hat{B}_g$ we can choose y so that $\langle g,[y,x]\rangle \neq 0$. This however implies $[x + b,y + b] = d/b$. Hence d/b is exactly the center of e/b.

We assert that to prove the theorem it suffices only to prove

Lemma 2

The center of e/d *is stable under* j.

Indeed assume Lemma 2 is true and let a be the center of e/d. Now S_g is non-singular on a since S_g is positive definite. But since a is stable under j it follows that $\hat{B}_g$ is also non-singular on a. Let v be the orthogonal complement to a in e/d relative to $\hat{B}_g$. We assert that v is a subalgebra. Indeed if $\hat{y},\hat{z} \in v$ and $\hat{x} \in a$ where $x,y,z \in e$ we must show

$$(\hat{x},[\hat{y},\hat{z}]) = 0 \quad . \tag{2.1}$$

But $(\hat{x},[\hat{y},\hat{z}]) = \langle g,[x[y,z]]\rangle = \langle g,[[x,y]z]\rangle + \langle g,[y,[x,z]]\rangle$. But $[x,y],[x,z] \in d$ since a is central in e/d. But then $[[x,y]z]$ and $[y,[x,z]]$ lie in b since $[d,e] \subseteq b$. This proves (2.1) so that v is a subalgebra. But it is obviously nilpotent so that if $\nu \neq 0$ then center $\nu \neq 0$. However, clearly center $\nu \subseteq$ cent $\mathcal{e}/d = \mathcal{a}$ which is a contradiction. Thus $\nu = 0$ so that $\mathcal{a} = \mathcal{e}/d$ is abelian. We proceed now to the

Proof of Lemma 2. Let $u \in$ center $\mathcal{e}/d$. We must prove ju is central in $\mathcal{e}/d$. Let $v \in \mathcal{e}/d$. We first observe that

$$j[ju,v] = [ju,jv] \quad . \tag{2.2}$$

That is j commutes with ad ju. Indeed $u + iju$ and $v + ijv$ lie in $h/d_{\mathbb{C}}$ and since u is central

$$[u + iju, v + ijv] = -[ju,jv] + i[ju,v] \quad .$$

However since $h/d_{\mathbb{C}}$ is an algebra it follows that $[ju,v] = -j[ju,jv]$. Applying j to both sides yields (2.2). Now let $B = \text{ad } ju$ so the problem is to show that $B = 0$. Let $A = B + B^t$ where superscript t denotes the transpose relative to S_g. Hence $A = A^t$ is a symmetric operator. We next establish the relation

$$\{Av,w\} = \{[jw,v],u\} \tag{2.3}$$

for any $v,w \in \mathcal{e}/d$. Indeed we first observe that for any $z_i \in \mathcal{e}/d$, $i = 1,2,3$ one has

$$([z_1,z_2],z_3) + ([z_2,z_3],z_1) + ([z_3,z_1],z_2) = 0 \quad . \tag{2.4}$$

This of course follows from the relation $([z_1,z_2],z_3) = \langle f,[y_3,[y_1,y_2]]\rangle$ where $y_i \in e$ and $\hat{y}_i = z_i$.

Now $\{Bv,w\} = \{[ju,v],w\} = (j[ju,v],w) = -([ju,v],jw)$ by (1.1). On the other hand $\{B^tv,w\} = \{v,Bw\} = (jv,[ju,w]) = -(v,j[ju,w])$ again by (1.1). But $j[ju,w] = [ju,jw]$ by (2.2) so that $\{B^tv,w\} = -([jw,ju],v)$ since $\tilde{B}_f$ is alternating. Thus

$$\{Av,w\} = -(([ju,v],jw) + (([jw,ju],v)) \quad .$$

Hence $\{Av,w\} = ([v,jw],ju)$ by (2.4). But then $\{Av,w\} = (j[jw,v],u) = \{[jw,v],u\}$ by (1.1) establishing (2.3).

As a consequence of (2.3) note that $Au = 0$ and since A is symmetric one therefore has, by (2.3),

$$0 = (Av,u) = \{[ju,v],u\} \tag{2.5}$$

for all $v \in \mathcal{e}/d$. We now assert that AB is skew-symmetric or that $AB + (AB)^t = 0$. That is since A is symmetric we assert

$$\{ABv,w\} + \{Av,Bw\} = 0 \tag{2.6}$$

for all $v,w \in \mathcal{e}/d$.

Indeed $\{ABv,w\} = \{A[ju,v],w\} = \{[jw,[ju,v]],u\}$ by (2.3) where $[ju,v]$ replaces v. On the other hand $\{Av,Bw\} = \{Av,[ju,w]\} = \{[j[ju,w],v],u\}$ by (2.3) where $[ju,w]$ replaces w. But $j[ju,w] = [ju,jw]$ by (2.2) so that

$$\{(AB + (AB)^t)v,w\} = \{([jw,[ju,v]] + [[ju,jw],v]),u\}$$
$$= \{[ju,[jw,v]],u\} \qquad (2.7)$$

by Jacobi. However, (2.7) vanishes by (2.5) where $[jw,v]$ replaces v. This proves AB is skew-symmetric.

Now $AB = (B + B^t)B = B^2 + B^tB$. But $AB = -(AB)^t = -B^tA = -((B^t)^2 + B^tB)$. Thus $B^2 + B^tB = -(B^t)^2 - B^tB$ or $B^2 + (B^t)^2 = -2B^tB$. Therefore, $A^2 = (B + B^t)^2 = B^2 + (B^t)^2 + BB^t + B^tB = BB^t - B^tB$. But then $\operatorname{tr} A^2 = 0$ since $\operatorname{tr} BB^t = \operatorname{tr} B^tB$. However, since A is symmetric A^2 is positive semi-definite so that $\operatorname{tr} A^2 = 0$ implies $A = 0$. Thus B is skew-symmetric. But B is clearly nilpotent. Hence $B = 0$. QED

One now deduces the following generalization of a result of Kirillov. (See [3]).

Theorem 3

Let G *be any simply connected nilpotent Lie group and let* g *be its Lie algebra. Let* $g \in n'$ *and let* h *be any positive polarization at* g. *Then* $\operatorname{ind}_G(\eta_g, h)$ *is irreducible and up to equivalence is independent of* h.

Proof. (Sketched). It follows from Theorem 2 that $\operatorname{ind}_E(\eta_g, h)$ is just the Bargmann-Segal (see e.g., [2]) holomorphic construction of an irreducible unitary representation of the Heisenberg group E/B. ($B \subseteq E$ is the subgroup corresponding to $b = \operatorname{Ker} g|d$.) One knows therefore that $\operatorname{ind}_E(\eta_g, h)$ is equivalent to $\operatorname{ind}_E \beta_g$ where $B \subseteq K \subseteq E$, K/B is a maximal commutative subgroup of E/B and β_g is the character on K whose differential is $2\pi i g|k$. Here k is the Lie algebra of K. But then $\operatorname{ind}_G(\eta_g, h)$ is equivalent to $\operatorname{ind}_G \beta_g$. However, since K is "half-way" between D and E it is also "half-way" between g_g and g. One thus has that k defines a real polarization at g. By Kirillov's result one knows that $\operatorname{ind}_G \beta_g$ is irreducible and that any real polarization gives rise to an equivalent representation. QED

Now returning to previous notation where g is solvable one is forced into considering complex polarizations of the nil-radical n of g since, in general, there exists no real polarization at $f = g|n$ which is stable under G_f. However, by the next lemma there exists complex polarizations and in fact positive polarizations stable under G_f. Since the commutator group $G' \subseteq N$ where $N \subseteq G$ corresponds to n it follows that $G_f' \subseteq N$ so that the hypothesis of the following lemma is satisfied where $F = G_f$.

Lemma 3

Let N *be a simply connected nilpotent Lie group and let* n *be its Lie algebra. Let* Aut n *be the group of all Lie algebra automorphisms of* n *so that* Ad N *is a subgroup of* Aut n.

Regard Aut n *as operating by contragredience on the dual* n'. *Let* $f \subseteq n'$. *Assume* F *is a group and a homomorphism* $F \to \text{Aut } n$ *(so that* F *operates on* n *and* n'*) such that* (1) *the commutator subgroup* F' *maps into* Ad N *and* (2) $F \cdot f = f$. *Then there exists a positive polarization* h_1 *at* f *which is stable under* F.

Proof. We assume inductively that the result is true for all simply connected nilpotent Lie groups of dimension smaller than dim n.

Let m = Ker f|center n. Assume this space has positive dimension. Clearly m is an ideal in n which is stable under F. Thus F operates on n/m inducing a map $F \to \text{Aut } n/m$ where $F' \to \text{Ad } N/M$ if M is the subgroup corresponding to m. Moreover, if $f_0 \in (n/m)'$ is induced by f then f_0 is fixed by F_0. Now by induction there exists $h_0 \subseteq (n/m)_{\mathbb{C}}$, a positive polarization at f_0 stable under F_0. But then $\pi^{-1}h_0 = h$ is clearly a positive polarization at f stable under F, where $\pi\colon n \to n/m$ is the quotient map (indeed $e = \pi^{-1}e_0$, $d = \pi^{-1}d_0$ and $e/d \cong e_0/d_0$). Thus we are done in this case so that we may assume dim $m = 0$ and hence center n is one-dimensional, spanned by an element z where $\langle f,z\rangle = 1$. Since f is fixed by F clearly z is also fixed under the action of F.

Now consider k = center $n/(z)$ so that $k = k_1/(z)$ where $k_1 \subseteq n$ is an ideal. Clearly Aut n operates on $n/(z)$ and k is clearly stable under the action of this group. However Ad N operates trivially on k since $[n,k_1] \subseteq \mathbb{R}z$. Thus the abelian group F/F' operates on k. Let $p \subseteq k$ be an irreducible subspace under the action of F/F' so that dim p is either 1 or 2. Now since $\langle f,z\rangle = 1$ we may write $k_1 = k_0 \oplus \mathbb{R}z$ where $k_0 = \text{Ker } f|k_1$. Since f is fixed under F and k_1 is stable under F it follows that k_0 is stable under F and that if $\pi\colon n \to n/(z)$ is the quotient map then π induces an F-isomorphism $k_0 \to k$. Let $p_0 \subseteq k_0$ be the F-irreducible subspace corresponding to $p \subseteq k$. Note then that F' must operate trivially on k_0.

Case 1. Assume dim $p_0 = 1$ so that $p_0 = \mathbb{R}w$. In this case we proceed along the lines used by Kirillov. That is, let $g \in n'$ be the linear functional defined by the relation $[y,w] = \langle g,y\rangle z$. One has $g \neq 0$ since otherwise w would be central in n contradicting the fact that center $n = \mathbb{R}z$. Thus there exists $x \in g$ such that $[x,w] = z$ and hence

$$n = \mathbb{R}x \oplus n_0$$

where n_0 = Ker g. But then n_0 is the centralizer of w and hence n_0 is a subalgebra stable under F. However, since n_0 has codimension 1 in n and n is nilpotent, n_0 is an ideal in n. In particular $N = XN_0$ where X and N_0 are the subgroups corresponding to $\mathbb{R}x$ and n_0.

Now the action of F on n_0 induces an epimorphism $F \to F_0 \subseteq \text{Aut } n_0$ where $F' \to F_0'$. However, $F' \to \text{Ad}_n N = \text{Ad}_n X \text{Ad}_n N_0$. But $\text{Ad}_n N_0$ operates trivially on $\mathbb{R}w$ since clearly $w \in$ center n_0. On the other hand F' operates trivially on $w \in P_0$ as observed above. But since $[x,w] = z$ no non-trivial element of $\text{Ad}_n X$ operates trivially on w so we must have $F' \to \text{Ad}_n N_0$ which implies $F_0' \subseteq \text{Ad}_{n_0} N_0$.

Now clearly $f_0 = f|n_0$ is invariant under F_0. Furthermore, we assert that

$$(n_0)_{f_0} = n_f \oplus \mathbb{R}w \quad . \tag{2.8}$$

Indeed $w \in (n_0)_{f_0}$ since $w_0 \in$ center n_0. To see that $n_f \subseteq (n_0)_{f_0}$ we have only to observe that $n_f \subseteq n_0$. But this is clear since otherwise there exists $y \in n_f$ such that $[y,w] = z$. But then

$$1 = \langle f,[y,w]\rangle = -\langle y \cdot f,w\rangle$$

contradicting the fact that $y \cdot f = 0$. Also one has $n_f \cap \mathbb{R}w = 0$ since $\langle w \cdot f,x\rangle = \langle f,[x,w]\rangle = \langle f,z\rangle = 1$. Finally if $y \in (n_0)_{f_0}$ let $c = \langle y \cdot f,x\rangle = \langle f,[x,y]\rangle$. But $\langle cw \cdot f,x\rangle = \langle f,cz\rangle = c$. Thus $\langle (y - cw) \cdot f,x\rangle = 0$. But $(y - cw) \cdot f|n_0 = (y - cw) \cdot f_0 = 0$ since $w \in (n_0)_{f_0}$. But then $y - cw = y_1 \in n_f$ so that $y \in n_f + \mathbb{R}w$. This establishes (2.8).

Now by induction there exists a positive polarization $h_0 \subseteq (n_0)_{\mathbb{C}}$ at f_0 which is stable under F_0. Clearly then one has

$$(n_f)_{\mathbb{C}} \subseteq ((n_0)_{f_0})_{\mathbb{C}} \subseteq h_0 \subseteq (n_0)_{\mathbb{C}} \subseteq n_{\mathbb{C}} \quad .$$

But since h_0 is "half-way" between $((n_0)_{f_0})_{\mathbb{C}}$ and $(n_0)_{\mathbb{C}}$ it is also "half-way" between $(n_f)_{\mathbb{C}}$ and $n_{\mathbb{C}}$ because n_f has codimension 1 in $(n_0)_{f_0}$ and n_0 has codimension 1 in n. Thus, if $h = h_0$ it follows that h is a positive polarization at f which is stable under the action of F.

Now if $\dim p_0 = 2$ we may write $p_0 = \mathbb{R}w_1 \oplus \mathbb{R}w_2$. If we define $g_j \in n'$, $j = 1,2$ by the relation $[y,w_j] = \langle g_j,y\rangle z$ then g_1 and g_2 are linearly independent since otherwise $p_0 \cap$ center $n \neq 0$. But of course $p_0 \cap$ center $n = 0$ since center $n = \mathbb{R}z$.

But then we may find elements x_1, $x_2 \in n$ such that

$$[x_i,w_j] = \delta_{ij}z \quad . \tag{2.9}$$

Clearly then

$$n = \mathbb{R}x_1 \oplus \mathbb{R}x_2 \oplus n_0 \tag{2.10}$$

where $n_0 = \text{Ker } g_1 \cap \text{Ker } g_2$ is the centralizer of the subspace p_0. Since p_0 is stable under F it follows that n_0 is a subalgebra stable under F. In fact since $[n,n]$ annihilates $k_1 \supseteq k_0 \supseteq p_0$, it follows that

$$[n,n] \subseteq n_0 \tag{2.11}$$

and hence n_0 is an ideal in n. The action of F on n_0 induces an epimorphism $F \to F_0 \subseteq \mathrm{Aut}\, n_0$ where F' maps into F_0'. But the map $X_1 \times X_2 \times N_0 \to N$ is bijective where $(a_1,a_2,b) \to a_1a_2b$ and where $N_0 \subseteq N$ is the subgroup corresponding to n and X_j is the subgroup corresponding to $\mathbb{R}x_j$, j = 1,2. But now N_0 operates trivially on $p_0 \subseteq k_0$. But since no non-trivial element of X_1X_2 operates trivially on p_0 by the relations (2.9) it follows that $F' \to \mathrm{Ad}_n N_0$ and hence $F_0' \subseteq \mathrm{Ad}_{n_0} N_0$.

Now let $f_0 = f|n_0$. By induction there exists a positive polarization h_0 at f_0 which is stable under the action of F_0.

As in the case where $\dim p_0 = 1$ one has $[n_f, p_0] = 0$ so that $n_f \subseteq n_0$ and hence

$$n_f \subseteq (n_0)_{f_0} \tag{2.12}$$

Next observe that

$$(n_0)_{f_0} \subseteq n_f + p_0 = n_f \oplus p_0 \tag{2.13}$$

Indeed if $y \in (n_0)_{f_0}$ and c_j, j = 1,2 are defined by $c_j = \langle y \cdot f, x_j \rangle$ then $g = (y - c_1w_1 - c_2w_2) \cdot f$ is orthogonal to $\mathbb{R}x_1 + \mathbb{R}x_2$ by the relations (2.9). However, clearly g is orthogonal to n_0 so that $g = 0$ which implies $y - c_1w_1 - c_2w_2 \in n_f$ and hence $y \in n_f + p_0$. Now $n_f \cap p_0 = 0$ since by the relation (2.9) any non-zero element $w \in p_0$ is such that $z \in \mathrm{Im}\,\mathrm{ad}\, w$. But since $\langle f,z\rangle \neq 0$ this implies $w \notin n_f$. Hence (2.13) is established.

Case 2. Assume $[w_1,w_2] = 0$. Then $p_0 \subseteq n_0$ and hence $p_0 \subseteq$ center n_0 which implies $p_0 \subseteq (n_0)_{f_0}$. Thus by (2.12) and (2.13) one has $(n_0)_{f_0} = n_f \oplus p_0$ so that n_f has codimension 2 in $(n_0)_{f_0}$. Since n_0 has codimension 2 in n this implies that h_0 is "half-way" between $(n_f)_{\mathbb{C}}$ and $n_{\mathbb{C}}$ and hence $h = h_0$ defines a positive polarization at f which is stable under F.

Case 3. Assume $[w_1,w_2] \neq 0$. Now since F' operates trivially on p_0 it follows that F operates, irreducibly, as an abelian group on the 2-dimensional space. The commuting ring in End p_0 is therefore isomorphic to $\mathbb{C}$ and hence w_1 and w_2 may be chosen in p_0 so that $\mathbb{C}u, \mathbb{C}\bar{u} \subseteq (p_0)_{\mathbb{C}}$ are stable under the action of F where $u = w_1 + \sqrt{-1}\, w_2$ and $\bar{u} = w_1 - \sqrt{-1}\, w_2$. Furthermore, it is clear that since they are necessarily independent we may choose w_1, w_2 so that $[w_1,w_2] = z$. But then we may choose x_1 and x_2 so that $x_1 = w_1$, $x_2 = -w_2$ and hence (2.10) becomes

$$\mathbb{R}w_1 \oplus \mathbb{R}w_2 \oplus n_0 = n \quad .$$

But then $p_0 \cap n_0 = 0$ so that, since $n_f \subseteq (n_0)_{f_0} \subseteq n_f + p_0$ by (2.12) and (2.13) one has $n_f = (n_0)_{f_0}$. But then since n_0 has codimension 2 in n, it follows that h_0 fails by one dimension of being a maximum isotropic subspace (m.i.s.) of $n_{\mathbb{C}}$ relative to B_f.

Now put

$$h = h_0 + \mathbb{C}u \quad .$$

Since $h_0 \subseteq (n_0)_{\mathbb{C}}$ and $u \in (p_0)_{\mathbb{C}}$ it follows that $[u,h_0] = 0$ so that not only h is a m.i.s. of $n_{\mathbb{C}}$ but h is a subalgebra stable under the action of F. Also since $n_f \subseteq h$ it follows that h is stable under Ad N_f. But now $h + \overline{h} = h_0 + \overline{h}_0 + \mathbb{C}u + \mathbb{C}\overline{u} = (h_0 + \overline{h}_0) + (p_0)_{\mathbb{C}}$. However, $h_0 + \overline{h}_0$ is a subalgebra since h_0 is a polarization at f_0. But $h_0 + \overline{h}_0 \subseteq (n_0)_{\mathbb{C}}$ and since $[p_0,n_0] = 0$ it follows that $h + \overline{h}$ is a subalgebra since $[(p_0)_{\mathbb{C}},(p_0)_{\mathbb{C}}] = \mathbb{C}z$ and $z \in n_f = (n_0)_{f_0} \subseteq h$. Thus h is a polarization at f. We have only to show that h is positive.

But now since $p_0 \cap n_0 = 0$ one has $d = h \cap n = h_0 \cap n = h_0 \cap n_0 = d_0$. But if $e = (h + \overline{h}) \cap n$ and $e_0 = (h_0 + \overline{h}_0) \cap n = (h_0 + \overline{h}_0) \cap n_0$ then one has

$$e/d = e_0/d_0 \oplus (d_0 \oplus p_0)/d_0 \quad .$$

But this is an orthogonal direct sum relative to both $\hat{B}_f$ and S_f. Indeed this is clear since e_0 and d_0 are orthogonal relative to B_{f_0} and hence relative to B_f. But also $[p_0,e_0] = 0$. Furthermore $(d_0 + p_0)/d_0$ is stable under j since $(p_0)_{\mathbb{C}} = (p_0)_{\mathbb{C}} \cap h \oplus (p_0)_{\mathbb{C}} \cap \overline{h} = \mathbb{C}u \oplus \mathbb{C}\overline{u}$. But by assumption S_f is positive definite on e_0/d_0. However, it is positive definite on $(d_0 + p_0)/d_0$ since if $[w_i] = w_i + d_0, i = 1,2$ one has $j[w_1] = [w_2]$ and $j[w_2] = -[w_1]$. Thus, $\{[w_1],[w_2]\} = 0$ and $\{[w_1],[w_1]\} = (j[w_1],[w_1]) = ([w_2],[w_1]) = \langle f,[w_1,w_2]\rangle = \langle f,z\rangle = 1$. Similarly $\{[w_2],[w_2]\} = 1$. Hence S_f is positive definite. QED

Lemma 3 shows that there exists a positive polarization h_1 at f which is stable under G_f. Now let $e = g|g_f$. We assert there exists a positive polarization h_2 at e (for the identity component $(G_f)_0$ of G_f) which is stable under G_g. To see this one cannot directly apply Lemma 3 since g_f is not necessarily nilpotent. However, if $n_f = g_f \cap n$ and $a = \text{Ker } f|n_f$ then a is ideal in g_f and g_f/a is indeed nilpotent. Furthermore e induces a linear functional on g_f/a and since $G_g' \subseteq N_f$, the subgroup corresponding to n_f, it follows from Lemma 3 that h_2 exists by passing to the quotient g_f/a. But now if we put $h = h_1 + h_2$ then it follows easily that h is an admissible polarization at g. But then we may form $\text{ind}_G(n_g,h)$ giving the most general irreducible unitary representation of a simply connected solvable Lie group of type I.

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