ATION AND SOLUTION OF AN INFINITE-COMPONENT WAVE EQUATION

FOR THE RELATIVISTIC COULOMB PROBLEM

by

I. T. Todorov*


## SUMMARY

he aim of these notes is to give a self-contained exposition of the der-
d solution of an infinite-component wave equation. They cover some of
s of recent work by C. Itzykson, V. Kadyshevsky, and the author [1,2,3].
First we sketch the derivation of a three-dimensional quasi-potential
in momentum space involving integration over the mass-shell hyperboloid
We show that for the relativistic Coulomb potential $V(p,q) = \frac{\alpha}{(p-q)^2}$
iation can be written in an equivalent algebraic form in terms of rational
ns of the generators of a degenerate ("metaplectic") representation of
). The solution of the bound-state eigenvalue problem is carried out by re-
, the representation of SO(4,2) with respect to the irreducible representa-
of its subgroup SO(3) ⊗ SO(2,1) and by an extensive use of the Bargmann
zation of the discrete series of unitary representations of SO(2,1).


* Institute for Advanced Study, Princeton, New Jersey. On leave from Joint Institute for Nuclear Research, Dubna, USSR and from Physical Institute of the Bulgarian Academy of Sciences, Sofia, Bulgaria.

TABLE OF CONTENTS

## INTRODUCTION

This paper consists of three parts. First, I will try to persuade you that the equation we are going to solve has something to do with physics. We will consider a class of relativistic quasi-potential equations for the two-body problem and will single out a simple equation of this class corresponding to the scalar Coulomb interaction. Second, we shall show that our simple equation is equivalent to an infinite-component wave equation written in terms of the generators of a unitary representation of the conformal group SO(4,2). Finally, we shall solve the arising eigenvalue problem by applying some known tools of the theory of representations of the pseudo-unitary group.

In Section 1 we will have to use, without much explanation, some of the physicists' jargon (which is introduced in the first few chapters of any textbook on quantum field theory). The rest of my talk (Sections 2,3) is practically self-contained and does not require any special knowledge of physics.

## 1. QUASI-POTENTIAL EQUATION FOR THE RELATIVISTIC TWO-BODY PROBLEM [1,2,3]

### 1.1 Old-fashioned Perturbation Theory and Feynman-Dyson Rules

We will be concerned in what follows with the scattering and bound-states problems of two relativistic particles.

Let us have two equal-mass particles of initial (4)-momenta $q_1, q_2$ and final momenta $p_1, p_2$. Taking into account the energy-momentum conservation ($p_1 + p_2 = q_1 + q_2$), we can express $p_i$ and $q_i$ in terms of three 4-vectors: the center-of-mass momentum

$$P = p_1 + p_2 = q_1 + q_2 , \tag{1.1}$$

and the relative momenta

$$p = \frac{1}{2}(p_1 - p_2), \quad q = \frac{1}{2}(q_1 - q_2) . \tag{1.2}$$

On the mass-shell, i.e., for $p_1^2 = p_2^2 = q_1^2 = q_2^2 = m^2$ we have the identities

$$pP = qP = 0, \; \frac{1}{4} P^2 + p^2 = \frac{1}{4} P^2 + q^2 = m^2.$$

(We use the system of units for which $c = \hbar = 1$ throughout these notes.) In the framework of quantum field theory, to each particle one usually makes correspond a local field operator. So, we associate with particles 1 and 2 the complex scalar fields $\psi_1(x)$ and $\psi_2(x)$, of mass $m$ and assume that their interaction is given by the local Hamiltonian density

$$\mathcal{H}(x) = -g(:\psi_1^*(x)\psi_1(x):+:\psi_2^*(x)\psi_2(x):)\varphi(x) \; , \tag{1.3}$$

where : : is the sign for the Wick "normal" product

$$:\psi^*(x)\psi(x): = \lim_{y\to 0} [\psi^*(x + y)\psi(x - y) - \langle 0|\psi^*(x + y)\psi(x - y)|0\rangle] ,$$

($|0\rangle$ is the "free vacuum") and $\varphi(x)$ is a hermitian field of mass $\mu$. Then, the scattering amplitude can be written as a (formal) power series in the coupling constant g. There have been two different presentations of this formal expansion: the old-fashioned (non-covariant) perturbation theory and the modern Feynman-Dyson covariant technique. The second one is much more familiar nowadays. Each term of the series is represented in this approach as a sum of multiple integrals corresponding to the so-called Feynman diagrams (see Figure 1).

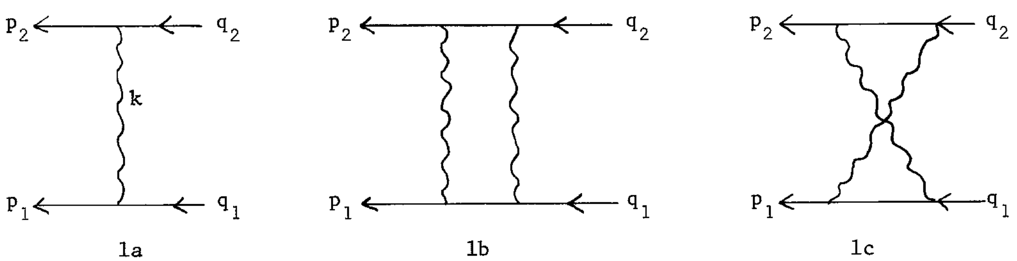


FIGURE 1

An important property of the Feynman rules is that they involve 4-momentum conservation in each vertex of the graph (a factor $g\delta(p + k - q)$ corresponding to a vertex with momentum $q$ on the incoming line and momenta $p$ and $k$ on the outgoing lines). This tempts the physicists to interpret individual Feynman graphs as multiple emission and absorption amplitudes (although, strictly speaking, only the sum of all graphs for a given process has a well-defined physical meaning). Such an interpretation, however, only makes sense for off-mass shell intermediate particles, since, according to the Feynman rules, to an internal (say wavy) line with mass $\mu$ and momentum $k$ corresponds a factor $\frac{1}{\mu^2 - k^2 - i0}$ (integration being carried out subsequently over all 4-dimensional internal momenta $k$), and this factor becomes infinite on the mass shell (i.e., for $k^2 = \mu^2$).

More recently [4] a graphic picture was also given for the old-fashioned perturbation expansion. To describe it, we associate with any Feynman graph with $N$ vertices $N!$ new graphs constructed in the following way. We start with the set of all oriented graphs with the same picture as the original one and with all possible enumerations of the vertices $1, \ldots, N$. Every internal line is oriented toward the vertex with smaller number. Further, we let a spurion (dotted) line enter vertex 1, connect 1 with 2, 2 with 3 and so on (always oriented toward the vertex with larger number), and finally go out of the vertex $N$. For instance, to the second order Feynman graph of Figure 1a correspond the two diagrams of Figure 2.

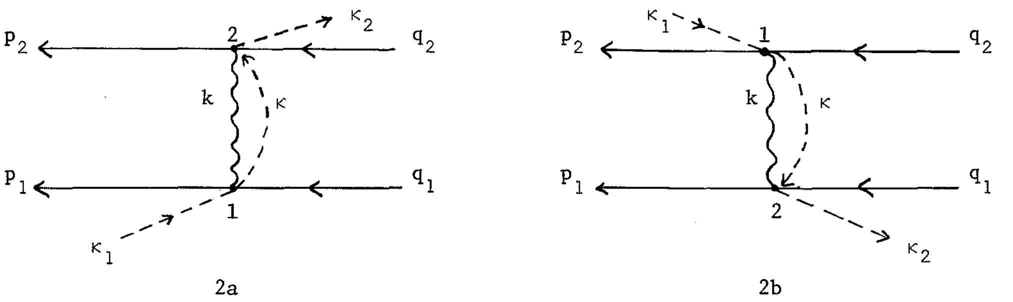


FIGURE 2

Here to the oriented wavy line with mass $\mu$ and momentum $k$ corresponds the "on-mass-shell propagator"

$$\delta^+_\mu(k) = \theta(k_0)\delta(k^2 - \mu^2), \text{ where } \theta(k_0) = \begin{Bmatrix} 1 & \text{for} & k_0 > 0 \\ 0 & \text{for} & k_0 < 0 \end{Bmatrix} . \tag{1.4}$$

However, the energy of the particles (represented by solid lines) is not conserved, the conservation law in each vertex taking into account the energies of the dotted lines. For instance, to vertex 1 of the diagram in Figure 2a corresponds a factor

$$-\frac{g}{\sqrt{2\pi}}\,\delta(q_1 + k - p_1 + (\kappa_1 - \kappa)n) ,$$

where $n$ is a 4-dimensional unit vector in the direction of the time axis. Finally, to an internal dotted line of "energy" $\kappa$ we make correspond the propagator

$$\frac{1}{2\pi}\,\frac{1}{\kappa - i0} . \tag{1.5}$$

Integration is carried out over $\kappa$ from $-\infty$ to $+\infty$ (along with the integration over the internal momenta $k$).

Remark. For those familiar with the formalism of quantum field theory we mention that the splitting of a Feynman graph of $N$ vertices into $N!$ non-covariant graphs (containing dotted lines) corresponds to the decomposition of a time-ordered product of $N$ local operators $H(x_1) \ldots H(x_N)$ into $N!$ ordinary products (with appropriate $\theta$-functions). On the energy shell, i.e., for $\kappa_1 = \kappa_2 = 0$, the sum of the contributions of these $N!$ graphs coincides with the (on-mass-shell) contribution of the original Feynman graph.

Example. The contribution from the two diagrams of Figure 2 is

$$\frac{1}{(2\pi)^2}\,\delta(p_1 + p_2 - q_1 - q_2 + (\kappa_2 - \kappa_1)n)T^{(2)} , \tag{1.6}$$

where

$$T^{(2)} = \frac{g^2}{2}\left(\frac{1}{\omega_{p_1-q_1}}\,\frac{1}{\kappa_1 + q_1^0 - p_1^0 + \omega_{p_1-q_1} - i0} + \frac{1}{\omega_{p_2-q_2}}\,\frac{1}{\kappa_1 + q_2^0 - p_2^0 + \omega_{p_2-q_2} - i0}\right), \tag{1.7}$$

where $\omega_k = \sqrt{\mu^2 + \underline{k}^2}$ . On the energy shell, for $\kappa_1 = \kappa_2 = 0$, $q_1 - p_1 = p_2 - q_2$ the right-hand side of Equation (1.7) reduces to the covariant Feynman rule for the on-shell amplitude $\underline{T}$:

$$\underline{T}^{(2)} = \frac{g^2}{\omega^2_{p_1-q_1} - (p_1^0 - q_1^0)^2 - i0} = \frac{g^2}{\mu^2 - (p_1 - q_1)^2 - i0} .$$

## 1.2. Off-mass-shell Bethe-Salpeter Equation and Off-energy-shell Quasi-potential Equation for the Scattering Amplitude

Two types of linear equations for the scattering amplitude have been considered corresponding to the two types of expansions discussed in the previous section. Historically, the first one is the Bethe-Salpeter (B-S) equation which was, actually, first proposed by Nambu (1950) (for a complete bibliography on the B-S equation see the recent review article [5]). It is an off-mass-shell equation which originates from the Feynman-Dyson rules. In order to write it down we need the notions of the "complete Feynman propagator" $\Delta'_F(p)$ and of the sum of all $\psi_1 + \psi_2$ irreducible graphs $I_p(p,q)$.

The complete (sometimes also called modified) Feynman propagator $\Delta'_F(p)$ is defined as the sum of the contributions of all Feynman graphs to the two point Green's function (see Figure 3).

FIGURE 3

$$\Delta'_F(p) = \frac{1}{m^2 - p^2 - i0} + \frac{g^2}{(2\pi)^4}\int_{(m+\mu)^2}^{\infty} \frac{f(x)\,dx}{(x - p^2)^2(x - p^2 - i0)} + O(g^4) , \tag{1.8}$$

where $f$ is defined by the phase-space integral

$$\frac{1}{\pi}\int \delta_m^+(p - k)\delta_\mu^+(k)d^4k = \theta(p_0)\theta(p^2 - (m + \mu)^2)f(p^2) ,$$

$$f(x) = \frac{1}{8x}\,[x^2 - 2(m^2 + \mu^2)x + (m^2 - \mu^2)^2]^{1/2} . \tag{1.9}$$

*Remark.* The graphs in Figure 3 correspond in general to divergent integrals (this is for instance the case with the second order term whose contribution is written explicitly in (1.8)). We choose the renormalization in such a way that the regularized integrals vanish for $p^2 = m^2$ together with their first derivatives. This permits cancellation of the pole terms $\frac{1}{(m^2 - p^2 - i0)^2}$ coming from the two external lines in all graphs of Figure 3 except the first one. Hence, according to our definition, only the first term in the expansion (1.8) has a pole-type singularity for $p^2 = m^2$.

A connected diagram D of the $\psi_1 + \psi_2$ (elastic) scattering process is called reducible (or more specifically $\psi_1 + \psi_2$-reducible) if it can be decomposed into two graphs D' and D" of the same process connected by one $\psi_1$ and one $\psi_2$ lines such that D' contains both incoming lines of D (with momenta $q_1, q_2$) and D" contains both outgoing lines of D (with momenta $p_1, p_2$) (see Figure 4).

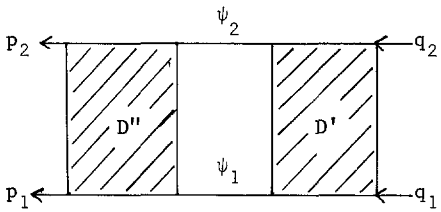


FIGURE 4

Otherwise, if this is not possible, the diagram is called $\psi_1 + \psi_2$-irreducible. According to this definition the graph shown in Figure 1b is reducible while the graphs of Figures 1a and 1c are irreducible. We denote the sum of the contributions of all irreducible graphs by $I_P(p,q)\delta(p_1 + p_2 - q_1 - q_2)$.

Let $T_P(p,q)$ be the off-mass-shell $\psi_1 + \psi_2$-scattering amplitude (in other words let $T_P(p,q)\delta(p_1 + p_2 - q_1 - q_2)$ be the sum of all connected Feynman graphs of the $\psi_1 + \psi_2$-elastic scattering without radiative corrections on the external lines). Then the B-S equation can be written in the form

$$T_P(p,q) = I_P(p,q) - \frac{i}{(2\pi)^2}\int I_P(p,k)\Delta_F'(\tfrac{1}{2}P + k)\Delta_F'(\tfrac{1}{2}P - k)T_P(k,q)d^4k \ . \qquad (1.10)$$

It can be checked directly that the iterative solution of Equation (1.10) coincides with the sum of all Feynman graphs for $T_P$. Equation (1.10) is a source of nontrivial approximations for $T_P$. Even if we restrict ourselves to the first terms of the expansions in $g^2$ for $\Delta_F'$ and $T_P$ we find that the solution of (1.10) has g dependent poles as a function of $P^2$ which never occurs in any finite order in perturbation theory. These poles are interpreted as the squares of the masses of the two-particle bound states.* They coincide with the eigenvalues of $P^2$ for which

* It should be realized that such an interpretation is not a consequence of the principles of quantum field theory. We shall discuss below the advantages of an alternative definition of the bound-state energy eigenvalues.

the homogeneous equation

$$[\Delta_F'(\frac{1}{2}P + p)\Delta_F'(\frac{1}{2}P - p)]^{-1}\Phi_P(p) = \frac{-i}{(2\pi)^2}\int I_P(p,k)\Phi_P(k)d^4k \ , \qquad (1.11)$$

(corresponding to (1.10)) has a non-trivial solution satisfying certain boundary conditions.

Equation (1.11) has a number of undesirable features as compared to the non-relativistic Schrödinger equation (for a concise discussion of the diseases of the B-S equation see the elegant paper by Wick [6]). First of all, it involves a fourth coordinate--the relative energy $p_0(k_0)$ (or the relative time in the original B-S formulation), which does not have a clear physical meaning. Its presence makes obscure the non-relativistic limit of the B-S equation and leads to extra (unphysical) solutions, the energy eigenvalues $(W^2 = P^2)$ being labeled by one more quantum number than in the Schrödinger equation. This point is clarified by the Wick-Cutkosky model [6]--the only exactly solvable example of the B-S equation we know. (In this example $[\Delta_F'(k)]^{-1}$ is replaced by $(\Delta_F(k))^{-1} = m^2 - k^2$ and $I_P(p,q)$ is given by (minus) the scalar Coulomb potential $I_P(p,q) = \frac{g^2}{-(p-q)^2}$ .) If $g^2$ belongs to a certain interval it has been shown that some extra energy eigenvalues do in fact appear (for more details see Reference [3]). In the lowest order approximation with respect to the coupling constant $g$ (which has only been considered in practice) the operator on the left-hand side of (1.11) is a fourth-order polynomial in $p$ (i.e., a fourth-order differential operator in coordinate space). This is another source of extra solutions of the B-S equation. No probabilistic interpretation is possible for the wave-function $\Phi$, since it is not normalizable.

The three-dimensional "quasi-potential" approach to the two-particle bound state problem, based on the off-energy shell old-fashioned perturbation theory (see [7,1]), seems free of all these difficulties of the B-S equation and our further discussion will be based on it.

First of all, we choose the unit vector $n$, of the time axis (which appeared in the formula of the old-fashioned perturbation theory) along the center of mass momentum $P$. In this frame, taking into account the conservation law

$$p_1 + p_2 - \kappa_1 n = q_1 + q_2 - \kappa_2 n \ , \qquad (1.12)$$

(see (1.6)) and the mass-shell condition

$$p_1^2 = p_2^2 = q_1^2 = q_2^2 = m^2 \ , \qquad (1.13)$$

we can write

$$\underline{p}_1 = -\underline{p}_2 = \underline{p}, \ \ \underline{q}_1 = -\underline{q}_2 = \underline{q}, \ \ p_1^0 = p_2^0 = p^0, \ \ q_1^0 = q_2^0 = q^0 \ ;$$

$$|p_0| = E_{\underline{p}} = \sqrt{m^2 + \underline{p}^2} \ , \ \ p_0 - 1/2\,\kappa_1 = q_0 - 1/2\,\kappa_2 \equiv E \ . \qquad (1.14)$$

Further, we introduce the notion of an irreducible graph in the Kadyshevsky diagram technique. We call a graph D, corresponding to the old-fashioned perturbation expansion of the $\psi_1\psi_2$-elastic scattering amplitude, irreducible if it cannot be split into two solid-line connected diagrams $D_1$ and $D_2$ in the way shown on Figure 5.

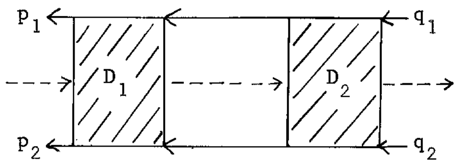


FIGURE 5. Reducible Graph

We denote the sum of all irreducible graphs (which do not contain radiative corrections on external lines) by

$$-V_E(p,q)\delta(p_1 + p_2 - \kappa_1 n - q_1 - q_2 + \kappa_2 n) \ .$$

Finally we define the total Green's function

$$2E_k G_E(k_0)\delta(\kappa_1 - \kappa_2)\delta(\underline{k} - \underline{k}') \ ,$$

as the sum of all solid line disconnected self-energy diagrams of the $(\psi_1\psi_2)$-scattering amplitude with the following property: the line $\psi_1$ with (incoming) momentum $k = (k_0,\underline{k})$ (and with all possible radiative corrections) may be connected with the line $\psi_2$ with (incoming) momentum $(k_0,-\underline{k})$ only by a dotted line. The first two terms in the expansion of $G_E(k_0)$ (with respect to $g^2$) are given by

$$2E_k G_E^{(2)}(k_0) = \frac{1}{4\pi}\left\{\frac{1}{k_0 - E - i0} + \left(\frac{g}{2\pi}\right)^2 \int_{x_0(k_0)}^{\infty} f(x^2 + m^2 - k_0^2)\,\frac{(x + 2k_0 - 2E)(k_0^2 + x) + xk_0^2}{(x^2 - k_0^2)^2[(\frac{1}{2}x + k_0 - E - i0)^2 - k_0^2]}\,dx\right\} , \tag{1.15}$$

where $x_0(k_0) = [(m + \mu)^2 + \underline{k}^2]^{1/2} = (2m\mu + \mu^2 + k_0^2)^{1/2}$ and f is defined by (1.9). The solid-line connected off-energy-shell scattering amplitude $T_E(p,q)$ (without radiative corrections on the external lines) satisfies the "quasi-potential" equation

$$T_E(p,q) + V_E(p,q) + \int V_E(p,k)G_E(k_0)T_E(k,q)\delta_m^+(k)d^4k = 0 \ . \tag{1.16}$$

In order to obtain the corresponding homogeneous equation we assume that there exists an r-fold degenerate ($r \geq 1$) bound state of mass $2B < 2m$ in the $\psi_1\psi_2$-system. Furthermore, in analogy with the Bethe-Salpeter equation we postulate that the scattering amplitude $T_E(p,q)$ has a simple pole for $E = B$. In the neighborhood of this pole we put

$$G_E(p_0)T_E(p,q)G_E(q_0) = \frac{1}{4\pi}\sum_{a=1}^{r}\frac{\phi_{Ba}(p)\overline{\phi}_{Ba}(q)}{B - E - i0} + \text{regular terms for } E \rightarrow B \ , \tag{1.17}$$

where $\phi_{Ba}(p)$ will be interpreted as the wave function of the bound state of mass 2B and other quantum numbers specified by a. Inserting (1.17) in Equation (1.16) and comparing the residues for the pole $E = B$, we obtain

$$\sum_{a=1}^{r} [\phi_{Ba}(p) + G_B(p_0)\int V_B(p,k)\phi_{Ba}(k)\delta_m^+(k)d^4k]\overline{\phi}_{Ba}(q) = 0 .$$

Taking into account that $\overline{\phi}_{Ba}(q)$ are linearly independent we find the following homogeneous equation for each of the wave functions $\phi_B(p)$:

$$[G_B(p_0)]^{-1}\phi_B(p) + \int V_B(p,k)\phi_B(k)\delta_m^+(k)d^4k = 0 . \tag{1.18}$$

The normalization condition for $\phi_B$ may be also obtained from Equation (1.16) by first applying to both sides the integral operator

$$(KF)(p) = \int T_E(p,p')G_E(p_0')F(p')\delta_m^+(p')d^4p'$$

and then inserting (1.17) and comparing the residues for $E = B$. The result is [8]:

$$\iint \overline{\phi}_{Ba}(k_1)\left\{-\frac{\partial}{\partial B}\left[\frac{\partial}{\partial B}\left(G_B(k_{10}\right)^{-1}2E_{k1}\delta(\underline{k}_1 - \underline{k}_2) + V_B(\underline{k}_1,\underline{k}_2)\right]\right\} \phi_{Bb}(k_2)\delta_m^+(k_1)\delta_m^+(k_2)d^4k_1d^4k_2 = \delta_{ab} . \tag{1.19}$$

Equation (1.18) does not have the defects of the Bethe-Salpeter equation discussed above. In particular, it has a straightforward (and transparent) non-relativistic limit.

## 1.3. A Simple Model: The Scalar Coulomb Problem

In the lowest order in $g$ the bound-state Equation (1.18) has the form

$$p_0(E - p_0)\phi_E(p) = \frac{1}{8\pi}\int V_E^{(2)}(p,k)\phi_E(k)\delta_m^+(k)d^4k , \tag{1.20}$$

where according to (1.7),

$$V_E^{(2)}(p,q) = \frac{g^2}{\omega_{p-q}(2E - p_0 - q_0 - \omega_{p-q} + i0)} , \tag{1.21}$$

$$\omega_k = \sqrt{\mu^2 + \underline{k}^2} .$$

The "potential" (1.21) is quite complicated so that Equation (1.20) does not allow an exact solution even in the limit of zero-mass exchange ($\mu = 0$). In what follows we shall study the model equation in which $V_E^{(2)}$ is replaced by the relativistic scalar Coulomb potential

$$V(p,q) = \frac{g^2}{(p - q)^2} \tag{1.22}$$

and the integration is carried over the two-sheeted hyperboloid $k^2 = m^2$ ($\theta(k_0)$ being replaced by $\varepsilon(k_0) = \theta(k_0) - \theta(-k_0)$ in the right-hand side of Equation (1.19)).

Let us make a few remarks about the place of this model in the study of the relativistic two-body problem.

Originally, back in 1963, Logunov and Tavkhelidze [9] have postulated the following three dimensional quasi-potential equation

$$T_{E_q}(\underline{p},\underline{q}) + V_{E_q}(\underline{p},\underline{q}) + \frac{1}{4\pi}\int V_{E_q}(\underline{p},\underline{k}) \frac{T_{E_k}(\underline{k},\underline{q})}{E_k^2 - (E_q + i0)^2} \frac{d^3k}{2E_k} = 0 \tag{1.23}$$

(we have changed the sign convention for V adopted in Reference [9] in order to be consistent with the non-relativistic limit for the potential). This equation differs from our Equation (1.16) both in the Green's function and in the potential (the second order off-shell amplitude and potential being defined by

$$T_{E_q}^{(2)}(\underline{p},\underline{q}) = -V_{E_q}^{(2)}(\underline{p},\underline{q}) = \frac{g^{(2)}}{\mu^2 + (\underline{p} - \underline{q})^2} \tag{1.24}$$

in [9]). However, the perturbative solutions of both Equations (1.16) and (1.23) coincide on the energy shell provided that we put the exact expressions for $G_E$ and $V_E$ (i.e., the sum of all irreducible graphs in our case), reproducing in both cases the on-mass-shell Feynman rules. The non-uniqueness of the quasi-potential equation originates in the non-uniqueness of the off-energy-shell extrapolation of the scattering amplitude. There exists in fact an infinite family of three dimensional equations of the type

$$T + V + VGT = 0 \tag{1.25}$$

which give the same on-shell amplitude and which ensure the elastic unitarity condition

$$T - T^* = T(G - G^*)T^* \tag{1.26}$$

for Hermitian potentials V. It is easy to see that our model equation with Green's function $G_E^{(0)} = [8\pi E_k(k_0 - E - i0)]^{-1}$ and potential (1.22) can be obtained in second order from an equation of this family (it is sufficient to check that on the energy shell, i.e., for $p_0 = q_0 = E_p = E$, the "relativistic Coulomb potential" (1.22) coincides with (1.21) and (1.24) for $\mu = 0$, and that the Green's functions of Equations (1.17) and (1.23) have the same discontinuity $G_E - G_E^*$). At the same time (1.22) provides a natural generalization of the non-relativistic Coulomb potential. The main approximation to the real electromagnetic interaction of two charged particles consists in the replacement of the vector potential (which gives rise to an angular momentum dependence of the energy eigenvalues) with a scalar potential (this is known to lead to an error of the order of $10^{-4}$). Another model equation of the same class (with $E_k$ replaced by E in $G_E^{(0)}$) is considered in [1].

## 2. ALGEBRAIZATION OF THE RELATIVISTIC COULOMB PROBLEM

### 2.1. Introductory Remarks

We shall deal from now on with Equations (1.20), (1.22). Noting that the coupling constant g has the dimension of mass and that m is the only mass in the Coulomb problem we introduce dimensionless variables by

$$\frac{1}{m^2} g^2 = \frac{2}{\pi}\alpha,\ \frac{1}{m} p \to p,\ \frac{1}{m} k \to k,\ \frac{1}{m} E \to E\ . \tag{2.1}$$

In these variables our quasi-potential equation assumes the form

$$p_0(E - p_0)\phi_E(P) = \frac{\alpha}{(2\pi)^2} \int \frac{\phi_E(k)}{(p - k)^2} \varepsilon(k_0)\delta(k^2 - 1)d^4k \ . \tag{2.2}$$

We are looking for the eigenvalues of $E$, for which Equation (2.2) has a non-trivial solution. Our first step to the solution of this problem will be its "algebraization". We will show that the free-particle energy operator $p_0$ and the integral operator on the right-hand side of Equation (2.2) can be expressed as simple rational functions of the generators of certain unitary representation of the conformal group $SO(4,2)$. A similar algebraization has been carried out for the Bethe-Salpeter equation (in terms of the generators of $SO(5,2)$) in Reference [10]. Before going into the technical details we would like to make a comment about the meaning of this step.

The advantage of the algebraic form of an equation is in its independence of the realization of the algebra under consideration. The representation of a given Lie algebra is specified by a set of identities in its enveloping algebra. It may have many different (though unitarily equivalent) realizations. The choice of the most appropriate realization for the given equation is suggested by the symmetry of the problem which is most easily seen in its algebraic, i.e., realization-independent formulation. A famous example of an algebraic presentation of a physical theory is the Dirac formulation of non-relativistic quantum mechanics which is given in terms of the generators $p$ and $q$ of the Heisenberg algebra. Some special problems of high symmetry such as the harmonic oscillator can be solved directly in the invariant formulation. For many others the algebraic picture, being the most flexible one, suggests a convenient choice of coordinates.

We will start with a brief description of the conformal group and of the peculiar degenerate unitary representation we are going to use.

## 2.2. A Remarkable Representation of the Conformal Group

The conformal group $SO(4,2)$ can be defined as the set of pseudo-orthogonal transformation in six dimensions which preserves a non-degenerate real symmetric quadratic form, $xgx$, with signature (2,4). For an appropriate choice of the basis we can write

$$xgx = x_A g^{AB} x_B = x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_5^2 + x_6^2 \tag{2.3}$$

(in order to be consistent with traditional notation in physics (where often $x_4 = ix_0$ is used) we omit the index 4 in labeling $x_A$ and $g^{AB}$). We will be interested actually in the restricted conformal group which consists of the connected component of the identity element of $SO(4,2)$ and is denoted by $SO_0(4,2)$.

The Lie algebra of $SO_0(4,2)$ is generated by the infinitesimal rotations $i\Gamma_{AB}$ (in the AB plane). They form an antisymmetric tensor ($\Gamma_{AB} = -\Gamma_{BA}$) with 15 independent components satisfying the commutation relations

$$[\Gamma_{AB},\Gamma_{CD}] = i(g_{AD}\Gamma_{BC} + g_{BC}\Gamma_{AD} - g_{AC}\Gamma_{BD} - g_{BD}\Gamma_{AC}) \ . \tag{2.4}$$

The lowest faithful representation of this Lie algebra is 4-dimensional and is given by the set of Dirac $\gamma$-matrices:

$$\Gamma_{a6} \to \gamma_{a6} = \frac{1}{2}\gamma_a, \ \Gamma_{ab} \to \gamma_{ab} = \frac{i}{4}[\gamma_a,\gamma_b], \ a, b = 0, 1, 2, 3, 5 \tag{2.5}$$

where $\gamma_a$ satisfy the identity

$$\{\gamma_a,\gamma_b\} \equiv \gamma_a\gamma_b + \gamma_b\gamma_a = 2g_{ab} \cdot \mathbb{1} \ . \tag{2.6}$$

The $\gamma$'s are in fact the generators of the defining representation of the pseudo unitary group SU(2,2) which is a two-fold covering group of $SO_0(4,2)$. In other words there exists a hermitian matrix $\beta$ with two positive and two negative eigenvalues such that

$$\beta\gamma_\mu = \gamma_\mu^*\beta \ . \tag{2.7}$$

We will also use the notation $\Gamma_a$ for $\Gamma_{a6}$.

Now we are going to describe the particular irreducible unitary representation $R_0$ of $SO_0(4,2)$ which we will use for the algebraization of Equation (2.2). This representation has been used for many years by physicists but has been usually omitted in the mathematical classification of the unitary representations of the pseudo unitary (or of the pseudo orthogonal) group (see, however, References [11,12] where the place of the "ladder" representations of SU(2,2) is indicated). The representation $R_0$ is characterized by the following properties: (i) it remains irreducible when restricted to any of the five-dimensional rotation subgroups $SO_0(3,2)$ and $SO_0(4,1)$ of $SO_0(4,2)$ as well as to its Poincaré subgroup; (ii) when restricted to the subgroup SO(4) the representation $R_0$ splits into the direct sum of tensor representations* $\bigoplus_{n=1}^{\infty}(n,n)$ each $(n,n)$ appearing with multiplicity one; (iii) the $n^2$-dimensional subspace $\mathcal{H}(n,n)$ (in which acts the representation $(n,n)$ of SO(4) is an eigen subspace for the generator $\Gamma_0(=\Gamma_{06})$ of the subgroup SO(2) which commutes with SO(4): $f_n \in \mathcal{H}(n,n) \Rightarrow \Gamma_0 f_n = nf_n$.

We will describe here a particular realization of the representation $R_0$ on the space $\mathcal{H}$ of functions $\phi(p)$ defined on the double sheeted hyperboloid $V_1 = \{p: p^2 = 1\}$

$$(\phi,\phi) \equiv \frac{1}{\pi^4}\iint \frac{\overline{\phi(p)}\phi(q)}{-(p-q)^2}\,\delta(p^2-1)\delta(q^2-1)d^4p\,d^4q < \infty \tag{2.8}$$

(cf. [13]).

First of all we introduce homogeneous coordinates on $V_1$:

$$p_\mu = \frac{u_\mu}{u_5}, \quad q_\mu = \frac{v_\mu}{v_5} \tag{2.9}$$

---

* We recall that the group SO(4) is locally isomorphic to the direct product SU(2) ⊗ SU(2). Accordingly, each (unitary, irreducible) representation of SO(4) can be characterized by two integers (k,l) equal to the dimensions of the corresponding representations of the two groups SU(2).

and consider $\phi(p)$ as a restriction to the manifold $\{u = (|p_0|, \underline{p} \in (p_0)), p \in V_1\}$ of a homogeneous function $F(u)$ of degree of homogeneity-2 defined on the light-cone $C^+_{1,4}$

$$F(\lambda u) = \lambda^{-2} F(u) \quad \text{for} \quad \lambda > 0,\ u \in C^+_{1,4} \tag{2.10}$$

$$C^+_{1,4} = \{u\colon u_0 = |\vec{u}| \equiv \sqrt{u_1^2 + u_2^2 + u_3^2 + u_5^2}\} . \tag{2.11}$$

Taking into account that

$$-(p - q)^2 = 2 \frac{uv}{u_5 v_5} \quad (uv \equiv u_0 v_0 - \vec{u}\vec{v})$$

we find that the scalar product (2.8) assumes the form

$$\begin{gathered}(F,F)_{-2} = \frac{1}{2\pi^4} \iint \overline{F}(u) \frac{1}{uv} F(v) \delta(u_0 - 1) \delta(v_0 - 1) \delta(u^2) \delta(v^2) d^5u d^5v \\ [\text{for} \quad \phi(p) = F(|p_0|, \underline{p}\ \varepsilon(p_0)) \quad \text{or} \quad F(u) = u_5^{-2} \phi(\frac{u_0}{u_5}, \frac{\underline{u}}{u_5})] .\end{gathered} \tag{2.12}$$

The restriction of the representation $R_0$ on the $SO_0(4,1)$ subgroup of $SO_0(4,2)$ is defined as a set of argument transformation

$$SO_0(4,1) \ni \Lambda \to [U(\Lambda)F](u) = F(\Lambda^{-1}u) . \tag{2.13}$$

That is the Majorana representation of the complementary series of unitary representations of $SO_0(4,1)$, i.e., the only representation of the complementary series which can be extended to a representation of $SO_0(4,2)$. To see this we first remark that the representation (2.13) in the space $\mathcal{H}_{-2}$ with scalar product (2.12) is equivalent to the representation given by the same formula (2.13) in the space $\mathcal{H}_{-1}$ of homogeneous functions of degree of homogeneity -1, equipped with scalar product

$$(F,G)_{-1} = \frac{-1}{2\pi^4} \int \overline{F(u)} \frac{1}{(uv)^2} G(v) \delta(u_0 - 1) \delta(v_0 - 1) \delta(u^2) \delta(v^2) d^5u d^5v . \tag{2.14}$$

We mention that the integral in (2.14) is in general divergent because of the singularity for $u = v$. It has to be defined by analytic continuation with respect to $N$ of the hermitian form $(F,G)_N$ (in which $-\frac{1}{2\pi^4}(uv)^{-2}$ in the integrand is replaced by $\frac{2^{N+1}\Gamma(-N)}{\pi^{7/2}\Gamma(-N - \frac{3}{2})}(uv)^{-3-N}$) (cf. [14]). The scalar product defined through this analytic continuation is positive-definite if and only if $N(N + 3) < 0$. The normalization is chosen in such a way that $(u_0^N, u_0^N)_N = 1$ ($F = u_0^N$ is the only $SO(4)$ invariant vector in $\mathcal{H}_N$ (up to a factor)). The intertwining operator $T$ which maps $\mathcal{H}_{-1}$ onto $\mathcal{H}_{-2}$ and its inverse are given by

$$\begin{aligned}(TF)(u) &= \frac{-1}{2\pi^2} \int F(v) \delta(v_0 - 1) \delta(v^2) \frac{d^5v}{(uv)^2} , \\ (T^{-1}F)(v) &= \frac{1}{2\pi^2} \int F(u) \delta(u_0 - 1) \delta(u^2) \frac{d^5u}{uv} .\end{aligned} \tag{2.15}$$

The action of the five additional generators $\Gamma_a$ $(a = 0,1,2,3,5)$ of the Lie algebra of $SO_0(4,2)$ in the space $\mathcal{H}_{-2}$ is defined by

$$(\Gamma_a F)(u) = [T(u_a F)](u) = \frac{-1}{2\pi^2} \int \frac{v_a}{(uv)^2} F(v)\delta(v_0 - 1)\delta(v^2)d^5v \,. \tag{2.16}$$

It can be verified by a straightforward computation that these operators satisfy (together with the generators $\Gamma_{ab}$ of $SO_0(4,1)$) the commutation relations (2.4). In particular,

$$i[\Gamma_a,\Gamma_b] = \Gamma_{ab} = \begin{cases} -i(u_a \frac{\partial}{\partial u_b} - u_b \frac{\partial}{\partial u_a}) & \text{for } a,b = 1,2,3,5 \\ -iu_0 \frac{\partial}{\partial u_b} & \text{for } a = 0,\ b = 1,2,3,5 \,. \end{cases} \tag{2.17}$$

It is easily seen also that the operators (2.16) are hermitian with respect to the scalar product (2.12). Some further property of the representation $\mathfrak{R}_0$ are given in the Appendix. (In particular, we show that $\mathfrak{R}_0$ defined so far as a representation of the Lie algebra of $SO_0(4,2)$ can be in fact integrated to a representation of the group; the global form of the representation coincides with the familiar realization of the conformal group in space-time which leaves invariant the D'Alembert equation $\Box f(x) = (\frac{\partial^2}{\partial x_0^2} - \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2})f(x) = 0.$)

## 2.3. Algebraic Form of Equation (2.2)

In the space $\mathcal{H}$ of functions $\phi(p)$ the operators $\Gamma_a$ (2.16) assume the form

$$(\Gamma_\mu \phi)(p) = -\frac{2}{\pi^2} \int \frac{q_\mu}{[(p-q)^2]^2} \phi(q)\varepsilon(q_0)\delta(q_0 - 1)d^4q \tag{2.18}$$

$$(\Gamma_5 \phi)(p) = -\frac{2}{\pi^2} \int \frac{1}{[(p-q)^2]^2} \phi(q)\varepsilon(q_0)\delta(q_0 - 1)d^4q \,. \tag{2.19}$$

Comparing (2.18) with (2.19) we see that

$$(p_\mu \phi)(p) = (\frac{1}{\Gamma_5} \Gamma_\mu \phi)(p) \,. \tag{2.20}$$

Taking into account that for any analytic function $F$ of $\Gamma_5$ we have

$$F(\Gamma_5)(\Gamma_\mu \pm \Gamma_{\mu 5}) = (\Gamma_\mu \pm \Gamma_{\mu 5})F(\Gamma_5 \pm i) \tag{2.21}$$

and using Equations (C.9), (C.10) (see Appendix C) we can verify that for $\lambda = 0$ the operators

$$p_\mu = \frac{1}{\Gamma_5} \Gamma_\mu \tag{2.22}$$

satisfy the identities $[p_\mu,p_\nu] = 0$, $p_\mu p^\mu = 1$.

On the other hand, one can check directly (or by using (2.15)) that

$$(\frac{1}{\Gamma_5} \phi)(p) = -\frac{1}{2\pi^2} \int \frac{1}{(p-q)^2} \phi(q)\,\varepsilon(q_0)\delta(q^2 - 1)d^4q \,. \tag{2.23}$$

Inserting (2.22) and (2.23) in the quasi-potential Equation (2.2) we find the following algebraic equation for the relativistic Coulomb problem

$$\frac{1}{\Gamma_5}\,[\Gamma_0(E - \frac{1}{\Gamma_5}\,\Gamma_0) + \frac{\alpha}{2}]\phi_E(p) = 0 \; . \tag{2.24}$$

Before going to the solution of Equation (2.24) we will make the following general comments.

(1) The prescription (2.22) for the algebraization of the (free) 4-momentum does not depend on the interaction under consideration.

(2) The simple algebraization of the potential based on Equation (2.23) is peculiar to the case of zero mass exchange. The relativistic Yukawa potential

$$V(p,q) = \frac{g^2}{(p - q)^2 - \mu^2} \tag{2.25}$$

leads already to considerable complications (see Section III.2 of Reference [2]). The reason is that the kernel in the scalar product (2.8) in $\mathcal{H}$ is closely related to the relativistic Coulomb potential. If on the other hand we adapt the scalar product in our representation space to the potential (2.25) for $\mu > 0$, the simplicity of the free Hamiltonian will be lost.

(3) We can use Equations (2.18-20) and (2.23) to solve the inverse problem: given *ad hoc* an infinite-component wave equation in the representation space $\mathcal{H}$ of $R_0$ (see References [13,15,16]) to reconstruct an equivalent integral equation in momentum space.

## 3. SOLUTION OF THE COULOMB EIGENVALUE PROBLEM

### 3.1. Group Theoretical Treatment of the Algebraic Equation

In order to get rid of the inverse powers of $\Gamma_5$ in Equation (2.24) we multiply it from the left by $\Gamma_5\Gamma_0^{-1}\Gamma_5$ and put

$$\phi_E = \Gamma_0 f_E \; . \tag{3.1}$$

This leads to the following equation for $f_E$:

$$[(\Gamma_0 - E\Gamma_5)\Gamma_0 - \frac{\alpha}{2}\,\Gamma_5]f_E = 0 \; . \tag{3.2}$$

First of all we observe that the operators $\Gamma_0$, $\Gamma_5$ and $\Gamma_{05}$ generate the Lie algebra of SO(2,1):

$$[\Gamma_0,\Gamma_{05}] = i\Gamma_5, \; [\Gamma_5,\Gamma_0] = i\Gamma_{05}, \; [\Gamma_{05},\Gamma_5] = -i\Gamma_0 \; . \tag{3.3}$$

Equation (C.12) of Appendix C shows that for the representation $R_0$ the Casimir operator of $\underline{SO}(2,1)$ is equal to the Casimir of $\underline{SO}(3)$. Hence, for fixed angular momentum $\ell$

$$\Gamma_0^2 - \Gamma_5^2 - \Gamma_{05}^2 = \underline{L}^2 = \ell(\ell + 1) \; . \tag{3.4}$$

Since Equation (3.2) is obviously SO(3) invariant, we will require that $f_E$ is an eigenvector of $\underline{L}^2$, say $f_{E\ell}$.

Equation (3.4) and the positivity of $\Gamma_0$ imply that we have to deal with one of the discrete series of unitary representations of SO(2,1) described by Bargmann[17] (see also [14] Chapter 7). Each irreducible representation $\mathcal{R}_0^{(\ell)}$ of this series can be realized as a group of coordinate transformations (with a suitable multiplier) in the space $\mathcal{H}_\ell$ of analytic functions on the unit disk

$$D_1 = \{z \in \mathbb{C},\ |z| < 1\} . \tag{3.5}$$

$\mathcal{H}_\ell$ is considered as a Hilbert space with scalar product

$$(g,f)_\ell = \frac{2\ell + 1}{\pi} \int_{D_1} (1 - z\bar{z})^{2\ell}\, \overline{g(z)} f(z) d^2 z . \tag{3.6}$$

The generators of the representation $\mathcal{R}_0^{(\ell)}$ are first order differential operators with respect to $z$:

$$\Gamma_0 = z \frac{d}{dz} + \ell + 1,\ \Gamma_5 = (\ell + 1)z + \frac{1}{2}(z^2 + 1)\frac{d}{dz}$$
$$\Gamma_{05} = i[(\ell + 1)z + \frac{1}{2}(z^2 - 1)\frac{d}{dz}] . \tag{3.7}$$

It is easily seen that the operators (3.7) satisfy the commutation relations (3.3) and the identity (3.4).

Inserting (3.7) in (3.2) we get the following second order (linear) differential equation for $f_{E\ell}(z)$:

$$\{zQ \frac{d^2}{dz^2} + [(\ell + 2 + \frac{\alpha}{2E})Q + (\ell + 1)Q'z + \frac{\alpha}{2E} z] \frac{d}{dz}$$
$$+ (\ell + 1)[(\ell + 1)Q' + \frac{\alpha}{2} z]\}f = 0 \tag{3.8}$$

where

$$Q = \frac{E}{2}(z^2 + 1) - z,\ Q' = Ez - 1 .$$

## 3.2. Calculation of the Energy Eigenvalues

The eigenvalues of $E$ have to be determined from the condition that $f_{E\ell}$ be regular in the unit disk. The possible singular points of any solution of (3.8) are $z = 0$, $z = \infty$ and

$$z = z_\pm = \frac{1}{E} \pm \frac{1}{E}\sqrt{1 - E^2} . \tag{3.9}$$

Among these four points only two $z = 0$ and $z = z_-$ belong to $D_1$. They are both "weak singularities" of the differential Equation (3.8) and there are regular solutions $f_0$ and $f_-$ in the neighborhood of any of them. In order to ensure that these two solutions are analytic continuation of one another, it is necessary to assume that the branch points at $z = z_+$ and $z = z_\infty$ are of the same type (so that one could consider a single-valued solution of (3.8) regular in the cut z-plane with a cut between $z_+$ and $\infty$ which does not cross the unit disk).

For $z \to z_+$ the asymptotic form of (3.8) is

$$[A(z - z_+) \frac{d^2}{dz^2} + B \frac{d}{dz} + C]f_+ = 0 \tag{3.10}$$

with $A = \sqrt{1 - E^2}\, z_+$, $B = z_+[\sqrt{1 - E^2}\,(\ell + 1) + \frac{\alpha}{2E}]$. For $z \to z_+$ the singular solution $f_+$ of (3.10) behaves like $(z - z_+)^{\nu_+}$ where

$$\nu_+ = 1 - \frac{B}{A} = -\ell - \frac{\alpha}{2E\sqrt{1 - E^2}} . \tag{3.11}$$

For $z \to \infty$ Equation (3.8) is equivalent to

$$[z^2 \frac{d^2}{dz^2} + (3\ell + 4 + \frac{\alpha}{2E}) z \frac{d}{dz} + 2(\ell + 1)(\ell + 1 + \frac{\alpha}{2E})]f_\infty = 0 . \tag{3.12}$$

The relevant solution of (3.12) is $f_\infty = z^{\nu_\infty}$ with

$$\nu_\infty = -\ell - 1 - \frac{\alpha}{2E} . \tag{3.13}$$

The branch points at $z = z_+$ and $z = \infty$ are of the same type if and only if $\nu_\infty - \nu_+$ is an integer. So, we put

$$\nu_\infty - \nu_+ = \frac{\alpha}{2E} (\frac{1}{\sqrt{1 - E^2}} - 1) - 1 = n - 1 . \tag{3.14}$$

Thus, the eigenvalues $E_n$ of $E$ are determined from the equation

$$\frac{\alpha}{2n} = (E_n + \frac{\alpha}{2n})\sqrt{1 - E_n^2} \tag{3.15}$$

or

$$E_n^3 + \frac{\alpha}{n} E_n^2 - (1 - \frac{\alpha^2}{4n^2})E_n - \frac{\alpha}{n} = 0 . \tag{3.16}$$

Only one of the three real roots of (3.16) satisfies (3.15). It can be written as an expansion in $\alpha_n \equiv \frac{\alpha}{2n}$:

$$\sqrt{1 - E_n^2} = \alpha_n - \alpha_n^2 + \frac{3}{2}\alpha_n^3 - 3\alpha_n^4 + \dots$$

$$E_n = 1 - \frac{1}{2}\alpha_n^2 + \alpha_n^3 - \frac{17}{8}\alpha_n^4 + \dots \quad . \tag{3.17}$$

In order to find the range* of the quantum number $n$ we look at the power series expansion of the solution of Equation (3.8):

$$f(z) = \sum_{\nu=0}^{\infty} f_\nu z^\nu . \tag{3.18}$$

In view of (3.8) the coefficients $f_\nu$ satisfy the following recurrence relation

$$(\nu + 1)(\nu + \ell + 2 + \beta)f_{\nu+1} - 2(\nu + \ell + 1)^2 \operatorname{ch}\lambda f_\nu + [(\nu - 1)(\nu + 3\ell + 2 + \beta) + 2(\ell + 1)(\ell + 1 + \beta)]f_{\nu-1} = 0,\ \nu = 0, 1, 2, \dots,\ \beta = \frac{\alpha}{2E} . \tag{3.19}$$

The radius of convergence of the power series (3.18) is determined by the behavior of the coefficients $f_\nu$ for large $\nu$. Dividing the left-hand side of (3.19) by

---

* This problem was not touched in Reference [2].

$\nu + 1$ and neglecting the terms of order $\frac{1}{\nu}$ we obtain the following asymptotic form for the recurrence relation

$$(\nu + \ell + 2 + \beta)f_{\nu+1} - \frac{2}{E}(\nu + 2\ell + 1)f_{\nu} + (\nu + 3\ell + \beta)f_{\nu-1} = 0. \quad (3.20)$$

It corresponds to a first order differential equation which can be obtained by multiplying by $z^{\nu}$ and summing over $\nu$. The result is

$$zQf' + \{(2\ell + 1)Q + \frac{E}{2}[\ell(z^2 - 1) + \beta(z^2 + 1)]\}f = \frac{E}{2}(\ell + 1 + \beta) . \quad (3.21)$$

(We have used the initial conditions $f_{-1} = 0$, $f_0 = f(0) = 1$.) The solution of (3.21) regular (and normalized to 1) for $z = 0$ is

$$f(z) = \frac{\ell + 1 + \beta}{z^{\ell + 1 + \beta}} \left(\frac{z - z_-}{z - z_+}\right)^{\beta/\sqrt{1-E^2}} \frac{1}{[(z - z_+)(z - z_-)]^{\ell}} \int_0^z \xi^{\ell+\beta} \left(\frac{\xi - z_+}{\xi - z_-}\right)^{\beta/\sqrt{1-E^2}} [(\xi - z_+)(\xi - z_-)]^{\ell-1} d\xi . \quad (3.22)$$

We can define $f(z)$ as analytic single valued function in the cut z-plane with a cut along the real semi axis $z \geq z_+$ provided that

$$\frac{\beta}{\sqrt{1 - E^2}} = \beta + n, \; n = 1, 2, \ldots \quad (3.23)$$

in accordance with (3.15) ($\beta$ is defined in (3.19)). It is regular for $z = z_-$ only if $n \geq \ell$. For $\ell = 0$ we actually have to require $n \geq 1$; it is easily verified that for $E = 0$, Equation (3.8) has no solution regular for $z = 0$. (This shows that contrary to the Wick-Cutkosky model [6] there is no limit of "maximal binding" in our quasi-potential equation.) The present argument cannot exclude however the values $n = \ell$ for $\ell \geq 1$. We observe that (3.22) gives the exact solution of Equation (3.8) for the s waves ($\ell = 0$) but not for $\ell \geq 1$ (however, it has for all $\ell$ the correct behavior $(z - z_+)^{\nu_+}$ as $z \to z_+$). We expect that the exact range of the quantum number n is always $n \geq \ell + 1$, which would give the familiar SO(4) degeneracy of the energy levels of the non-relativistic hydrogen atom (as well as of the Wick-Cutkosky model). We mention that the second order term in Equation (3.17) reproduces precisely the Balmer formula for the non-relativistic Coulomb energy levels as it should be in any consistent relativistic generalization of the Coulomb problem.

# APPENDIX A

## DIFFERENT REALIZATIONS AND PROPERTIES OF THE EXCEPTIONAL REPRESENTATION $\mathcal{R}_0$ OF $SO_0(4,2)$

### A. The Set of Conformal Transformations in Space-time as a Global Realization of $\mathcal{R}_0$

Consider the space $X$ of negative frequency solutions

$$f(x) = \frac{1}{(2\pi)^{3/2}} \int \mathfrak{f}(\xi) e^{-ix\xi} \delta_0^+(\xi) d^4\xi, \quad \delta_0^+(\xi) = \theta(\xi_0)\delta(\xi^2) \tag{A.1}$$

of the D'Alembert equation

$$\Box f(x) \equiv \left(\frac{\partial^2}{\partial x_0^2} - \Delta\right) f(x) = 0 \tag{A.2}$$

with scalar product

$$\begin{aligned}(f,g) &= i \int_{x_0=t} \left(\overline{f}(x) \frac{\partial g(x)}{\partial x_0} - \frac{\partial \overline{f}(x)}{\partial x_0} g(x)\right) d^3\underline{x} \\ &= \int \overline{\mathfrak{f}}(\xi) \mathfrak{g}(\xi) \delta_0^+(\xi) d^4\xi \; .\end{aligned} \tag{A.3}$$

The representation of the conformal group acting in $X$ which leaves Equation (A.2) and the scalar product (A.3) invariant is generated by the following transformations:

(i) Poincaré transformations

$$[U(a,\Lambda)f](x) = f(\Lambda^{-1}(x - a)) \tag{A.4}$$

(ii) Dilations

$$(U(\lambda)f)(x) = \lambda^{-1} f(\lambda^{-1}x) \tag{A.5}$$

(iii) Inversion

$$[U(R)f](x) = \frac{1}{x^2} f\left(\frac{-x}{x^2}\right) \; . \tag{A.6}$$

The inversion $(Rx)_\mu = -\frac{x_\mu}{x^2}$ does not actually belong to the connected component of the identity of the conformal group, but the set of non-linear transformations

$$[R\{b,1\}Rx]_\mu = \frac{x_\mu - x^2 b_\mu}{d(b,x)}, \quad d(b,x) = 1 - 2bx + b^2x^2 \tag{A.7}$$

belongs to $SO_0(4,2)$ and generates the so-called special conformal transformations

$$[U(R\{-b,1\}R)f](x) = \frac{1}{d(b,x)} f\left(\frac{x_\mu - x^2 b_\mu}{d(b,x)}\right) . \tag{A.8}$$

The (hermitian) infinitesimal operators of the subgroups (A.4), (A.5), and (A.8) are given by

$$P_\mu = i\partial_\mu,\ M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu),\ (\partial_\mu \equiv \frac{\partial}{\partial x^\mu}) ,$$
$$D = -i(1 + x_\mu\partial^\mu),\ K_\mu = i(2x_\mu + 2x_\mu x_\nu\partial^\nu - x^2\partial_\mu) . \qquad (A.9)$$

These operators are related to the generators $\Gamma_{ab}$ and $\Gamma_a$ used in Section 2.2 by

$$M_{\mu\nu} \Leftrightarrow \Gamma_{\mu\nu},\ P_\mu \Leftrightarrow \Gamma_\mu + \Gamma_{\mu 5},\ D \Leftrightarrow \Gamma_5,\ K_\mu \Leftrightarrow \Gamma_\mu - \Gamma_{\mu 5} . \qquad (A.10)$$

This well-known representation of the conformal group (related to the 0-spin 0-mass particles) is equivalent to the representation $R_0$ defined in Section 2.2. The intertwining operator $V$ which maps $\mathcal{H}$ onto $X$ can be written down explicitly:

$$\mathcal{H} \ni \phi(p) \overset{V}{\to} f(x) = \frac{1}{\pi}\int D_0^{(-)}(p + x)\varepsilon(p_0)\delta(p^2 - 1)\phi(p)d^4p \qquad (A.11)$$

where

$$D_0^{(-)}(x) = \frac{i}{(2\pi)^3}\int e^{-ix\xi}\delta_0^+(\xi)d^4\xi = \frac{-i}{(2\pi)^2}\frac{1}{(x_0 - i0)^2 - \underline{x}^2} \qquad (A.12)$$

is the Lorentz invariant negative frequency solution of Equation (A.2). (The distribution $D_0^{(-)}(x)$ appears in quantum field theory as the two-point function of a zero mass field.) The realization of the representation $R_0$ in $X$ displays its irreducibility with respect to the Poincaré subgroup of the conformal group.

## B. $R_0$ As One of the Metaplectic Representations of SU(2,2)

The metaplectic series of unitary representations of SU(2,2) can be constructed in infinitesimal form starting with the 4-dimensional representation (2.5) of the Lie algebra. To do this, we introduce the 4-component operator valued spinor $\varphi$ satisfying the canonical commutation relations

$$[\varphi^\alpha,\varphi^\beta] = 0,\ [\varphi^\alpha,\tilde{\varphi}_\beta] = \delta^\alpha_\beta,\ \alpha,\beta = 1,2,3,4,\ \tilde{\varphi} = \varphi^*\beta ; \qquad (B.1)$$

here $\beta$ is the hermitian matrix satisfying (2.7) and normalized by the requirements $\det\beta = 1$, $\beta\gamma_0$ is positive definite. It is easy to verify that the set of operators

$$\Gamma_{AB} = \tilde{\varphi}\gamma_{AB}\varphi \qquad (B.2)$$

obeys the commutation relations (2.4) since (B.1) implies that

$$[\Gamma_{AB},\Gamma_{CD}] = \tilde{\varphi}[\gamma_{AB},\gamma_{CD}]\varphi . \qquad (B.3)$$

The metaplectic series of the so-called ladder representations of SU(2,2) corresponds to the (star) representation of the canonical commutation relations (B.1) in the Fock space $F$ defined in the following way. There exists a unit vector $|0)$ in $F$ (defined up to a phase factor) for which

$$(\gamma_0 + 1)\varphi|0) = \tilde{\varphi}(\gamma_0 - 1)\ |0) = 0,\ (\Gamma_0\ |0) = |0)) . \qquad (B.4)$$

The vector $|0)$ so defined is $SU(2) \times SU(2)$ invariant.

In order to label the irreducible representations of the metaplectic series, it is convenient to extend the representation defined by (B.2) to a representation of U(2,2) by introducing a 16th generator,

$$C = \frac{1}{2} \tilde{\varphi}\varphi \, . \qquad (B.5)$$

C belongs to the center of the enveloping algebra of the Lie algebra $\underline{U}(2,2)$ and hence, should be a multiple of the identity in each irreducible subspace of F. It is easy to verify that the spectrum of C in F is given by

$$C = \lambda - 1, \ \lambda = 0, \ \pm\frac{1}{2}, \ \pm 1, \ \ldots \ . \qquad (B.6)$$

It can be proved that for fixed C (or λ) the ladder representation $R_\lambda$ acting in the corresponding invariant subspace $F_\lambda$ of F is already irreducible. All elements of the center of the enveloping algebra of the metaplectic series are functions of λ. In particular, the second order Casimir operator $C_2$ of SU(2,2) is given by

$$C_2 = \frac{1}{2} \Gamma_{AB}\Gamma^{AB} = 3(\lambda^2 - 1) \ . \qquad (B.7)$$

It has been shown explicitly in Reference [18] that the metaplectic representations $R_\lambda$ so defined are equivalent to the representation of the conformal group in space-time, corresponding to zero-mass particles of helicity λ. In particular, for λ = 0, we recover the representation $R_0$ described in Section 2.2 and Appendix A.

The ladder representations $R_\lambda$ are closely related to the two metaplectic representations of the real symplectic group Sp(4,R) in 8-dimension described in References [19,20]. Namely, if $R^{(0)}$ is the single-valued and $R^{(1)}$ the double-valued representation of Sp(4,R) acting in the same Fock space F, then

$$\begin{aligned} R^{(0)} &= \sum_{\lambda=0,\pm 1,\pm 2,\ldots} \oplus R_\lambda \\ R^{(1)} &= \sum_{\lambda=\pm\frac{1}{2},\pm\frac{3}{2},\ldots} \oplus R_\lambda \ . \end{aligned} \qquad (B.8)$$

More about the different realizations of the ladder representations and their equivalence is said in Appendix to Reference [2]. The term metaplectic and the first mathematical description of the metaplectic representations of Sp(n,R) is due to Weil [21]. (See also Mackey [22].) The description of the metaplectic representations of $\underline{U}(2,2)$ in terms of creation and annihilation operators was first given by Kurşunoglu [23].

## C. Quadratic Identities in the Enveloping Algebra of the Metaplectic Representations

We shall collect in this section a set of quadratic identities which hold in the enveloping algebra of the metaplectic representation of $\underline{U}(2,2)$. They can be derived by using (B.1), (B.2), and the identity

$$\sum_{a=0,1,2,3,5} (\gamma_a)^\alpha_\beta (\gamma^a)^\delta_\delta = \delta^\alpha_\beta \delta^\delta_\delta + 2\varepsilon^{\alpha\delta\sigma\tau} B_{\sigma\beta} B_{\tau\delta} \ , \qquad (C.1)$$

where $\varepsilon^{\alpha\delta\sigma\tau}$ is the completely antisymmetric unit tensor in 4-dimension

($\varepsilon^{1234} = 1$) and B is defined (up to an irrelevant sign) by

$$B\gamma_{ab}B^{-1} = -{}^{t}\gamma_{ab}, \; (a,b = 0,1,2,3,5), \; {}^{t}B = -B, \; (B^{-1})^{\beta\alpha} = \frac{1}{2}\,\varepsilon^{\alpha\beta\sigma\tau}B_{\sigma\tau} \qquad (C.2)$$

(the superscript t to the left of a matrix stands for transposition).

Each of the metaplectic representations $R_\lambda$ remains irreducible when restricted to any of the 5-dimensional rotation subalgebras of $\underline{SO}(4,2)$. Hence, their second order Casimir operators are functions of $\lambda$ only. A direct calculation gives

$$\frac{1}{2}\,\Gamma_{ab}\Gamma^{ab} = 2(\lambda^2 - 1) = \frac{1}{2}\,\Gamma_{\mu\nu}\Gamma^{\mu\nu} + \Gamma_\mu\Gamma^\mu \qquad (C.3)$$

(repeated upper and lower indices have to be summed over the range $a,b = 0,1,2,3,5$; $\mu,\nu = 0,1,2,3$). Comparing (C.3) with (B.7), we find

$$\Gamma_\mu\Gamma^\mu = \Gamma_{5\mu}\Gamma^{5\mu} = \lambda^2 + \Gamma_5^2 - 1 \; . \qquad (C.4)$$

We also have

$$\frac{1}{2}\,\Gamma_{\mu\nu}\Gamma^{\mu\nu} = \underline{L}^2 - \underline{N}^2 = \lambda^2 - 1 - D^2, \; \underline{LN} = -\lambda\Gamma_5 \qquad (C.5)$$

(with $\Gamma_{ij} = \varepsilon_{ijk}L_k$, $\Gamma_{oj} = N_j$, $i,j,k = 1,2,3$). More generally, the following tensor identities hold:

$$\{\Gamma_{\mu 5},\Gamma_\nu\} - \{\Gamma_\mu,\Gamma_{\nu 5}\} = 2\Gamma_5\Gamma_{\mu\nu} - \lambda\varepsilon_{\mu\nu\sigma\tau}\Gamma^{\sigma\tau} \qquad (C.6)$$

$$\{\Gamma_{CA},\Gamma^{CB}\} = (\lambda^2 - 1)\delta_A^B (A,B = 0,1,2,3,5,6) \; . \qquad (C.7)$$

As mentioned before, each of the representations $R_\lambda$ remains also irreducible when restricted to the Poincaré subgroup generated by $P_\mu$ and $M_{\mu\nu}$ (or $K_\mu$ and $M_{\mu\nu}$; see (A.10)). This gives

$$P_\mu P^\mu = K_\mu K^\mu = 0, \; \underline{P}\,\underline{L} = P_0\lambda \; . \qquad (C.8)$$

Equation (C.6) implies

$$P_\mu K_\nu - P_\nu K_\mu = 2(\Gamma_5 - i)\Gamma_{\mu\nu} - \lambda\varepsilon_{\mu\nu\sigma\tau}\Gamma^{\sigma\tau} \; . \qquad (C.9)$$

The scalar product of P and K is a function of $\lambda$ and $\Gamma_5$:

$$KP = (PK)^* = 2[\lambda^2 + (\Gamma_5 + i)^2] \; . \qquad (C.10)$$

The Casimir operators of the $\underline{SO}(4)$ subalgebra are expressed in terms of $\Gamma_0$ and $\lambda$:

$$\sum_{j=1}^{3} (L_j^2 + \Gamma_{j5}^2) = \Gamma_0^2 + \lambda^2 - 1, \; L_j\Gamma_{j5} = \lambda\Gamma_0 \; . \qquad (C.11)$$

From (C.4) and (C.11), it follows that

$$\underline{L}^2 = \Gamma_0^2 - \Gamma_{05}^2 - \Gamma_5^2 \; . \qquad (C.12)$$

## REFERENCES

[1] Itzykson, C., Kadyshevsky, V. G., and Todorov, I. T., *Three Dimensional Formulation of the Relativistic Two-Body Problem and Infinite Component Wave Equations*, Institute for Advanced Study, Princeton, preprint (1969) and *Phys. Rev.* (to be published).

[2] Itzykson, C., and Todorov, I. T., "An Algebraic Approach to the Relativistic Two-Body Problem" in *Proceedings of the Coral Gables Conference on Fundamental Interactions on High Energy*, T. Gudehus *et al.* editors, Gordon and Breach, New York (1969).

[3] Todorov, I. T., "On the Three Dimensional Formulation of the Relativistic Two-Body Problem", *Lectures Presented at the Theoretical Physics Institute*, University of Colorado, Boulder (1969).

[4] Kadyshevsky, V. G., "Relativistic Equations for the S-Matrix in the p-Representation", I "Unitarity and Causality Conditions"; II, *Soviet Phys. JETP*, 19, 443, 597 (1964).

[5] Nakanishi, N., "A General Survey of the Theory of the Bethe-Salpeter Equation", *Prog. Theor. Phys. Suppl.*, No. 43, 1 (1969).

[6] Wick, G. C., "Properties of the Bethe-Salpeter Wave Functions", *Phys. Rev.*, 96, 1124 (1954).

Cutkosky, R. E., "Solutions of a Bethe-Salpeter Equation", *Phys. Rev.*, 96, 1135 (1954).

[7] Kadyshevsky, V. G., "Quasi-potential Equation for the Relativistic Scattering Amplitude", *Nucl. Phys.*, 136, 125 (1968).

Kadyshevsky, V. G., and Mateev, M. D., "On a Relativistic Quasi-potential Equation in the Case of Particles with Spin", *Nuovo Cimento*, 55A, 233 (1968).

[8] Faustov, R. N., and Helashvili, A. A., "Normalization Condition for Simultaneous Wave Function of the Bound State of Two Particles", *JINR*, Dubna, preprint P2-4345 (1969).

[9] Logunov, A. A., and Tavkhelidze, A. N., "Quasi-optical Approach in Quantum Field Theory", *Nuovo Cimento*, 29, 380 (1963).

Logunov, A. A., Tavkhelidze, A. N., Todorov, I. T., and Khrustalev, O. A., "Quasi-potential Character of the Scattering Amplitude", *Nuovo Cimento*, 30, 134 (1963).

[10] Kyriakopoulos, E., "Dynamical Groups and the Bethe-Salpeter Equation", *Phys. Rev.*, 174, 1846 (1968).

[11] Todorov, I. T., "Discrete Series of Hermitian Representations of the Lie Algebra of U(p,q)", Int. Centre Theoret. Phys., Trieste, preprint IC/66/71 (1966).

[12] Yao, Tsu, "Unitary Irreducible Representations of SU(2,2), I and II", *J. Math. Phys.*, 8, 1931 (1967) and 9, 1615 (1968).

[13] Fronsdal, C., "Infinite Multiplets and the Hydrogen Atom", *Phys. Rev.*, 156, 1665 (1967).

[14] Gel'fand, I. M., Graev, M. I., and Vilenkin, N. Ya., "Integral Geometry and Representation Theory" in *Generalized Functions*, Vol. 5, Academic Press,

New York (1966). See also "Properties and Operations", Appendix B to Vol. 1, Academic Press, New York (1964).

[15] Nambu, Y., "Infinite-component Wave Equations with Hydrogen-like Mass Spectra", *Phys. Rev.*, 160, 1171 (1967).

[16] Barut, A. O., and Kleinert, H., "Current Operators and Majorana Equation for the Hydrogen Atom from Dynamical Groups", *Phys. Rev.*, 157, 1180 (1967).

[17] Bargmann, V., "Irreducible Unitary Representations of the Lorentz Group", *Annals of Math.*, 48, 568 (1947).

[18] Mack, G., and Todorov, I. T., "Irreducibility of the Ladder Representations of U(2,2) When Restricted to Its Poincaré Subgroup", *J. Math. Phys.*, 10, 2078 (1969).

[19] Itzykson, C., "Remarks on Boson Commutation Rules", *Commun. Math. Phys.*, 4, 92 (1967).

[20] Bargmann, V., "Group Representations on Hilbert Spaces of Analytic Functions" in *Lectures at the International Symposium on Analytic Methods in Mathematical Physics, Indiana University 1968*, Gordon and Breach, New York (1970).

[21] Weil, A., "Sur Certains Groupes d'Operateurs Unitaires", *Acta Math.*, 111, 143 (1964).

[22] Mackey, G., "Some Remarks on Symplectic Automorphisms", *Proceedings Amer. Math. Soc.*, 16, 393 (1965).

[23] Kurşunoglu, B., *Modern Quantum Theory*, W. H. Freeman and Co., San Francisco (1962), p. 257.