TENSOR OPERATORS FOR THE GROUP SL(2,C)

by

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INTRODUCTION

This talk consists of three parts: some selected topics of a purely mathematical theory of irreducible tensor operators, the adaptation of this theory to the decomposition of the current density operators of elementary particle physics restricted to single-particle spaces, and an application of this formalism to a phenomenological analysis of certain scattering experiments.

1. IRREDUCIBLE TENSOR OPERATORS

1.1. Notations and Some Known Facts About the Representations of SL(2,C)

We shall mainly adhere to the notations of Gel'fand and Naimark. [1] In particular we make explicit use of matrices like the following ones

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SL(2,C); \; u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in SU(2) \equiv K, \; k = \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix}, \; \zeta = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$$

where λ, μ, z are complex numbers and K stands for "maximal compact subgroup" of SL(2,C). One possibility to realize the principal series of representations of SL(2,C) is on a space of measurable functions $f(z)$ which have finite norm with respect to the scalar product

$$(f_1,f_2) = \int \bar{f}_1(z) f_2(z) dz \; . \tag{1.1}$$

We denote this space $L^2(Z)$. The group operations are introduced by

$$T_a^\chi f(z) = \alpha^\chi(z,a) f(z_a) \tag{1.2}$$

with

$$\zeta a = k\zeta_a ; \; \zeta_a = \begin{pmatrix} 1 & 0 \\ z_a & 1 \end{pmatrix}, \; k = \begin{pmatrix} \lambda^{-1}(z,a) & \mu \\ 0 & \lambda(z,a) \end{pmatrix},$$

$$\alpha^\chi(z,a) = |\lambda(z,a)|^{i\rho-2} \left(\frac{\lambda(z,a)}{|\lambda(z,a)|} \right)^{-m} \tag{1.3}$$

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where ρ is real and m is an integer. We characterize the representation χ by the pairs of numbers

$$\chi = (m,\rho) = \{n_1,n_2\};\; n_{1,2} = \frac{-m}{+2} + \frac{i}{2}\rho \tag{1.4}$$

and use

$$-\chi = (-m,-\rho) \quad \text{if} \quad \chi = (m,\rho) \;.$$

We call this realization of the principal series the "noncompact picture".

Another realization of the principal series is obtained in a space $L_m^2(K)$ of measurable functions $\varphi(u)$ on K satisfying the constraint

$$\varphi(u(\psi)u) = e^{im\psi}\varphi(u)$$
$$u(\psi) = \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix} \tag{1.5}$$

which have finite norm with respect to the scalar product

$$(\varphi_1,\varphi_2) = \int \overline{\varphi_1(u)}\varphi_2(u)d\mu(u) \;. \tag{1.6}$$

We introduce the operators T_a^χ by the definition

$$T_a^\chi\varphi(u) = \alpha^\chi(u,a)\varphi(u_a);\; ua = ku_a,\; k = \begin{pmatrix} \lambda^{-1}(u,a) & \mu \\ 0 & \lambda(u,a) \end{pmatrix} \tag{1.7}$$

and a relation between λ and α^χ as in (1.3). This realization is denoted the "compact picture". We arrive at the principal series in this compact picture if we use the technique of induced representations and induce from one-dimensional unitary representations $\xi(k)$ of the subgroup of triangular matrices k

$$\xi(k) = |\lambda|^{i\rho}\left(\frac{\lambda}{|\lambda|}\right)^{-m}$$

and identify the cosets of this subgroup in $SL(2,C)$ with the cosets of the subgroup $U(1)$ in $SU(2)$ by

$$a = ku \;.$$

The equivalence of the compact and the noncompact picture is easily established.

Following Gel'fand [1] we consider a set of closed topological vector spaces D_χ which are dense in the Hilbert spaces $L_m^2(K)$. They consist of infinitely differentiable functions $\varphi(u)$ satisfying (1.5) and possess a topology typical for a space of type K in Gel'fand's notation [2]. In the noncompact picture the corresponding spaces D_χ consist of infinitely differentiable functions $f(z)$ (considered as functions of two real variables) which possess an asymptotic expansion

$$f(z) \cong |z|^{i\rho-2}\left(\frac{z}{|z|}\right)^{-m} \sum_{k,\ell=0}^{\infty} C_{k\ell}\, z^{-k}\bar{z}^{-\ell} \tag{1.8}$$

around $z = \infty$. The topology is simply carried over from the compact picture. The spaces D_χ are invariant subspaces under operation of T_a^χ, and the operators T_a^χ are continuous. We emphasize that in the compact picture the definition of the spaces D_χ is independent of the parameter ρ. One space D_χ may therefore be used simultaneously for the definition (1.7) of operators T_a^χ with fixed m but variable complex ρ. Completing the spaces D_χ for fixed m with respect to the scalar product norm (1.6), we obtain the original Hilbert spaces $L^2_m(K)$. In this fashion we can make $L^2_m(K)$ to carry representations χ for arbitrary complex ρ, which are nonunitary if $\mathrm{Im}\rho \neq 0$. Translating this construction into the noncompact picture we obtain representations in Hilbert spaces $L^2_{\rho_1}(Z)$ with the scalar product defined by

$$(f_1,f_2) = \int \overline{f_1(z)} f_2(z)(1 + |z|^2)^{\rho_1} dz, \ \rho_1 = \mathrm{Im}\rho \ .$$

For any fixed χ with arbitrary complex ρ there is an isometric mapping from $L^2_{\rho_1}(Z)$ onto $L^2_m(K)$ which intertwines the bounded operators T^χ_a.

The spaces D_χ possess invariant closed subspaces E_χ if both n_1 and n_2 (1.4) are positive integers. In the noncompact picture the spaces E_χ are spanned by polynomials in z and $\bar{z}$ of maximal order $n_1 - 1$, respectively $n_2 - 1$. Therefore

$$\dim E_\chi = n_1 n_2 \ . \tag{1.9}$$

In addition the space D_χ possesses an invariant subspace if both n_1, n_2 are negative integers. This subspace is denoted F_χ and consists of all functions of D_χ whose momenta

$$\int z^k \bar{z}^\ell f(z) dz \tag{1.10}$$

vanish for all orders

$$0 \leqq k \leqq -n_1 - 1$$

$$0 \leqq \ell \leqq -n_2 - 1 \ .$$

Again we find

$$\dim D_\chi / F_\chi = n_1 n_2 \ . \tag{1.11}$$

Next we recall Gel'fand's results [1] on bilinear invariant functionals on spaces D_χ. We define such a functional $B(f_1,f_2)$ for two functions $f_{1,2} \in D_{\chi_{1,2}}$ requiring

a) linearity

$$B(\sum_i \alpha_i f_i, \sum_j \beta_j h_j) = \sum_{ij} \alpha_i \beta_j B(f_i,h_j) \ ; \tag{1.12}$$

b) continuity in each argument;

c) invariance for all $a \in SL(2,C)$

$$B(T^{\chi_1}_a f_1, T^{\chi_2}_a f_2) = B(f_1,f_2) \ . \tag{1.13}$$

In the case that both D_{χ_1} and D_{χ_2}, $\chi_1 = \{n_1,n_1'\}$, $\chi_2 = \{n_2,n_2'\}$, are such that neither pair consists of nonnegative or nonpositive integers (we call χ regular in such a case) the functional B can be proved to be generated by a homogeneous distribution $M(z)$ in the form

$$B(f_1,f_2) = \int (\int M(z_1) f_1(z_2 + z_1) dz_1) f(z_2) dz_2 \tag{1.14}$$

where $M(z)$ is nontrivial only in the following two cases

$$\chi_1 = -\chi_2 : \; M(z) = C\delta(z) \tag{1.15}$$

$$\chi_1 = \chi_2 = (m,\rho) : \; M(z) = C|z|^{-i\rho-2}\left(\frac{z}{|z|}\right)^{+m} . \tag{1.16}$$

Because of (1.15) we call two representations χ_1, χ_2 dual to each other if $\chi_1 = -\chi_2$. The kernel (1.16) serves also as an intertwining operator for a dual pair of representations. A convolution with $M(z)$ can be shown to establish a one-to-one and bicontinuous map from D_χ onto $D_{-\chi}$ if χ is regular, which intertwines T_a^χ and $T_a^{-\chi}$.

In order to treat also the nonregular cases which were so far excluded, we recall the properties of the distribution

$$p_{\sigma,m}(z) = |z|^\sigma \left(\frac{z}{|z|}\right)^m$$

(σ complex, m integral). Considered in its analytic dependence on σ it is a meromorphic function with simple poles at

$$\sigma_n = -2n - 2 - |m|, \; n = 0, 1, 2, \cdots$$

and the residues

$$\operatorname*{Res}_{\sigma_n} p_{\sigma,m}(z) = \frac{2\pi(-1)^m}{n!(n+|m|)!} \frac{\partial^{\sigma_1}}{\partial z^{\sigma_1}} \frac{\partial^{\sigma_2}}{\partial \bar{z}^{\sigma_2}} \delta(z) \tag{1.17}$$

where

$$\begin{aligned} \sigma_1 &= n + \frac{1}{2}(|m| - m) \\ \sigma_2 &= n + \frac{1}{2}(|m| + m) . \end{aligned} \tag{1.18}$$

Formally the distribution $M(z)$ (1.16) is identical with $p_{-i\rho-2,m}(z)$, only the domains on which they operate differ. However, a look at the compact picture or at the asymptotic expansion (1.8) convinces us that the distribution $p_{-i\rho-2,m}(z)$ can be extended from test functions $f(z)$ with compact support (say) onto D_χ by continuity. Analytic continuations in ρ can still be given a rigorous meaning in the compact picture, where the domain D_χ is independent of the parameter ρ. The results on the analytic structure of $p_{-i\rho-2,m}(z)$ in ρ apply therefore also to $M(z)$, the only source of singularities is the behavior at $z = 0$.

The residue or the constant term in the Laurent expansion of $p_{-i\rho-2,m}(z)$ around a pole give new distribution kernels $M(z)$ which lead to bilinear invariant functionals and corresponding intertwining operators, that establish one-to-one and bicontinuous mappings between the spaces of the following pairs (we set $\chi = \{n,n'\}$)

a) D_χ and $D_{-\chi}$, if one of the numbers n, n' is zero, whereas the other is an arbitrary integer;

b) E_χ and $D_{-\chi}/F_{-\chi}$ if n, n' are both positive integers;

c) F_χ and $D_{-\chi}/E_{-\chi}$ if n, n' are both negative integers;

d) F_χ and $D_{\pm\chi'}$, if n, n' are both negative integers and χ' is defined by
$$\chi' = \{n,-n'\} \quad \text{for} \quad \chi = \{n,n'\} .$$

Together with the regular case, this is a complete list of intertwining operators for the spaces D_χ, E_χ, F_χ, and their quotient spaces.

1.2. Trilinear Invariant Functionals

The same relation which exists between intertwining operators and bilinear invariant functionals holds true between irreducible tensor operators and trilinear invariant functionals. Trilinear invariant functionals for three arbitrary representations can be obtained by analytic continuation in the three ρ's from a trilinear invariant functional for three representations of the principal series using essentially the same method as for the bilinear invariant functional. The kernel which generates the trilinear invariant functional for three representations of the principal series is the same as the kernel which was used by Naimark to decompose the tensor product of two representations of the principal series [3]. We start our discussion with his results.

We refer to the noncompact picture. We define a Hilbert space $L^2(Z \times Z)$ of measurable functions $f(z_1,z_2)$ which have finite norm

$$(f_1,f_2) = \int \overline{f_1(z_1,z_2)} f_2(z_1,z_2) dz_1 dz_2 \ . \tag{1.19}$$

This space carries the unitary representation $\chi_1 \times \chi_2$ defined by

$$T_a^{\chi_1 \times \chi_2} f(z_1,z_2) = \alpha^{\chi_1}(z_1,a)\alpha^{\chi_2}(z_2,a) f((z_1)_a,(z_2)_a) \tag{1.20}$$

with the notations (1.3). We call this representation the tensor product of the representations χ_1 and χ_2 which are both assumed to belong to the principal series. The issue solved by Naimark is the decomposition of this tensor product into a direct integral of irreducible representations.

We consider the set of Hilbert spaces $L^2(Z,\chi)$ each of which consists of measurable functions $f(z,\chi)$ in z, which have finite norm with respect to the scalar product

$$(f_1,f_2)_\chi = \int \overline{f_1(z,\chi)} f_2(z,\chi) dz \ . \tag{1.21}$$

χ runs over the principal series, and each $L^2(Z,\chi)$ is assumed to carry the representation χ. We form the direct integral

$$H = \int^{\oplus} L^2(Z,\chi) d\chi \tag{1.22}$$

where $d\chi$ is the Plancherel measure of $SL(2,C)$ normalized as

$$d\chi = (m^2 + \rho^2) d\rho \ . \tag{1.23}$$

We sum over all m and integrate over the real ρ axis, thus we count two representations χ and $-\chi$ out of almost each equivalence class. Actually we want to consider the spaces $L^2(Z,\chi)$ and $L^2(Z,-\chi)$ as isometric images of each other via the intertwining operator, such that the double counting is only a symmetric way of writing

$$H = 2 \sum_m^{\oplus} \int_0^{\infty \oplus} d\rho (m^2 + \rho^2) L^2(Z,\chi) \ .$$

The Hilbert space H can be decomposed into two orthogonal subspaces H_+ and H_- which are obtained by restricting the integration (1.22) to even respectively odd m.

We define Naimark's kernel by

$$N(z_1,z_2,z_3|\chi_1,\chi_2,\chi_3) = \frac{1}{8\pi^2}|z_1-z_2|^{\sigma_3}|z_2-z_3|^{\sigma_1}|z_3-z_1|^{\sigma_2}$$
$$\left(\frac{z_1-z_2}{|z_1-z_2|}\right)^{\mu_3}\left(\frac{z_2-z_3}{|z_2-z_3|}\right)^{\mu_1}\left(\frac{z_3-z_1}{|z_3-z_1|}\right)^{\mu_2} \quad (1.24)$$

if $\sum_i m_i$ is even and by zero if this sum is odd. The parameters σ_i and μ_i are linear combinations of the m_i and ρ_i

$$\begin{aligned} \sigma_i &= -\frac{i}{2}(\rho_1+\rho_2+\rho_3-2\rho_i) - 1 \\ \mu_i &= +\frac{1}{2}(m_1+m_2+m_3-2m_i) \ . \end{aligned} \quad (1.25)$$

Naimark has proved the following assertion [3]: The integral transformations

$$\begin{aligned} f(z,\chi) &= \int N(z,z_1,z_2|-\chi,\chi_1,\chi_2)f(z_1,z_2)dz_1dz_2 \\ f(z_1,z_2) &= \int N(z,z_1,z_2|\chi,-\chi_1,-\chi_2)f(z,\chi)dzd\chi \end{aligned} \quad (1.26)$$

which can be made to converge in the sense of the respective image spaces by an appropriate regularization procedure, establish an isometric mapping of $L^2(Z \times Z)$ on H_s, $s = (-1)^{m_1+m_2}$, such that for fixed $f(z_1,z_2)$ and its image $f(z,\chi)$ the vectors

$$T_a^{\chi_1\times\chi_2}f(z_1,z_2),\ T_a^{\chi}f(z,\chi)$$

are mapped onto each other for all $a \in SL(2,C)$.

Naimark [3] proved this theorem by reducing it to the Plancherel theorem for $SL(2,C)$.

We can now define irreducible tensor operators. Since we want later to continue analytically, we restrict ourselves to the spaces D_χ from the outset. Let three spaces D_{χ_i}, $i = 1, 2, 3$, be given such that $\sum_i m_i$ is even. We consider an operator A on the tensor products $f_2 \times f_1$, $f_{1,2} \in D_{\chi_{1,2}}$,

$$A(f_2 \times f_1) \in D_{\chi_3} \ .$$

First we require that A has to be continuous in its arguments f_1 and f_2 separately. In the case that all three χ_i are regular we call A an irreducible tensor operator if it satisfies in addition the "covariance" relation

$$A(T_a^{\chi_2}f_2 \times T_a^{\chi_1}f_1) = T_a^{\chi_3}A(f_2 \times f_1) \ . \quad (1.27)$$

If all three χ_i belong to the principal series, A is, up to a constant, uniquely determined and is given by Naimark's kernel (1.24) in the form

$$A(f_2 \times f_1)(z_3) = \int N(z_3,z_2,z_1|-\chi_3,\chi_2,\chi_1)f_2(z_2)f_1(z_1)dz_2dz_1 \ . \quad (1.28)$$

This operator is obviously related with the trilinear invariant functional

$$B(f_3,f_2,f_1) = \int N(z_3,z_2,z_1|\chi_3,\chi_2,\chi_1)f_3(z_3)f_2(z_2)f_1(z_1)dz_3dz_2dz_1 \ . \quad (1.29)$$

If all three $f_i(z_i)$ in (1.29) are in D_{χ_i} this functional can be continued off the principal series.

In particular we are interested in the cases where D_{χ_3} possesses an invariant subspace E_{χ_3} or F_{χ_3}, such that applying an appropriate limiting process to (1.28) yields an operator A whose range is in the invariant subspace. If we keep $\chi_{1,2}$ at regular positions, a nontrivial operator obtained in this fashion with image in E_{χ_3} or in the quotient space D_{χ_3}/F_{χ_3} is denoted a finite irreducible tensor operator. These are the operators of major physical interest.

In the special case that

$$n_3 = n_3' = 2, \ \dim E_{\chi_3} = 4$$

so that E_{χ_3} carries the four-vector representation, we call A a vector operator or a generalized Dirac matrix. We use the same notation when

$$n_3 = n_3' = -2, \ \dim D_{\chi_3}/F_{\chi_3} = 4$$

such that the four-vector representation appears on the quotient space. This approach to the generalized Dirac matrices, which are known since the work of Gel'fand and Yaglom [4] and Naimark [1], is due to Wess [5]. Our presentation is an extension of Wess's work but still by no means complete.

1.3. Finite Tensor Operators

If we continue the trilinear invariant functional off the principal series, singularities arise due to the behavior of Naimark's kernel on the manifolds $z_1 = z_2$ etc. We consider first the case that we reach a point

$$\chi_3 = \{n_3, n_3'\}, \ n_3, n_3' > 0 \ \text{integral} \ . \tag{1.30}$$

We want to investigate the condition under which the integral (1.28) lies entirely in the invariant subspace E_{χ_3}. With the notation

$$\begin{aligned} A_i &= \frac{1}{2}(\sigma_i + \mu_i) \\ B_i &= \frac{1}{2}(\sigma_i - \mu_i) \end{aligned} \tag{1.31}$$

a necessary condition for this to happen is

$$\left.\begin{aligned} A_3 &= -\nu - 1 \\ B_3 &= -\mu - 1 \end{aligned}\right\} \nu,\mu = 0,\ 1,\ 2,\ \cdots \ . \tag{1.32}$$

If in (1.29) $f_3(z_3)$ lies not entirely in $F_{-\chi_3}$ (replace χ_3 by $-\chi_3$ in (1.29)), the trilinear invariant functional has a pole in σ_3 with the residue

$$\begin{aligned} \operatorname{Res} B(f_3,f_2,f_1) &= \frac{1}{4\pi\nu!\mu!} \int dz_3 f_3(z_3) \\ &\times \int f_2(z)|z - z_3|^{\sigma_1}\left(\frac{z - z_3}{|z - z_3|}\right)^{\mu_1} \frac{\partial^\nu}{\partial z^\nu}\frac{\partial^\mu}{\partial \bar{z}^\mu}|z_3 - z|^{\sigma_2}\left(\frac{z_3 - z}{|z_3 - z|}\right)^{\mu_2} \\ &\times f_1(z)dz \ . \end{aligned} \tag{1.33}$$

In order that the inner integral in (1.33) defines an element of E_{χ_3}, i.e., a polynomial in z, $\bar{z}$, we must implement (1.32) by the requirement

$$\nu \leqq n_3 - 1 \qquad\qquad (1.34)$$
$$\mu \leqq n_3' - 1$$

which can be obtained by inspection from

$$\sigma_1 + \sigma_2 = i\rho_3 - 2 = n_3 + n_3' - 2$$
$$\mu_1 + \mu_2 = -m_3 = n_3 - n_3' \ .$$

As an example we consider the four-vector with

$$n_3 = n_3' = 2$$

in which case ν, μ range over the values 0 and 1 only. We write the inner integral in (1.33) as

$$\int f_2(z)A(\nu,\mu|z,z_3)f_1(z)dz \qquad\qquad (1.35)$$

and get the following operators $A(\nu,\mu|z,z_3)$ (in a new normalization)

$$A(0,0|z,z_3) = (z - z_3)(\bar{z} - \bar{z}_3)$$
$$A(1,0|z,z_3) = (z - z_3)(\bar{z} - \bar{z}_3)\frac{\partial}{\partial z} - (n_1 - 1)(\bar{z} - \bar{z}_3)$$
$$A(0,1|z,z_3) = (z - z_3)(\bar{z} - \bar{z}_3)\frac{\partial}{\partial \bar{z}} - (n_1' - 1)(z - z_3) \qquad\qquad (1.36)$$
$$A(1,1|z,z_3) = (z - z_3)(\bar{z} - \bar{z}_3)\frac{\partial^2}{\partial z\partial\bar{z}} - (n_1 - 1)(\bar{z} - \bar{z}_3)\frac{\partial}{\partial\bar{z}}$$
$$- (n_1' - 1)(z - z_3)\frac{\partial}{\partial z} + (n_1 - 1)(n_1' - 1) \ .$$

The representations χ_1 and χ_2 are restricted by (1.32) to

$$m_1 = -m_2 - 2(\nu - \mu) \qquad\qquad (1.37)$$
$$\rho_1 = -\rho_2 - 2i(\nu + \mu - 1)$$

which relation we abbreviate as $\chi_1 = (-\chi_2)_{\nu\mu}$.

Finally we consider the case that both n_3, n_3' are negative integers, so that D_{χ_3} possesses an invariant subspace F_{χ_3}. In this case the integrals

$$\int dz_3 z_3^{\ell}\bar{z}_3^{k} \int N(z_3,z_2,z_1|-\chi_3,\chi_2,\chi_1)f_2(z_2)f_1(z_1)dz_2dz_1 \qquad\qquad (1.38)$$

can easily be evaluated by elementary methods and shown to be zero for any pair of functions $f_{1,2}$ belonging to regular representations $\chi_{1,2}$ and for all

$$0 \leqq \ell \leqq -n_3 - 1$$
$$0 \leqq k \leqq -n_3' - 1$$

provided A_3 and B_3 (1.31) are not simultaneously integers on the half axis

$$A_3 \leqq -n_3 - 1 \qquad\qquad (1.39)$$
$$B_3 \leqq -n_3' - 1 \ .$$

This means that with the sole exception of the cases (1.39) the integral (1.28) defines an element of F_{χ_3}. Only in the exceptional cases (1.39) the components in the finite dimensional space D_{χ_3}/F_{χ_3} are nonzero. If we implement (1.39) by the further condition

$$A_3 \geqq 0,\ B_3 \geqq 0 \tag{1.40}$$

we obtain finite irreducible tensor operators. In fact, it is possible in this case to decompose Naimark's kernel into two intertwining operators that map $D_{\chi_{1,2}}$ each onto $D_{-\chi_{1,2}}$, the finite irreducible tensor operator obtained earlier which maps the tensor product of $D_{-\chi_1}$ and $D_{-\chi_2}$ into $E_{-\chi_3}$, and the intertwining operator from $E_{-\chi_3}$ into D_{χ_3}/F_{χ_3}.

2. CURRENT DENSITY OPERATORS

2.1. Vertex Functions

First we introduce a special realization of a unitary irreducible representation of the group $SL(2,C) \times T_4$ for a particle of mass M and spin S and positive energy. We define a Hilbert space of measurable, vector valued functions on $SL(2,C)$ $\Phi_q(a)$, $-S \leqq q \leqq S$, $S - q$ integral, which have finite norm with respect to the scalar product

$$(\Phi^1,\Phi^2) = \int d\mu(a) \sum_q \overline{\Phi^1_q(a)}\Phi^2_q(a) \tag{2.1}$$

$d\mu(a)$ is the Haar measure on $SL(2,C)$ normalized (in the notations of Section 1.1) as

$$a = \zeta k,\ d\mu(a) = (2\pi)^{-4} dz d\lambda d\mu \ .$$

In addition, we require that the functions $\Phi_q(a)$ be covariant on right cosets of SU(2), i.e.,

$$\Phi_q(ua) = \sum_{q'} D^S_{qq'}(u)\Phi_{q'}(a) \ . \tag{2.2}$$

The matrix D^S describes a unitary irreducible representation of SU(2) of spin S. We call this Hilbert space $L^2(M,S)$. In this space we define the representation $\{M,S\}$ by

$$\begin{aligned} U_a\Phi_q(a_1) &= \Phi_q(a_1 a) \\ U_x\Phi_q(a) &= \exp\{\tfrac{1}{2}\, iM\mathrm{Tr}(\underset{\sim}{x} a^+ a)\}\Phi_q(a) \end{aligned} \tag{2.3}$$

where $\underset{\sim}{x}$ is a two-by-two matrix constructed from the translation four-vector x as

$$\underset{\sim}{x} = \sum_{\mu=0,1,2,3} x_\mu \sigma_\mu \ .$$

Here σ_o is the unit matrix and σ_k, k = 1, 2, 3, are the familiar Pauli matrices. We mention that the four-momentum vector used in physics is related with the argument $a \in SL(2,C)$ by

$$\begin{aligned} P_o &= \tfrac{1}{2}\, M\mathrm{Tr}(a^+ a) \\ P_k &= -\tfrac{1}{2}\, M\mathrm{Tr}(\sigma_k a^+ a) \ . \end{aligned} \tag{2.4}$$

The unitary irreducible representation $\{M,S\}$ of $SL(2,C) \times T_4$ is unitary but reducible on the subgroup $SL(2,C)$. In order to reduce this representation we embed the space $L^2(M,S)$ in the space $L^2(SL(2,C))$ which carries the right regular representation. A canonical way of doing this is

$$\Phi_q(a) = \int d\mu(u) D^S_{q'}(u)\tilde{\Phi}(u^{-1}a)$$

where $\tilde{\Phi}(a)$, is any element of $L^2(SL(2,C))$. If we apply the Plancherel theorem of $SL(2,C)$ to the right regular representation, we obtain the direct integral decomposition

$$L^2(M,S) = \sum_{m=-2S}^{2S}{}^{\oplus} \int_0^\infty{}^{\oplus} d\rho\,(m^2 + \rho^2) L^2(\chi) \tag{2.5}$$

where $L^2(\chi)$ carries the principal series representation χ of $SL(2,C)$. As a realization we may use for example the space $L^2_m(K)$ discussed in Section 1. We emphasize that this decomposition is free of degeneracies. If we restrict a current density operator to single particle spaces, say its domain is in $L^2(M_1,S_1)$ and its range in $L^2(M_2,S_2)$, it decomposes together with the two spaces, and it is this decomposition which we are interested in.

In order to define suitable vertex functions for a given current density operator $j_\mu(x)$ acting between the spaces $L^2(M_{1,2},S_{1,2})$, we consider the matrix elements

$$\begin{aligned}\langle\Phi^2|j_\mu(0)|\Phi^1\rangle &= (\Phi^2, j_\mu(0)\Phi^1) \\ &= N_1N_2 \sum_{q_1q_2} \int d\mu(a_1)d\mu(a_2)\overline{\Phi^2_{q_2}(a_2)}\Gamma_\mu(a_2,a_1)_{q_2q_1}\Phi^1_{q_1}(a_1)\end{aligned} \tag{2.6}$$

where $\Phi^i \in L^2(M_i,S_i)$. The N_i are normalization constants. The normalization customarily used in physical literature is such that for the matrix element of the electromagnetic current between proton states we have

$$\Gamma_\mu(e,e)_{q_2q_1} = \delta_{\mu o}\delta_{q_1q_2} \text{ times charge of the proton}$$

(e is the unit element of $SL(2,C)$) which is achieved by

$$N = (2S + 1)\left(\frac{2M}{(2\pi)^3}\right)^{\frac{1}{2}} 8\pi^2M^2 .$$

Of course the domain of $j_\mu(0)$ is not the whole Hilbert space $L^2(M_1,S_1)$ in general, but at least it is not smaller than the space $C^\infty_c(M_1,S_1)$ of infinitely differentiable functions with compact support on $SL(2,C)$ that satisfy the constraint (2.2). Under the Fourier decomposition (2.5) this space goes over into a space of functions (for the realizations $L^2_m(K)$ these functions can be written $\varphi(u,m,\rho)$ satisfying the constraint (1.5)) which are entire in ρ.

The definition (2.6) is not yet unique, we complete it by requiring covariance on right cosets of $SU(2)$ in (2.12). The vertex function $\Gamma_\mu(a_2,a_1)_{q_2,q_1}$ is then a vector valued function. Let us define

$$|a|^2 = \mathrm{Tr}(a^+a) .$$

The "four-momentum transfer" q

$$q = p_2 - p_1,\ p_{1,2} = p(a_{1,2})$$

(see (2.4)) lies in the domain

$$q^2 = q_0^2 - q_1^2 - q_2^2 - q_3^2 = M_1^2 + M_2^2 - M_1M_2|a_1a_2^{-1}|^2 \leqq (M_1 - M_2)^2 .$$

From field theory we know that below a "threshold mass" M_{th}

$$-\infty < q^2 < M_{th}^2$$

(M_{th} may equal two pion masses, for example) the vertex function is analytic as a function of the real variables on $SL(2,C) \times SL(2,C)$. In the worst case, namely when

$$M_{th}^2 < (M_1 - M_2)^2$$

there is a finite q^2 interval on which we have no analyticity. But a physicist's intuition lets us expect that in this interval we have at most a finite number of singular points due to additional thresholds with continuity at these points and continuous differentiability in between. The harmonic analysis of the vertex functions is consequently beset with at most a complication due to their behavior if q^2 tends to infinity. We may try to handle this complication by means of a regularization procedure.

In order to formulate the four-vector covariance of the vertex function and the covariance on the right cosets of $SU(2)$ it is advantageous to introduce a basis in $L_m^2(K)$, the "canonical basis". We use the functions

$$\varphi_q^j(u) = (2j + 1)^{\frac{1}{2}}\ D^j_{\frac{1}{2}m,q}(u);\ -j \leqq q \leqq j,\ j = \frac{1}{2}\ |m| + n,\ n = 0,\ 1,\ 2,\ \cdots \qquad (2.7)$$

where D^S is the same unitary matrix as in (2.2). The orthonormality and completeness of this basis in $L_m^2(K)$ follows from the theorem of Peter and Weyl. This basis lies in the spaces D_χ and a subbasis can be used to span the invariant subspaces E_χ and F_χ. It can be carried over to the noncompact picture where we denote its elements by $f_q^j(z)$. If the operator T_a^χ in D_χ acts on a basis element $f_q^j(z)$ we obtain the "coordinate functions"

$$T_a^\chi f_q^j(z) = \sum_{j'q'} D^\chi_{j'q'jq}(a) f^{j'}_{q'}(z) . \qquad (2.8)$$

In particular we have

$$\begin{aligned} D^\chi_{j_1q_1j_2q_2}(u) &= \delta_{j_1j_2}\, D^{j_1}_{q_1q_2}(u) \\ D^\chi_{j_1q_1j_2q_2}(d) &= \delta_{q_1q_2}\, d^\chi_{j_1j_2q}(\eta) \end{aligned} \qquad (2.9)$$

where the matrix d is defined by

$$d = \begin{pmatrix} e^{\frac{1}{2}\eta} & 0 \\ 0 & e^{-\frac{1}{2}\eta} \end{pmatrix},\ \eta \geqq 0 . \qquad (2.10)$$

Finally, we switch from the vector labels $\mu = 0,\ 1,\ 2,\ 3$ to components with respect to the canonical basis in the space E_χ, $\chi = (0,-4i)$, which carries the vector representation

$$\Gamma^J_Q(a_2,a_1)_{q_2q_1} : J = Q = 0 \quad \text{and} \quad J = 1,\ Q = +1,\ 0,\ -1$$

$$\Gamma^0_0 = \pi^{\frac{1}{2}}\,\Gamma_0,\ \Gamma^1_0 = -(\tfrac{\pi}{3})^{\frac{1}{2}}\,\Gamma_3,\ \Gamma^1_{\pm 1} = \pm(\tfrac{\pi}{6})^{\frac{1}{2}}(\Gamma_1 \mp i\Gamma_2) \ . \tag{2.11}$$

Then the covariance properties of the vertex functions are expressed by the formulae

$$\Gamma^J_Q(u_2a_2,u_1a_1)_{q_2q_1} = \sum_{q_1'q_2'} D^{S_2}_{q_2q_2'}(u_2)D^{S_1}_{q_1'q_1}(u_1^{-1})\Gamma^J_Q(a_2,a_1)_{q_2'q_1'} \tag{2.12}$$

$$\Gamma^J_Q(a_2a^{-1},a_1a^{-1})_{q_2q_1} = \sum_{\substack{J'Q' \\ J'\leq 1}} D^{(0,-4i)}_{JQJ'Q'}(a)\Gamma^{J'}_{Q'}(a_2,a_1)_{q_2q_1} \ . \tag{2.13}$$

2.2. The Decomposition of a Vertex Function with Covariance in the Principal Series

In (2.12), (2.13) the covariance was formulated in so general terms that we may immediately modify these equations and study vertex functions which transform as a representation χ of the principal series. To avoid confusion we add the label χ to the arguments of the vertex functions, the coordinate functions in (2.13) $D^{(0,-4i)}$ are replaced by D^χ. The main tool of the Fourier decomposition of the vertex function obtained in this fashion is Naimark's theorem. At the end we continue in χ analytically until we arrive at the point $(0,-4i)$ again.

For χ in the principal series complex conjugation maps D_χ onto $D_{-\chi}$ (independently of the two pictures), in particular

$$\overline{f^j_q(z)^\chi} \in D_{-\chi} \ . \tag{2.14}$$

Denoting analytic continuations of the complex conjugate off the principal series by $(\cdots)^*$, we have from (2.14) and (1.2)

$$(f^j_q(z)^\chi)^* \in D_{-\chi}$$
$$(T^\chi_a f^j_q(z)^\chi)^* = T^{-\chi}_a(f^j_q(z)^\chi)^* \ . \tag{2.15}$$

The unitarity of the principal series representations implies

$$(D^\chi_{j_1q_1j_2q_2}(a))^* = D^\chi_{j_2q_2j_1q_1}(a^{-1}) \ . \tag{2.16}$$

The bilinear invariant functional (1.14), (1.15) enables us to introduce a matrix calculus by

$$B((f^{j_2}_{q_2})^*, Af^{j_1}_{q_1}) = \langle \chi_2;j_2q_2|A|\chi_1;j_1q_1\rangle \tag{2.17}$$

provided

$$f^{j_1}_{q_1} \in D_{\chi_1},\ f^{j_2}_{q_2} \in D_{\chi_2},\ AD_{\chi_1} \subset D_{\chi_2} \ .$$

For $A = \mathbb{1}$ this gives

$$\langle\chi;j_2q_2|\mathbf{1}|\chi;j_1q_1\rangle = \delta_{j_1j_2}\delta_{q_1q_2} \tag{2.18}$$

whereas $A = T^\chi_a$ leads us back to the coordinate functions (2.8)

$$\langle\chi;j_2q_2|T^\chi_a|\chi;j_1q_1\rangle = D^\chi_{j_2q_2j_1q_1}(a) \ . \tag{2.19}$$

A similar notation can be used for the trilinear invariant functional (1.29) for the representations $\chi_1, -\chi_2, -\chi_3$

$$B((f^J_Q)^*,(f^{j_2}_{q_2})^*,f^{j_1}_{q_1}) = \langle\chi_2;j_2q_2|A^J_Q(\chi_3)|\chi_1;j_1q_1\rangle \tag{2.20}$$

for any

$$f^{j_1}_{q_1} \in D_{\chi_1},\ f^{j_2}_{q_2} \in D_{\chi_2},\ f^J_Q \in D_{\chi_3}$$

and χ_3 in the principal series, say. The linearity and continuity of the functional implies

$$B((f^J_Q)^*,(T^{\chi_2}_{a_2^{-1}}f^{j_2}_{q_2})^*,T^{\chi_1}_{a_1^{-1}}f^{j_1}_{q_1}) = \sum_{j_1'q_1'j_2'q_2'} D^{\chi_1}_{j_1'q_1'j_1q_1}(a_1^{-1})D^{\chi_2}_{j_2q_2j_2'q_2'}(a_2)$$
$$\times \langle\chi_2;j_2'q_2'|A^J_Q(\chi_3)|\chi_1;j_1'q_1'\rangle \ .$$

With (2.16), (2.19) and matrix calculus we can continue this equation

$$= \langle\chi_2;j_2q_2|T^{\chi_2}_{a_2}A^J_Q(\chi_3)T^{\chi_1}_{a_1^{-1}}|\chi_1;j_1q_1\rangle \ . \tag{2.21}$$

From (2.15), (2.16) and the invariance of the trilinear functional we have

$$\langle\chi_2;j_2q_2|T^{\chi_2}_{a^{-1}}A^J_Q(\chi_3)T^{\chi_1}_a|\chi_1;j_1q_1\rangle = \sum_{J'Q'} D^{\chi_3}_{JQJ'Q'}(a)\langle\chi_2;j_2q_2|A^{J'}_{Q'}(\chi_3)|\chi_1;j_1q_1\rangle \ . \tag{2.22}$$

Comparing (2.21), (2.22) with (2.12), (2.13) we recognize that the vertex function $\Gamma^J_Q(a_2,a_1|\chi)_{q_2q_1}$ has the same covariance properties as the matrix element

$$\langle\chi_2;S_2q_2|T^{\chi_2}_{a_2}A^J_Q(\chi)T^{\chi_1}_{a_1^{-1}}|\chi_1;S_1q_1\rangle$$

where χ_1 and χ_2 are arbitrary. This fact suggests that we decompose the vertex functions into such matrix elements.

In fact we define a Fourier transform by

$$M(\chi_2,\chi_1;\chi) = \int d\mu(a_2a_1^{-1}) \sum_{JQ} \Gamma^J_Q(a_2,a_1|\chi)_{q_2q_1}$$
$$\times \langle\chi_2;S_2q_2|T^{\chi_2}_{a_2}A^J_Q(\chi)T^{\chi_1}_{a_1^{-1}}|\chi_1;S_1q_1\rangle^* \tag{2.23}$$

when all three $\chi_{1,2}$ and χ are in the principal series. The left-hand side of (2.23) can be verified to be independent of q_1 and q_2. The main tool in the inversion of this Fourier transformation is Naimark's theorem in the form (note that the product $\chi \times \chi_2$ is decomposed into $\int^\oplus d\chi_1$)

$$\sum_{j_1 q_1} \int d\chi_1 \langle \chi_2; j_2' q_2' | A^{J'}_{Q'}(\chi) | \chi_1; j_1 q_1 \rangle^* \langle \chi_2; j_2 q_2 | A^J_Q(\chi) | \chi_1; j_1 q_1 \rangle = \delta_{j_2 j_2'} \delta_{q_2 q_2'} \delta_{JJ'} \delta_{QQ'} \tag{2.24}$$

and the Plancherel theorem for SL(2,C) in a similar matrix version

$$\frac{1}{2} \int d\chi \sum_{J_1 Q_1 J_2 Q_2} D^\chi_{J_1 Q_1 J_2 Q_2}(a_1) (D^\chi_{J_1 Q_1 J_2 Q_2}(a_2))^* = \delta(a_1, a_2) \ . \tag{2.25}$$

Here $\delta(a_1, a_2)$ is the delta-function on SL(2,C) normalized with respect to the Haar measure (see (2.1)). From (2.12) and (2.13) we have

$$\begin{aligned} &\frac{1}{2} \int d\chi_1 d\chi_2 M(\chi_2, \chi_1; \chi) \langle \chi_2; S_2 q_2 | T^{\chi_2}_{a_2} A^J_Q(\chi) | \chi_1; S_1 q_1 \rangle \\ &= \frac{1}{2} \int d\chi_1 d\chi_2 \sum_{j_1 j_2 q_1' q_2'} \langle \chi_2; j_2 q_2' | T^{\chi_2}_{a_2} A^J_Q(\chi) | \chi_1; j_1 q_1' \rangle \\ &\quad \times \int d\mu(a_2') \sum_{J'Q'} \langle \chi_2; j_2 q_2' | T^{\chi_2}_{a_2'} A^{J'}_{Q'}(\chi) | \chi_1; j_1 q_1' \rangle^* \Gamma^{J'}_{Q'}(a_2', e | \chi)_{q_2 q_1} \end{aligned}$$

where we exploited the covariance to set $a_1' = e$. Inserting the two formulae (2.24), (2.25) into this expression we obtain the desired result $\Gamma^J_Q(a_2, e | \chi)_{q_2 q_1}$. By means of covariance we can extend this result to the general inversion formula

$$\Gamma^J_Q(a_2, a_1 | \chi)_{q_2 q_1} = \frac{1}{2} \int d\chi_1 d\chi_2 M(\chi_2, \chi_1; \chi) \times \langle \chi_2; S_2 q_2 | T^{\chi_2}_{a_2} A^J_Q(\chi) T^{\chi_1}_{a_1^{-1}} | \chi_1; S_1 q_1 \rangle \ . \tag{2.26}$$

It does not make much sense to discuss the convergence of (2.23) and (2.26) using the information on the vertex function supplied by field theory. We mention only that a sufficient condition for the proper convergence of both (2.23) and (2.26) is infinite differentiability and rapid decrease (that is, faster decrease than any power of $|a|$) of $\Gamma^J_Q(a, e | \chi)_{q_2 q_1}$ on SL(2,C). We call such vertex functions "smooth".

2.3. The Decomposition of the Four-vector vertex Function

For a smooth vertex function we write (2.23), (2.26) in the form

$$\begin{aligned} \Gamma^J_Q(a_2, a_1 | \chi)_{q_2 q_1} &= \frac{1}{2} \int d\chi_1 d\chi_2 \int d\mu(a_2' a_1'^{-1}) \sum_{J'Q'} \\ &\quad \times \Gamma^{J'}_{Q'}(a_2', a_1' | \chi)_{q_2 q_1} \langle \chi_2; S_2 q_2 | T^{\chi_2}_{a_2} A^J_Q(\chi) T^{\chi_1}_{a_1^{-1}} | \chi_1; S_1 q_1 \rangle \\ &\quad \times \langle \chi_2; S_2 q_2 | T^{\chi_2}_{a_2'} A^{J'}_{Q'}(\chi) T^{\chi_1}_{a_1'^{-1}} | \chi_1; S_1 q_1 \rangle^* \end{aligned} \tag{2.27}$$

which we want to continue in χ. Because of the smoothness of the vertex function we may handle the expression (2.27) rather freely without facing problems of convergence. When we reach the point

$$\chi = \{n, n'\}, \ n = n' = 2$$

we define $\Gamma^J_Q = 0$ for $J > 1$. The first bracket $\langle\cdots\rangle$ in (2.27) with J restricted to $J \leqq 1$ is finite everywhere except that a pole occurs whenever

$$\chi_1 = (\chi_2)_{\nu\mu}$$

(see (1.37), remember that χ_2 has been replaced by $-\chi_2$ in (2.20)). The bracket $\langle\cdots\rangle^*$ of (2.27) is in turn zero everywhere except at the same position, where it assumes a finite limit. Both the residue of the pole and the finite limit are matrix elements of vector operators (1.36). A careful analysis shows that the integration over χ_1 and χ_2 picks up just the residue of the pole, and of the two integrations only one is left. One of the two representations $\chi_{1,2}$ is pushed off the principal series, for convenience we choose χ_1. The smoothness of the vertex function accounts for the nonunitarity (polynomial increase) of χ_1. With the new variable $\lambda = \frac{1}{2} i\rho$ and $S = \min(S_1, S_2)$ we get

$$\Gamma^J_Q(a_2, a_1)_{q_2 q_1} = \frac{8i}{\pi} \sum_{\nu,\mu=0,1} \sum_{m=-2S}^{2S} \int_{-i\infty}^{+i\infty} d\lambda \times M_{\nu\mu}(m,\lambda)\langle\chi; S_2 q_2 | T^{\chi}_{a_2} A^J_Q(\nu,\mu) T^{\chi_{\nu\mu}}_{a_1^{-1}} | \chi_{\nu\mu}; S_1 q_1\rangle \tag{2.28}$$

and the Fourier transform

$$M_{\nu\mu}(m,\lambda) = \sum_{JQ} \int d\mu(a_2 a_1^{-1}) \Gamma^J_Q(a_2, a_1)_{q_2 q_1} (-1)^J (2J+1) \times \langle\chi; S_2 q_2 | T^{\chi}_{a_2} A^J_Q(1-\mu, 1-\nu) T^{\chi_{1-\mu,1-\nu}}_{a_1^{-1}} | \chi_{1-\mu,1-\nu}; S_1 q_1\rangle^* . \tag{2.29}$$

As long as χ is in the principal series $(\cdots)^*$ means the complex conjugate.

Formula (2.29) for the Fourier transform can easily be simplified. We set first

$$\begin{aligned} a_2 &= u_2 d(\eta_2) a \\ a_1 &= u_1 d(-\eta_1) a \end{aligned} \qquad \eta_{1,2} \geqq 0,\ \eta = \eta_1 + \eta_2$$

($d(\eta_{1,2})$ as in (2.10)) and can decompose the measure correspondingly

$$d\mu(a_2 a_1^{-1}) = (4\pi)^{-1} d\mu(u_1) d\mu(u_2) \operatorname{sh}^2\eta\, d\eta .$$

Inserting this into (2.29) we obtain

$$M_{\nu\mu}(\chi) = [4\pi(2S_1+1)(2S_2+1)]^{-1} \int_0^\infty d\eta \operatorname{sh}^2\eta \times \sum_{JQJ_1J_2q_1q_2} (-1)^J (2J+1) \Gamma^J_Q(d(\eta_2), d(\eta_1)^{-1})_{q_2 q_1} \times d^{\chi^*}_{S_2 J_2 q_2}(\eta_2) d^{(\chi^*)_{1-\mu,1-\nu}}_{J_1 S_1 q_1}(\eta_1) \langle\chi^*; J_2 q_2 | A^J_Q(1-\mu, 1-\nu) | (\chi^*)_{1-\mu,1-\nu}; J_1 q_1\rangle \tag{2.30}$$

where we used the notation

$$\chi^* = (m, -\rho) \quad \text{for} \quad \chi = (m, \rho) .$$

In (2.30) we may set η_1 or η_2 equal to zero in which case one of the two d-functions drops out. We recall that by (2.4) the matrix $d(\eta)$ corresponds to the four-momentum

$$p = (M\mathrm{ch}\eta, 0, 0, -M\mathrm{sh}\eta) \ .$$

The vertex function entering (2.30) has therefore been brought into a "collinear" frame of inertia. In these frames it is easy to express vertex functions by some conventional kind of form factors. The momentum transfer q^2 is

$$q^2 = M_1^2 + M_2^2 - 2M_1M_2\mathrm{ch}\eta \ .$$

In physical applications the vertex functions have to be regularized to fulfill the assumption of smoothness. One of the basic premises in standard applications to physics is that the removal of the regularization can be accounted for by a mere change of the integration contours in (2.28). Typical statements arrived at in such applications involve asymptotic expansions of vertex functions. The derivation of such asymptotic expansions is always based on the following "Weyl symmetry relations" which reflect the existence of an intertwining operator for two representations χ and $-\chi$:

$$d^{\chi}_{j_1 j_2 q}(\eta) = \beta^{j_1}(-\lambda)\beta^{j_2}(\lambda) d^{-\chi}_{j_1 j_2 q}(\eta) \tag{2.31}$$

$$\begin{aligned} \langle \chi; S_2 q_2 | A^J_Q(\nu,\mu) | \chi_{\nu\mu}; S_1 q_1 \rangle &= -\beta^{S_2}(-\lambda)\beta^{S_1}(\lambda_{\nu\mu}) \\ &\times \langle -\chi; S_2 q_2 | A^J_Q(1-\nu, 1-\mu) | (-\chi)_{1-\nu,1-\mu}; S_1 q_1 \rangle \end{aligned} \tag{2.32}$$

$$M_{\nu\mu}(\chi) = -\beta^{S_2}(\lambda)\beta^{S_1}(-\lambda_{\nu\mu}) M_{1-\nu,1-\mu}(-\chi) \tag{2.33}$$

with

$$\beta^S(\lambda) = \frac{\Gamma(S+1+\lambda)}{\Gamma(S+1-\lambda)} \ . \tag{2.34}$$

We close this section with the remark that the principal series is not sufficient for an expansion of vertex functions with tensorial covariance of higher rank, e.g., with a covariance like that of antisymmetric or symmetric traceless tensors of rank two. In addition, we have then contributions from a "discrete" series. Details on the material presented in this section can be found in [6].

3. PHENOMENOLOGICAL ANALYSIS OF THE ELECTROMAGNETIC VERTEX FUNCTION OF HADRONS

3.1. Asymptotic Expansions of Form Factors

We assume that the Fourier transformation (2.29) yields an analytic function of λ. Moreover, we assume (only for simplicity) that the singularities closest to the imaginary axis are simple poles. At present there is in fact no justification of this hypothesis other than an *a posteriori* verification of its implications by experiment. The situation is even worse than in the formally related case of Regge poles, since nonrelativistic quantum mechanics cannot serve as a heuristic guide.

We set $a_1 = e$, $a_2 = d(\eta)$ in (2.28) and have

$$\Gamma_Q^J(d(\eta),e)_{q_2q_1} = \frac{8i}{\pi}\sum_{\nu,\mu}\sum_m\int_{-i\infty}^{+i\infty}d\lambda M_{\nu\mu}(\chi) \times \sum_{J_2} d^{\chi}_{S_2J_2q_2}(\eta)\langle\chi;J_2q_2|A_Q^J(\nu,\mu)|\chi_{\nu\mu};S_1q_1\rangle \equiv \Gamma_Q^J(\eta)_{q_2q_1} \; . \tag{3.1}$$

Obviously J_2 is restricted to $|J_2 - S_1| \leq 1$. The Weyl symmetry (2.31) of the coordinate function can be made explicit if we introduce "coordinate functions of the second kind" (or e-functions) by

$$d^{\chi}_{j_1j_2q}(\eta) = e^{\chi}_{j_1j_2q}(\eta) + \beta^{j_1}(-\lambda)\beta^{j_2}(\lambda)e^{-\chi}_{j_1j_2q}(\eta) \; . \tag{3.2}$$

These e-functions are defined uniquely by their asymptotic property

$$\overline{\lim_{\mathrm{Re}\,\lambda\leq 0,|\lambda|\to\infty}}|\lambda e^{-\lambda\eta}e^{\chi}_{j_1j_2q}(\eta)| < \infty \quad \text{for} \quad \eta > 0 \; .$$

The Weyl symmetry (2.32), (2.33) of both the Fourier transforms and the matrix elements of the vector operators $A(\nu,\mu)$ allows us to rewrite (3.1) as

$$\Gamma_Q^J(\eta)_{q_2q_1} = \frac{16i}{\pi}\sum_{\nu,\mu}\sum_m\int_{-i\infty}^{+i\infty}d\lambda M_{\nu\mu}(\chi) \times \sum_{J_2} e^{\chi}_{S_2J_2q_2}(\eta)\langle\chi;J_2q_2|A_Q^J(\nu,\mu)|\chi_{\nu\mu};S_1q_1\rangle \; . \tag{3.3}$$

If $M_{\nu\mu}(\chi)$ has the properties assumed at the beginning, the asymptotic behavior of $\Gamma_Q^J(\eta)_{q_2q_1}$ in η for $\eta \to \infty$ is

$$\Gamma_Q^J(\eta) \cong -32 \operatorname*{Res}_{\lambda=\lambda'} M_{\nu'\mu'}(m',\lambda) \times \sum_{J_2} e^{(m',\lambda')}_{S_2J_2q_2}(\eta)\langle m',\lambda';J_2q_2|A_Q^J(\nu',\mu')|\chi'_{\nu\mu};S_1q_1\rangle$$

where the dominant pole is assumed to appear in the Fourier transform labelled ν', μ', m' and at the position λ'. The asymptotic behavior of the e-function in η is

$$e^{\chi}_{j_1j_2q}(\eta) = Ce^{(\lambda-1-|q+\frac{1}{2}m|)\eta}(1 + O(e^{-2\eta})) \tag{3.5}$$

where C is independent of η.

3.2. Electromagnetic Form Factors of the Nucleon

We want now to investigate in detail the case that the asymptotic behavior of the form factors is caused by a simple pole in the Fourier transforms. We neglect the other singularities, mention, however, that the known analytic structure of the form factors is not reproduced by a finite number of poles.

As long as we consider only one process, elastic electron proton scattering, for example, the physical meaning of such pole is difficult to describe. Quite the same situation arises in the case of Regge poles. The interpretation of the Regge pole as an exchanged object is connected with the possibility to identify this Regge pole in a whole class of processes and to characterize it by a set of quantum numbers. A meaningful interpretation of the poles we are considering here necessitates the simultaneous discussion of a whole class of processes as well. Since we shall do this in the next section we postpone the further discussion.

In an actual physical application we have to extend the group SL(2,C) to parity, time reversal, and isospin. In addition, we know that the electromagnetic current is conserved and that its restriction to single-particle spaces is selfadjoint. The group extensions are in fact trivial generalizations, but current conservation imposes a subsidiary condition on the Fourier transforms which deserves a detailed study. We can show that it implies linear difference equations whose coefficients depend on the masses, the spins, and the intrinsic parities. In the case of the nucleon form factors [7] the isospin invariance is taken into account by treating the isoscalar and isovector parts of the current independently but in an analogous fashion. Since $S_1 = S_2 = \frac{1}{2}$ in this case, the representations of SL(2,C) occurring have $m = \pm 1$. The following symmetry relations

$$\begin{aligned} M_{00}(1,\lambda) &= M_{00}(-1,\lambda) \\ M_{01}(-1,\lambda) &= -M_{10}(1,\lambda) \\ M_{11}(1,\lambda) &= M_{11}(-1,\lambda) \end{aligned} \tag{3.6}$$

are due to parity invariance; Weyl symmetry relations imply

$$\begin{aligned} M_{01}(-1,\lambda) &= -M_{10}(1,-\lambda) \\ M_{00}(1,\lambda) &= (\lambda + \tfrac{1}{2})(\lambda - \tfrac{3}{2})M_{11}(-1,-\lambda) \ . \end{aligned} \tag{3.7}$$

Current conservation and time reversal invariance yield finally

$$M_{11}(1,\lambda) = M_{11}(1,-\lambda - 1) \ . \tag{3.8}$$

Due to the selfadjointness all $M_{\nu\mu}(\chi)$ are real for real λ.

The unnormalized form factors of Sachs type [7] are given by

$$\begin{aligned} \Gamma^0_0(\eta)_{\pm\frac{1}{2},\pm\frac{1}{2}} &= e\sqrt{\pi}\ \mathrm{ch}\ \frac{1}{2}\ \eta G_E(q^2) \\ \Gamma^1_0(\eta)_{\pm\frac{1}{2},\pm\frac{1}{2}} &= e\sqrt{\frac{\pi}{3}}\ \mathrm{sh}\ \frac{1}{2}\ \eta G_E(q^2) \\ \Gamma^1_{\pm 1}(\eta)_{\mp\frac{1}{2},\pm\frac{1}{2}} &= e\sqrt{\frac{2\pi}{3}}\ \mathrm{sh}\ \frac{1}{2}\ \eta G_M(q^2) \ . \end{aligned} \tag{3.9}$$

The most appealing ansatz seems to be a dominant pole in M_{01} or M_{10} of the isovector current (we denote such pole "isovector class one") at $\lambda = \lambda_1$ on the real axis close to $-\frac{1}{2}$. It implies

$$\left.\begin{aligned} G_E &\cong C_E\left(-\frac{q^2}{M^2}\right)^{\lambda_1-\frac{3}{2}} \\ G_M &\cong C_M\left(-\frac{q^2}{M^2}\right)^{\lambda_1-\frac{3}{2}} \end{aligned}\right\} q^2 \to -\infty$$

$$C_E : C_M = (\lambda_1 - \tfrac{1}{2})^{-1} \ . \tag{3.10}$$

We can describe by this ansatz a positive definite proton magnetic form factor, a negative neutron magnetic form factor, and a proton electric form factor which has to change sign before the asymptotic domain is reached (somewhere between 5 and 10 GeV^2). These findings are in agreement with the experimental data.

3.3. Form Factors for the Electroproduction of Nucleon Resonances

Inelastic scattering of electrons off a proton serves to analyze the electromagnetic transition matrix elements for a proton going into nucleon resonances. We consider an irreducible representation of SL(2,C) extended by parity and isospin. We call a set of resonances a tower, if their spin-parity and isospin quantum numbers allow us to fit them into one such representation. Towers are therefore labelled by the invariants of the extended SL(2,C). Neither need all places in such representations be occupied by resonances, nor is the number of resonances occupying one state in a representation or the number of towers to which one resonance belongs bounded by one (we allow for an arbitrary "representation mixing"). We substantiate this definition by the assumption that the Fourier transforms of the electromagnetic transition elements from the proton to the resonances of one tower exhibit a pole at the same position. The quantum numbers and the SL(2,C) invariants of the tower of resonances are coupled with the corresponding quantum numbers of the proton tower via the vector operator and isospin Clebsch-Gordon coefficients. In particular is the location of the pole in the Fourier transforms identical with the SL(2,C) invariant λ of the tower of resonances. Since as we mentioned an infinite set of poles is necessary to reproduce the known analytic properties of the form factors, one resonance has contributions in an infinite set of towers, we have "infinite representation mixing".

The concept thus arrived at is best compared with Barut's notion of dynamical groups [8]. It deviates mainly from it by weakening most of its premises. In particular we need not specify

a) The noncompact group. Such group must always include SL(2,C) as a subgroup because of relativistic invariance. We use the minimal group SL(2,C) itself.

b) The representations by their unitarity, irreducibility, degeneracy, etc. We note that a dynamical group model based on a simple Lie group which is strictly bigger than SL(2,C) implies Fourier transforms on SL(2,C) which exhibit sequences of equally spaced poles. This behavior is analogous to the reduction of a Toller pole into infinite families of Regge poles.

c) The form of the current.

We lose by this generalization a global representation of the form factors and are left only with asymptotic expansions. However, the generality of our ansatz lets us hope that our scheme might prove useful for a phenomenological analysis of the electron scattering data. The tower hypothesis requires an experimental verification.

We want to illustrate finally how this tower hypothesis correlates data for different production processes. We consider a tower with isospin $\frac{1}{2}$ and spin parity content $S^P = \frac{1^\pm}{2}, \frac{3^\pm}{2}, \cdots$ which is connected with the proton via the isovector part of the electromagnetic current, and whose SL(2,C) invariant λ corresponds to the position of the isovector class one pole (see Section 3.2). This is justified by the fact that the proton itself fits into this tower. Other candidates for this tower are

$$N(1518),\ S^P = \tfrac{3}{2}^-,\quad N(1680),\ S^P = \tfrac{5}{2}^-,\quad N(1688),\ S^P = \tfrac{5}{2}^+ .$$

If such pole dominates the form factors, we obtain for the ratio of the cross sections in the laboratory frame and fixed electron scattering angle

$$\lim_{\substack{q^2 \to -\infty \\ \theta \text{ fixed}}} \left(\frac{d\sigma}{d\Omega}\right)_{res} : \left(\frac{d\sigma}{d\Omega}\right)_{elast} = \text{const.}$$

where the const. does not depend on θ. Its value remains unknown, since we did not specify the residues corresponding to the different members of one tower in our model. We can use this additional freedom to adjust these residues for current conservation without restricting the mass spectrum.

3.4. Conclusion

The use which can be made of our mathematical formalism is certainly not restricted to the analysis of the phenomenology of electromagnetic processes as sketched in Section 3. The priority given to this application is historical and due to the simple fact that this application is formally the simplest one. Further details on this kind of application are contained in the original articles [9].

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