LIE ALGEBRAS OF LOCAL CURRENTS AND THEIR REPRESENTATIONS*

by

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1. INTRODUCTION

In these lectures, our aim is to describe some problems concerning representations of infinite dimensional Lie algebras, whose solution would be of considerable interest to physicists. These problems arise quite generally in trying to implement the "current algebra" approach to elementary particle physics. However, the specific topics we shall discuss here have to do with recent suggestions that one might be able to write relativistic theories of hadrons exclusively in terms of local observables such as currents [1-4].

The talks are organized as follows. First, we shall try to explain briefly how our approach fits in with what physicists usually call "current algebra". Secondly, we shall rewrite ordinary non-relativistic quantum mechanics in terms of local currents, and present the mathematical framework for discussing representations of the current algebra thus obtained. This discussion will provide a nontrivial example where the idea of working exclusively with local currents can be carried out in an explicit and mathematically rigorous way.

Next, we shall display a representation of the current algebra for a nonrelativistic system having infinitely many degrees of freedom. This representation is obtained by taking the limit of a theory with N identical non-interacting bosons in a volume V, as the number of particles and the volume become infinite, while the average density (N/V) remains fixed. Finally, we shall briefly discuss a relativistic model for charged scalar mesons based on local currents, and mention a few of the many questions which remain unanswered in the non-relativistic and relativistic theories.

2. BACKGROUND [5]

The "currents" which usually appear in relativistic current algebras are the weak and electromagnetic currents of the strongly interacting particles or, as they are called, the hadrons.

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The existence and properties of these currents are inferred from experimental studies of the hadronic weak and electromagnetic interactions. While we have been familiar with the basic properties of the electromagnetic four-vector current $J^{\mu}_{EM}(x)$ for quite a while, the nature of the vector and axial vector currents which play a fundamental role in the weak interactions has become reasonably clear only within the past fifteen years or so.

One of the interesting consequences of the approximate SU(3) invariance of the strong interactions is that it allows a certain unification in the description of the weak and electromagnetic currents. This is achieved by combining the various parts of the electromagnetic and vector weak currents into a single object having eight components, which we write as

$$F^{\mu}_{j}(x);\ \mu = 0,1,2,3,\ j = 1,\dots,8\ . \tag{2.1}$$

This "vector octet" of currents behaves like a 4-vector under Lorentz transformations and transforms like an octet under SU(3) rotations. The pieces of the axial vector weak currents can likewise be combined into a second eight-component object

$$F^{5\mu}_{j}(x)\ , \tag{2.2}$$

which is an axial vector and which also transforms like an SU(3) octet.

In the vector octet, for example, the strangeness-conserving part of the vector weak current is proportional to $F^{\mu}_{1} + iF^{\mu}_{2}$, the electromagnetic current $J^{\mu}_{EM}(x) = e(F^{\mu}_{3}(x) + \frac{1}{\sqrt{3}} F^{\mu}_{8}(x))$, while $F^{\mu}_{4}(x),\dots,F^{\mu}_{7}(x)$ are related to the strangeness-changing weak currents.

The space integrals of the time components of the local current densities $F^{\mu}_{j}(x)$ and $F^{5\mu}_{j}(x)$ define a set of charges, $F_{j}(x_{\circ})$ and $F^{5}_{j}(x_{\circ})$. For $j = 1,2,3$, $F_{j}(x_{\circ}) = I_{j}$, which is the isotopic spin; the hypercharge $Y = \frac{2}{\sqrt{3}} F_{8}$, and the electric charge $Q = \int J^{\circ}_{EM}(x)d^{3}x = e(I_{3} + \frac{1}{2} Y)$. We remark that the charges F_{1}, F_{2}, F_{3}, and F_{8} arise from conserved currents and are thus constants of the motion, whereas the other charges may vary with time.

The local currents $F^{\mu}_{j}(x)$ and $F^{5\mu}_{j}(x)$ and their associated charges are the basic objects of study in "current algebra".

The fundamental hypothesis of current algebra, due to Gell-Mann [6,7], states that the time components of the physical vector and axial vector octet currents satisfy the <u>equal-time</u> commutation relations:

$$[F^{\circ}_{k}(\vec{x}),F^{\circ}_{\ell}(\vec{y})]\big|_{x^{\circ}=y^{\circ}} = i\delta(\vec{x} - \vec{y})f_{k\ell m}F^{\circ}_{m}(\vec{x}) \tag{2.3a}$$

$$[F_k^{\circ}(\vec{x}),F_\ell^{5\circ}(\vec{y})]|_{x^\circ=y^\circ} = i\delta(\vec{x}-\vec{y})f_{k\ell m}F_m^{5\circ}(\vec{x}) \tag{2.3b}$$

$$[F_k^{5\circ}(\vec{x}),F_\ell^{5\circ}(\vec{y})]|_{x^\circ=y^\circ} = i\delta(\vec{x}-\vec{y})f_{k\ell m}F_m^{\circ}(\vec{x}) \ , \tag{2.3c}$$

where the numbers $f_{k\ell m}$ are the structure constants of SU(3). We remark that these commutation relations define an infinite-dimensional Lie algebra of local currents when integrated with a suitable class of testing functions.

Integration of Equations (2.3) over $\vec{x}$ and $\vec{y}$ leads to the equal-time charge algebra

$$[F_k(x^\circ),F_\ell(x^\circ)] = if_{k\ell m}F_m(x^\circ) \tag{2.4a}$$

$$[F_k(x^\circ),F_\ell^5(x^\circ)] = if_{k\ell m}F_m^5(x^\circ) \tag{2.4b}$$

$$[F_k^5(x^\circ),F_\ell^5(x^\circ)] = if_{k\ell m}F_m(x^\circ) \ . \tag{2.4c}$$

This weaker version of Gell-Mann's hypothesis is what has actually been used in many of the most successful applications of current algebra, as in the derivation of the famous Adler-Weisberger relation [8,9].

To the physicist, Gell-Mann's hypothesis is very beautiful. The reason for this is that it captures so much of what we really think is correct in our understanding of the weak and electromagnetic interactions of hadrons in the form of simple, possibly exact, relationships between experimentally observable quantities. For example, this idea allows one to formulate the notion of universality of strength of the weak interactions in a way that does not require a detailed description of how the hadronic weak current is built up out of particle fields. Furthermore, the commutation relations (2.3) and (2.4) specify a mathematical sense in which the group SU(3) × SU(3) acts in the strong interactions, even though it is not an invariance group.

These ideas of Gell-Mann are the foundation on which we would like to build. An obvious extension of Equations (2.3) is to try to find the commutation relations satisfied by the other components of the octet currents, and to extract the physics contained in them. But we wish to discuss the possibility that one can go further, and write complete relativistic theories in which all of the fundamental dynamical variables in the theory are local observables, such as the vector and axial vector currents mentioned above.

To clarify the question, let us recall the canonical field theory of neutral scalar mesons. As discussed in Todorov's lectures [10], one has fields $\varphi(\vec{x},t)$ and $\pi(\vec{x},t)$ which satisfy the equal-time commutation relations

$$[\varphi(\vec{x},t),\pi(\vec{y},t)] = i\delta(\vec{x}-\vec{y}) \ . \tag{2.5}$$

It is assumed that $\varphi(x)$ and $\pi(x)$ form a complete set of operators in the sense that the manifold of all states available to the system spans a single irreducible representation of the local algebra (2.5). The dynamics of the theory is contained in the Hamiltonian

$$H = \int d^3\vec{x}[\pi^2(x) + \nabla\varphi(x) \cdot \nabla\varphi(x) + \mu^2\varphi^2(x)] + H_I \ , \qquad (2.6)$$

where H_I is the interaction Hamiltonian, usually taken to be a polynomial in $\varphi(x)$ Thus H is explicitly a function of $\varphi(x)$ and $\pi(x)$. We are asking whether one can repeat this pattern using as "coordinates" local observables such as the currents themselves, with a local current algebra replacing Equation (2.5) and with the Hamiltonian an explicit function of the currents.

Remarks. (i) The analogue of the canonical commutation relations in a theory based on currents is an equal-time current algebra, such as Equations (2.3). In studying the mathematical structure of local current algebras, one is already studying relationships between observable quantities which are subject, in principle to direct experimental tests.

(ii) Another familiar point is that, among the hundred-odd known hadrons, there are presently no candidates to play the role of "elementary particle", quarks not yet having been observed. Since relativistic theories have traditionally been written in terms of canonical fields whose quanta may be considered as the building blocks of matter, one may be at a loss, when presented with the hadron spectrum, to know where to start.

Local currents treat all particles on an equal footing in the sense that, if one starts with the physical current, and postulates various commutation relations between its components, one does not have to say anything at the beginning about what kinds of particles are present in the theory. All of the different charged particles will make their contribution to the electromagnetic current, for example, but instead of trying to specify at the outset how the current is constituted in terms of particle fields, one can learn this in the process of solving the theory. Thus one might hope that the currents could define a theory in which no hadron plays a special role.

(iii) We expect that in theories written in terms of currents, the local currents themselves will be fields which satisfy Wightman's axioms [11].

(iv) It is hardly necessary to emphasize how far we are today from being able to implement these ideas in situations of immediate relevance to high energy particle physics. We face not only the problem of writing down the correct current commutation relations and the proper Hamiltonian; we have not even identified with any degree of certainty a complete set of local currents in terms of which to describe the hadron system.

To explore the basic ideas, we therefore take various canonical field theory models and rewrite them in terms of currents, obtaining a current algebra and a formula for the Hamiltonian as a function of the currents. Once one has abstracted these relationships from the underlying field theory, one is entitled to take them as a new starting point for the description of the physical system. In the following, we outline some results on representations of the current algebras which arise in non-relativistic quantum mechanics and in a relativistic model for charged scalar mesons.

3. NON-RELATIVISTIC CURRENT ALGEBRA

3.1. n-Particle Representations of the Current Algebra [12-16]

Our starting point is the second-quantized formulation of the quantum mechanics of a system of spinless particles. In this formalism we introduce a Hilbert space $H = \bigoplus_{n=0}^{\infty} H_n$, where H_n is the Hilbert space of symmetric (or antisymmetric) L^2 functions of n vector variables. An element $\Psi \in H$ may be written as $\Psi = (\Psi_0, \Psi_1, \ldots)$ with $(\Psi,\Psi) = \sum_{n=0}^{\infty} (\Psi_n,\Psi_n) < \infty$.

For the commutation relations $[\psi(\vec{x}),\psi^*(\vec{y})]_- = \delta(\vec{x} - \vec{y})$, the equations

$$(\psi(\vec{x})\Psi)_n(\vec{x}_1,\ldots,\vec{x}_n) = \sqrt{n+1}\; \Psi_{n+1}(\vec{x}_1,\ldots,\vec{x}_n,\vec{x})$$

and

$$(\psi^*(\vec{x})\Psi)_n(\vec{x}_1,\ldots,\vec{x}_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \delta(\vec{x} - \vec{x}_j)\Psi_{n-1}(\vec{x}_1,\ldots,\hat{\vec{x}}_j,\ldots,\vec{x}_n) \tag{3.1}$$

define operator-valued distributions $\psi(\vec{x})$ and $\psi^*(\vec{x})$ in the Hilbert space of symmetric functions. Likewise fields satisfying anticommutation relations, $[\psi(\vec{x}),\psi^*(\vec{y})]_+ = \delta(\vec{x} - \vec{y})$, are defined by the equations

$$(\psi(\vec{x})\Psi)_n(\vec{x}_1,\ldots,\vec{x}_n) = \sqrt{n+1}\; \Psi_{n+1}(\vec{x}_1,\ldots,\vec{x}_n,\vec{x})$$

and

$$(\psi^*(\vec{x})\Psi)_n(\vec{x}_1,\ldots,\vec{x}_n) = \frac{(-1)^{n+1}}{\sqrt{n}} \sum_{j=1}^{n} (-1)^{j+1}\delta(\vec{x} - \vec{x}_j)\Psi_{n-1}(\vec{x}_1,\ldots,\hat{\vec{x}}_j,\ldots,\vec{x}_n) \tag{3.2}$$

in the Hilbert space of antisymmetric functions.

Defining the number density of particles as

$$\rho(\vec{x}) = \psi^*(\vec{x})\psi(\vec{x}) \;,$$

and the particle flux density by

$$\vec{J}(\vec{x}) = \frac{1}{2i}\,[\psi^*(\vec{x})\nabla\psi(\vec{x}) - (\nabla\psi^*(\vec{x}))\psi(\vec{x})] \;, \tag{3.3}$$

one can obtain by direct calculation from either (3.1) or (3.2) that

$$(\rho(f)\Psi)_n = \sum_{j=1}^{n} f(\vec{x}_j)\Psi_n$$

$$(J(\vec{g})\Psi)_n = \frac{1}{2i} \sum_{j=1}^{n} [\vec{g}(\vec{x}_j) \cdot \nabla_j + \nabla_j \cdot \vec{g}(\vec{x}_j)]\Psi_n \tag{3.4}$$

for the smeared currents $\rho(f) = \int \rho(\vec{x})f(\vec{x})d\vec{x}$ and $J(\vec{g}) = \int \vec{J}(\vec{x}) \cdot \vec{g}(\vec{x})d\vec{x}$.

Restricted to H_n, Equation (3.4) defines an irreducible representation, called the n-particle representation, of the non-relativistic current algebra [1]:

$$[\rho(f),\rho(g)] = 0$$
$$[\rho(f),J(\vec{g})] = i\rho(\vec{g}\cdot\nabla f) \qquad (3.5)$$
$$[J(\vec{g}),J(\vec{h})] = iJ(\vec{h}\cdot\nabla\vec{g} - \vec{g}\cdot\nabla\vec{h}) \ .$$

In the representation (3.4), the number operator N^{op} is a super-selecting operator.

3.2. Exponentiating the Lie Algebra of Currents [12,13]

The currents in (3.5) are in general unbounded operators; thus they are defined only on a dense domain D which may depend on the testing function. Under these circumstances, current commutators might not always make sense. Therefore we look for a group, which we can represent by unitary operators.

Let $\vec{\varphi}_t^{\vec{g}}: \mathbb{R}^s \to \mathbb{R}^s$ denote the flow for time t by the vector field $\vec{g}$; i.e.,

$$\frac{\partial \vec{\varphi}_t^{\vec{g}}}{\partial t}(\vec{x}) = \vec{g}(\vec{\varphi}_t^{\vec{g}}(\vec{x}))$$

with $\vec{\varphi}_{t=0}^{\vec{g}}(\vec{x}) = \vec{x}$. If $\vec{g}$ has components in Schwartz' space S, then $\vec{\varphi}_t^{\vec{g}}$ exists and is C_∞ for all t.

It turns out that the correct objects to define are $U(f) = e^{i\rho(f)}$ and $V(\vec{\varphi}_t^{\vec{g}}) = e^{itJ(\vec{g})}$ where

$$U(f_1)V(\vec{\psi}_1)U(f_2)V(\vec{\psi}_2) = U(f_1 + f_2 \circ \vec{\psi}_1)V(\vec{\psi}_2 \circ \vec{\psi}_1) \ . \qquad (3.6)$$

One can prove this by studying the n-particle representation, which becomes

$$U(f)\Psi_n = \exp[i\sum_{j=1}^{n} f(\vec{x}_j)]\Psi_n$$
$$V(\vec{\psi})\Psi_n(\vec{x}_1,\ldots,\vec{x}_n) = \Psi_n(\psi(\vec{x}_1),\ldots,\psi(\vec{x}_n))\prod_{j=1}^{n}\sqrt{\det\left(\frac{\partial\psi^k}{\partial x^\ell}(\vec{x}_j)\right)} \ . \qquad (3.7)$$

From (3.7) one can verify (3.4) and hence (3.5) using Stone's theorem; therefore (3.6) is the correct group law to study.

Thus we must consider representations of the semidirect product $S \wedge K$, where S is Schwartz' space, and K is the group of C_∞ diffeomorphisms from $\mathbb{R}^s \to \mathbb{R}^s$ generated by the flows $\vec{\varphi}_t^{\vec{g}}$ under composition. K may be appropriately topologized. It may be pointed out that S is needed in order to be able to take successive derivatives in (3.5).

3.3. The Gel'fand-Vilenkin Formalism [12,13]

The Gel'fand-Vilenkin formalism [17] is suitable for the study of representations of groups such as $S \wedge K$, in which an abelian subgroup is a nuclear space. Such groups also occur in relativistic models [1-4,12]. We assume familiarity with the topology of S and remark that S' denotes the continuous dual of S, with (F,f) the value of $F \in S'$ at $f \in S$.

A cylinder set in S' is a set of the form $\{F \in S' | ((F,f_1),\ldots,(F,f_n)) \in A\}$ for $A \subseteq \mathbb{R}^n$. A is called the base of the cylinder set.

A cylindrical measure μ on S' is a countably additive normalized measure μ on the σ-algebra generated by all cylinder sets with Borel base.

An important result is expressed in the following Theorem (Bochner's theorem for nuclear spaces):

If $L(f)$ is a continuous functional on S, with $L(0) = 1$, which satisfies the "positivity condition"

$$\sum_{j,k=1}^{n} \overline{C}_k C_j L(f_j - f_k) \geq 0 \tag{3.8}$$

for $f_j \in S$ and $C_j \in \mathbb{C}$, then there exists a unique cylindrical measure μ such that

$$L(f) = \int_{S'} e^{i(F,f)} d\mu(F) \ . \tag{3.9}$$

If U is a strongly continuous cyclic representation of S in H with cyclic vector Ω, we can let $L(f) = (\Omega, U(f)\Omega)$ define a cylindrical measure μ according to Equation (3.9). Then H can be realized as $L^2_\mu(S')$ with $\Omega(F) \equiv 1$, and

$$U(f)\Psi(F) = e^{i(F,f)}\Psi(F) \ . \tag{3.10}$$

If $U(f)V(\vec{\psi})$ is a representation of $S \wedge K$, with $\Omega \in H$ cyclic for U, then μ is quasi-invariant for K in the following sense: if we define $(\vec{\psi}^* F, f) = (F, f \circ \vec{\psi})$ and $\mu^{\vec{\psi}}(X) = \mu(\vec{\psi}^* X)$, then $\mu^{\vec{\psi}}$ and μ have the same sets of measure zero. Along with (3.10), we have

$$V(\vec{\psi})\Psi(F) = \chi_{\vec{\psi}}(F)\Psi(\vec{\psi}^* F)\sqrt{\frac{d\mu^{\vec{\psi}}}{d\mu}}\,(F) \tag{3.11}$$

where $\frac{d\mu^{\vec{\psi}}}{d\mu}(F)$ is the Radon-Nikodym derivative.

The "multiplier" $\chi_{\vec{\psi}}(F)$ is a complex-valued function of modulus one. While $\chi_{\vec{\psi}}(F) \equiv 1$ is always a possibility, one can obtain nontrivial inequivalent representations with the same μ from different families of χ's. The χ's satisfy

$$\chi_{\vec{\psi}_2}(F)\chi_{\vec{\psi}_1}(\vec{\psi}_2^* F) = \chi_{\vec{\psi}_1 \circ \vec{\psi}_2}(F) \ . \tag{3.12}$$

Many deep parallels with Mackey's theory [18] of representations of semidirect products of locally compact groups inhere in the Gel'fand-Vilenkin formalism.

For the n-particle representation (3.7), μ is concentrated on the set $F = \{F_{\vec{x}_1} + \ldots + F_{\vec{x}_n} ; \vec{x}_j \neq \vec{x}_k\}$, where $(F_{\vec{x}}, f) = f(\vec{x})$, with $d\mu(F_{\vec{x}_1} + \ldots + F_{\vec{x}_n}) \propto e^{-\vec{x}_1^2} \ldots e^{-\vec{x}_n^2} d\vec{x}_1 \ldots d\vec{x}_n$. In the symmetric case, $\chi_{\vec{\psi}}(F) \equiv 1$, while in the antisymmetric case, this is no longer true. The two cases are unitarily inequivalent in more than one spatial dimension. This method of describing particle statistics is discussed in detail in [13-15].

3.4. Representations of the Non-Relativistic Current Algebra in the "N/V" Limit [19]

The n-particle representations of $\mathcal{S} \wedge \mathcal{K}$ are of course a mere restatement of the ordinary quantum mechanics of n identical particles; i.e., a system of finitely many degrees of freedom. Here we see how this reformulation leads to some particularly simple expressions in the limit of infinitely many noninteracting identical particles at constant average density.

To consider N bosons in a volume V, we impose periodic boundary conditions on the wave functions $\Psi(\vec{x}_1,\ldots\vec{x}_n)$, which are symmetric with respect to interchange of particle coordinates. This corresponds to a representation of $C_\infty(T^s) \wedge \mathcal{K}(T^s)$ where T^s is the s-torus, $C_\infty(T^s)$ is topologized like a nuclear space, and $\mathcal{K}(T^s)$ is the group of C_∞ diffeomorphisms from $T^s \to T^s$.

We know that the state of lowest energy is $\Omega_{N,V}(\vec{x}_1,\ldots,\vec{x}_N) = \left(\frac{1}{\sqrt{V}}\right)^N$. Thus $L_{N,V}(f) = (\Omega_{N,V}, e^{i\rho(f)}\Omega_{N,V})$ becomes

$$\left[\frac{1}{V}\int d\vec{x}e^{if(\vec{x})}\right]^N = \left[1 + \frac{1}{V}\int d\vec{x}[e^{if(\vec{x})} - 1]\right]^N .$$

Setting $\bar{\rho} = N/V$ and taking the limit as $N,V \to \infty$, one obtains

$$L(f) = \exp[\bar{\rho}\int (e^{if(\vec{x})} - 1)d\vec{x}] . \tag{3.13}$$

One can check that if $L(f)$ is given by Equation (3.13) it is continuous, positive, and satisfies $L(0) = 1$. Thus, $L(f)$ is the Fourier transform of a cylindrical measure μ in $\mathcal{S}'$, and defines a representation of $\mathcal{S}$.

By the same procedure, one can obtain

$$L(f,\vec{\psi}) \equiv (\Omega,U(f)V(\vec{\psi})\Omega) = \exp[\bar{\rho}\int (e^{if(\vec{x})}\sqrt{J_{\vec{\psi}}(\vec{x})} - 1)d\vec{x}] , \tag{3.14}$$

where $J_{\vec{\psi}}(\vec{x}) = \det(\frac{\partial\psi^k}{\partial x^j}(\vec{x}))$ is the Jacobian of $\vec{\psi}$. From (3.14), one can compute all of the n-point ground-state expectation functions of the currents. For example,

$$\begin{aligned}
&\langle\rho(f)\rangle = \bar{\rho}\int f(\vec{x})d\vec{x} , \\
&\langle J(\vec{g})\rangle = 0 , \\
&\langle\rho(f)\rho(g)\rangle = \langle\rho(fg)\rangle + \langle\rho(f)\rangle\langle\rho(g)\rangle , \\
&\langle\rho(f)J(\vec{g})\rangle = -\frac{1}{2i}\langle\rho(\vec{g}\cdot\nabla f)\rangle , \\
&\langle J(\vec{g})J(\vec{h})\rangle = \frac{1}{4}\langle\rho(\nabla\cdot\vec{g}\nabla\cdot\vec{h})\rangle ,
\end{aligned} \tag{3.15}$$

and so on. Equation (3.14) is equivalent to (3.13) together with the commutation relations (3.5) and the equation

$$(\nabla\rho + 2i\vec{J})(\vec{x})\Omega = 0 , \tag{3.16}$$

which is true for every N,V. Since the kinetic energy piece of the Hamiltonian density can be written in terms of currents as [1,14,15]

$$H(\vec{x}) = \frac{1}{8M} (\nabla\rho - 2i\vec{J})(\vec{x}) \frac{1}{\rho(\vec{x})} (\nabla\rho + 2i\vec{J})(\vec{x}) \quad , \tag{3.17}$$

Equation (3.16) implies that $H\Omega = 0$. We shall discuss apparently singular Hamiltonians such as (3.17) in Section (3.5).

It is also possible to carry out an "N/V" limit for non-interacting particles satisfying Fermi statistics. The results are described in [19].

3.5. "Singular" Hamiltonians

The Hamiltonian density (3.17) seems to contain the singular expression $\rho^{-1}(\vec{x})$. It has been shown [14,15] that in an irreducible n-particle representation of (3.5) the factor $(\nabla\rho + 2i\vec{J})(\vec{x})$ appearing in $H(\vec{x})$ is proportional to $\rho(\vec{x})$. Thus the factor $\rho^{-1}(\vec{x})$ is explicitly cancelled in (3.17) with the result that $H(f) = \int f(\vec{x})H(\vec{x})d\vec{x}$ is actually a well-defined operator in Hilbert space. Here we shall indicate how the quantity $\rho^{-1}(\vec{x})$ can be given a direct mathematical definition.

Suppose we are given a Hilbert space $\mathcal{H}$, an operator valued distribution $\rho(\vec{x})$ in $\mathcal{H}$ and a dense domain D for ρ with $\rho(f)D \subseteq D$ for all $f \in \mathcal{S}$. Define $\mathcal{V}$ to be the linear span of $\{f(\vec{x})\rho(\vec{x})\Phi | \Phi \in D, f \in \mathcal{O}_M\}$, where $\mathcal{O}_M$ denotes the real-valued C_∞ functions which, together with all derivatives, are of polynomial growth at ∞. Thus $\mathcal{V}$ is a family of vector-valued distributions. It may well be the case that for distinct choices of Φ and f, e.g., Φ_1, Φ_2, and f_1, f_2, one can have $f_1(\vec{x})\rho(\vec{x})\Phi_1 = f_2(\vec{x})\rho(\vec{x})\Phi_2$. Then $\rho^{-1}(\vec{x})\colon \mathcal{V} \times \mathcal{V} \to \mathcal{S}'$ is given by

$$(f(\vec{x})\rho(\vec{x})\Phi, \rho^{-1}(\vec{x})g(\vec{x})\rho(\vec{x})\Psi) = (\Phi, f(\vec{x})\rho(\vec{x})g(\vec{x})\Psi) \quad , \tag{3.18}$$

extended sesqui-linearly to $\mathcal{V} \times \mathcal{V}$. It is now an easy lemma to show that $\rho^{-1}(\vec{x})$ is well-defined. One should note that $\rho^{-1}(\vec{x})$ is <u>not</u> well-defined by the requirement that "$\rho^{-1}(\vec{x})\colon \mathcal{V} \to \mathcal{V}$" be given by $f(\vec{x})\rho(\vec{x})\Phi = f(\vec{x})\Phi$. Thus $\rho^{-1}(\vec{x})$ is a map from $\mathcal{V} \times \mathcal{V} \to \mathcal{S}'$, although of course it is not an operator-valued distribution in $\mathcal{H}$.

Let $K(\vec{x})$ be a (not necessarily Hermitian) operator-valued distribution in $\mathcal{H}$ on D, with $K(\vec{x})D$ and $K^*(\vec{x})D$ contained in $\mathcal{V}$. Then K is <u>related</u> to ρ in a certain sense, and one can define the matrix elements of $H(\vec{x}) = K^*(\vec{x})\rho^{-1}(\vec{x})K(\vec{x})$ by

$$(\Psi, H(\vec{x})\Phi) = (K(\vec{x})\Psi, \rho^{-1}(\vec{x})K(\vec{x})\Phi) \quad . \tag{3.19}$$

In the "N/V" example above, $\vec{K}(\vec{x}) = (\nabla\rho + 2i\vec{J})(x)$ is related to ρ by the commutation relations together with Equation (3.16).

4. A RELATIVISTIC MODEL FOR CHARGED SCALAR MESONS [2,20]

The charged scalar model was originally defined [2] in terms of the operators

$$\begin{aligned} j_\mu(x) &= i[\varphi^*(x)\partial_\mu\varphi(x) - (\partial_\mu\varphi^*(x))\varphi(x)] \\ S(x) &= \varphi^*(x)\varphi(x) \\ \dot{S}(x) &= \varphi^*(x)\pi^*(x) + \pi(x)\varphi(x) \;, \end{aligned} \tag{4.1}$$

where $\partial_0\varphi = \pi^*$. The fields are assumed to satisfy the canonical equal-time commutation relations

$$[\varphi(\vec{x}),\pi(\vec{y})] = [\varphi^*(\vec{x}),\pi^*(\vec{y})] = i\delta(\vec{x}-\vec{y}) \;, \tag{4.2}$$

with all of the other commutators vanishing. This leads to the current commutation relations

$$[j_0(f),j(\vec{g})] = -2iS(\vec{g}\cdot\nabla f) \tag{4.3a}$$

$$[S(f),\dot{S}(g)] = 2iS(fg) \tag{4.3b}$$

$$[j(\vec{g}),\dot{S}(f)] = 2ij(f\vec{g}) \;. \tag{4.3c}$$

All of the other commutators vanish.

Setting $K_\mu(x) = \partial_\mu S(x) - ij_\mu(x)$, the energy-momentum tensor in this model is (without interactions)

$$\theta_{\mu\nu}(x) = \frac{1}{4}K^*_\mu \frac{1}{S} K_\nu + \frac{1}{4}K^*_\nu \frac{1}{S} K_\mu - g_{\mu\nu}[\frac{1}{4}K^*_\alpha \frac{1}{S} K^\alpha - m^2 S] \;. \tag{4.4}$$

Let us emphasize that we do not actually know any representations in which (4.1) and (4.3) together make literal sense. We are indeed considering a situation where, having guessed the current algebra, it is taken as the fundamental starting point of a theory based solely on currents.

One may choose to look at the subalgebra of (4.1) consisting of j_μ and S. It is then consistent to represent S by a multiple of the identity [20]: $S(\vec{x}) = \frac{c}{2}I$. Of course $\dot{S}$ then equals zero, and the commutation relations (4.3b) and (4.3c) must be abandoned. If $\dot{S} = 0$, Equation (4.4) implies that $[H,S(f)] = 0$, so this is at least a consistent model. It is in fact Sugawara's model [4] for the case of a trivial internal symmetry group. The Hamiltonian density becomes [20]

$$H(\vec{x}) = \frac{1}{2c}[j_0(\vec{x})j_0(\vec{x}) + \vec{j}(\vec{x})\cdot\vec{j}(\vec{x})] + m^2 \tag{4.5}$$

which is the same as in the Sugawara model.

The choice $S(\vec{x}) = cI$ might at first be regarded as natural for it makes unambiguous sense out of $\frac{1}{S(\vec{x})}$ in (4.4) and the Hamiltonian becomes bilinear in the currents. But we know that products of distributions at a point rarely make mathematical sense, while we have seen in the non-relativistic model how the "inverse of an operator-valued distribution" <u>can</u> make sense when appropriately sandwiched between vector-valued distributions. In fact (4.4) may be less singular than a bilinear expression; the factor $\frac{1}{s}$ might cancel something in the numerator.

5. QUESTIONS

Now it is time to reveal the extent of our ignorance by mentioning a few of the questions to which we don't have answers.

A complete classification of the irreducible representations of $S \wedge K$ would presumably amount to solving the many-body problem, at least in the "N/V" limit, and is therefore very likely a forlorn hope. However, any examples of representations beyond those mentioned would be extremely interesting. To construct such examples, it would be helpful to know something about the measurability of the orbits in S' under the action of K.

We would like to have a way to determine the functional $L(f)$ in the N/V limit directly, without first having to start from the form of the functional in a box. Preliminary results in this direction have been obtained, using functional differential equations [19,21]. Furthermore, one would like to have techniques for the approximate determination of $L(f)$, in view of the fact that it is unlikely that this functional can be calculated exactly in most situations of practical interest.

Finally, we reiterate that we have no concrete representations of the charged scalar algebra, or any other interesting local, relativistic current algebras, at this time. To construct such representations may be a crucial step in extending the results described here to the domain of particle physics.

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