INFINITE DIMENSIONAL LIE ALGEBRAS AND CURRENT ALGEBRA*

by

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ABSTRACT

The "current algebras" of elementary particle physics and quantum field theory are interpreted as infinite dimensional Lie algebras of a certain definite kind. The possibilities of algebraic structure and certain types of representations of these algebras by differential operators on manifolds are investigated, in a tentative way. The Sugawara model is used as a typical example. A general differential geometric method (involving jet spaces) for defining currents associated with classical field theories is presented. In connection with the abstract definition of current algebras as modules, a purely module-theoretic definition of a "differential operator" is presented and its properties are studied.

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1. INTRODUCTION

In the sense used in this paper, "current algebra" means a program of studying elementary particle physics and quantum field theory from the viewpoint of Lie algebra theory. Specifically, we are concerned with the existence and mathematical properties of certain infinite dimensional Lie algebras whose representations might serve to define the states of physically interesting field-theoretic dynamical systems. As proposed by M. Gell-Mann [6], this study seems to offer the simplest and most natural method for understanding the observed elementary particle symmetries and using them to derive further, deeper facts about the elementary particles. We refer to the books by Adler and Dashen [1] and Renner [16] for further motivation concerning the "physics" of current algebras. Here, we will mainly be concerned with various mathematical questions which are suggested by the broad program. This paper will report on work in progress.

To give a quick idea of what is involved, proceed as follows: Choose the following range of indices;

$$1 \le a,b \le n; \qquad 1 \le i,j \le 3$$

Let $x = (x_i)$, $y = (y_i)$ denote 3-vectors, i.e., elements of R^3; Consider "symbols" $v_a(x)$ satisfying relations of the following form:

$$[v_a(x),v_b(y)] = c_{abc}v_c(x)\delta(x - y) + d_{abci}\partial_i(v_c(x)\delta(x - y)) + \dots \tag{1.1}$$

(The terms ... will mean terms involving higher order derivatives.)

Now, the "Lie algebra" defined symbolically by (1.1) can be defined in a more precise mathematical way as follows. Introduce the set of C^∞, real-valued functions $f\colon R^3 \to R$, denoted by $:F:$. Since such functions can be added, multiplied, and multiplied by real scalars, F is a commutative, associative algebra, with the real numbers, R, as field of scalars. For $f \in F$, introduce the following symbol:

$$v_a(f) = \int v_a(x)f(x)dx \quad . \tag{1.2}$$

Then, the rules (1.1) transcribe following the usual calculational rules for generalized functions into the following expressions:

$$[v_a(f_1),v_b(f_2)] = c_{abc}v_c(f_1f_2) - d_{abc}v_c(\partial_i(f_1)f_2) + \dots \quad . \tag{1.3}$$

We can now give mathematical structure to these formulas. Let Γ be the real vector space spanned by the symbols $:v_a(f):$. Then (1.3) defines a skew-symmetric, real bilinear map $:\Gamma \times \Gamma \to \Gamma:$ that defines a Lie algebra like structure on Γ. (We do not necessarily require that it satisfy the Jacobi identity; typically, however, a quotient algebra will satisfy the Jacobi identity. See Section 6 for further comments on this point.)

Further, Γ is an F-module, with multiplication by an $f \in F$ defined as follows:

$$f(v_a(f')) = v_a(ff') \quad . \tag{1.4}$$

Now, the bracket [,] defined by (1.3) is not an arbitrary R-bilinear map. Roughly, it involves a differential expression in the F-module structure. To make this precise, we will, in Section 2, give an abstract algebraic definition of a "differential operator" purely within the category of F-modules.

Now, in the "currents" of Lagrangian quantum field theory, one finds among the "$v_a(x)$" expressions labeled as follows:

$$v^{\alpha}_{\mu}(x), \ 1 \le \alpha,\beta \le m; \ 0 \le \mu,\nu \le 3 \tag{1.5}$$

α is an "internal symmetry" index μ is a "space-time" index. Typically, these objects are determined--at least in a formal way--by well-known formulas from the Lagrangian and the Lie algebra of an internal symmetry transformation group. (See [9] for a discussion of the algebraic properties of these rules.) For example, for the "Sugawara model", [3, 9, 18, 21], the following relations are satisfied:

$$[V^{\alpha}_0(x), \ V^{\beta}_0(y)] = c_{\alpha\beta\gamma} V^{\gamma}_0(x)\delta(x - y) \tag{1.6}$$

$$[V^{\alpha}_j(x), \ V^{\beta}_j(y)] = 0 \tag{1.7}$$

$$[V^{\alpha}_0(x), \ V^{\beta}_i(y)] = c_{\alpha\beta\gamma} V^{\alpha}_i(x)\delta(x - y) + \lambda\delta_{\alpha\beta}\partial^x_i\delta(x - y) \quad . \tag{1.8}$$

In (1.5-1.7), "$c_{\alpha\beta\gamma}$" are the structure constants of a semisimple compact Lie algebra (with respect to a Lie algebra basis that is orthonormal with respect to the Killing form), and λ is a free parameter.

2. DIFFERENTIAL OPERATORS ON MODULES

As indicated in the introduction, in order to have a "definition" of current algebras as mathematical objects, independently of their usual association with quantum field theory, it is desirable to have a definition of "differential operator" valid for arbitrary modules. (There is, in the mathematical literature, a definition for sections of vector-bundles. See [15].) Indeed, this is a question of independent mathematical interest. In this section we will give such a definition.*

Let F be an arbitrary commutative, associative algebra with the real numbers as field of scalars, and with an identity element denoted by "1". Denote

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* This definition is also known to M. Atiyah.

F-modules by $\Gamma, \Gamma', \ldots$. What is desired, for each integer $r \geq 0$, is a "functor" assigning to each pair (Γ, Γ') another F-module $D^r(\Gamma, \Gamma')$, which may be thought of as the "r-th order differential operators from Γ to Γ'." We will, in fact, define $D^r(\Gamma, \Gamma')$ by induction on r.

First, for $r = 0$, let $D^0(\Gamma, \Gamma')$ be the set of F-linear maps: $\Gamma \to \Gamma'$, i.e. an element $D \in D^0(\Gamma, \Gamma')$ is an R-linear map: $\Gamma \to \Gamma'$ such that:

$$D(f\gamma) = fD(\gamma) \quad \text{for} \quad f \in F,\ \gamma \in \Gamma \tag{2.1}$$

Suppose now that D is an arbitrary R-linear map: $\Gamma \to \Gamma'$. Define an R-bilinear map: $F \times \Gamma \to \Gamma'$ as follows.

$$D(f,\gamma) = D(f\gamma) - fD(\gamma) \quad \text{for} \quad f \in F,\ \gamma \in \Gamma \tag{2.2}$$

For fixed $f \in F$, define D_f as a R-linear map: $\Gamma \to \Gamma'$ as follows

$$D_f(\gamma) = D(f,\gamma) \quad . \tag{2.3}$$

Definition

Suppose that $D^{r-1}(\Gamma, \Gamma')$ is defined. Then, $D^r(\Gamma, \Gamma')$ consists of the R-linear maps $D\colon \Gamma \to \Gamma'$ such that, for each $f \in F$, the map D_f belongs to $D^{r-1}(\Gamma, \Gamma')$.

We must now show that $D^r(\Gamma, \Gamma')$ defined in this way has the usual properties one would expect to justify calling it the "F-module of r-th order differential operators".

Theorem 2.1

If $D \in D^r(\Gamma, \Gamma')$, $D' \in D^s(\Gamma', \Gamma'')$, then

$$D'D \in D^{r+s}(\Gamma, \Gamma'') \quad .$$

Proof. Proceed by induction on $r + s$. For $r + s = 0$, it is evident. Let

$$D'' = D'D \quad .$$

Then,

$$D''_f(\gamma) = D''(f\gamma) - fD''(\gamma) = D'(D(f\gamma)) - fD'D(\gamma)$$

$$= D'(D_f(\gamma) + fD(\gamma)) - fD'D(\gamma) = D'(D_f(\gamma)) + D'_f D(\gamma) \quad .$$

This proves the following basic formula:

$$(D'D)_f = D'D_f + D'_f D \quad . \tag{2.4}$$

By induction hypothesis, the right hand side of (2.4) belongs to $D^{r+s-1}(\Gamma,\Gamma')$, hence $D'D$ belongs to $D^{r+s}(\Gamma,\Gamma')$.

Now, let us determine $D^1(F,\Gamma')$. (Note that F may be considered as an F-module.) Given $D \in D^1(F,\Gamma')$, set

$$f_1 = D(1) \quad . \tag{2.5}$$

Define $D' \in D(F,\Gamma')$ as follows:

$$D'(f) = D(f) - f_1 = D_f(1) \tag{2.6}$$

Theorem 2.2

D' *is a derivation of* F, *into* Γ', *i.e.*

$$D'(ff') = D'(f)f' + fD'(f') \quad \text{for} \quad f,\, f' \in F \quad . \tag{2.7}$$

Proof. By assumption, D_f is a zero-th order operator, i.e., an F-linear map: $F \to \Gamma$, hence:

$$D_f(f') = D_f(1)f' \quad . \tag{2.8}$$

Then,

$$D_f(f') = (D(f) - fD(1))f' = D_f(1)f' \quad .$$

But also,

$$D_f(f') = D(ff') - fD(f') = D_{ff'}(1) + ff'D(1) - f(D_{f'}(1) + f'D(1)) \quad .$$

Combining these two formulas gives:

$$D_{ff'}(1) = D_f f' D_f(1) + fD_{f'}(1) \quad . \tag{2.9}$$

In view of (2.6), this proves (2.7).

Theorem 2.3

$D^1(F,\Gamma')$ is a direct sum of the subspace $D^0(F,\Gamma')$ and the space of derivations of F into Γ', i.e., an "inhomogeneous" first order operator can be written in a unique way as a sum of a zero-th order operator and a "homogeneous" first order operator.

Proof. Theorem 2.2 shows that the sum of these two spaces spans $D^1(F,\Gamma')$. We must show that they have no non-zero elements in common. Suppose then that $D \in D^0(F,\Gamma')$ is a derivation. Then,

$$D(ff') = fD(f') + f'D(f) = ff'D(1) = 2ff'D(1) \quad ,$$

forcing :$D(1) = 0$:, which forces :$D = 0$: .

Suppose now that Γ,Γ' are F-modules, and that $D: \Gamma \to \Gamma'$ is a differential operator. For $\gamma \in \Gamma$, set:

$$D^{\gamma}(f) = D(f\gamma) \quad . \tag{2.10}$$

Thus, D^{γ} can be considered on an R-linear map: $F \to \Gamma'$.

Theorem 2.4

If $D \in D^{r}(\Gamma,\Gamma')$, then, for fixed γ, D^{γ} belongs to $D(F,\Gamma')$.

Proof. Again, by induction on r. For $f' \in F$,

$$(D^{\gamma})_{f}(f')^{\gamma} = D^{\gamma}(ff') - fD^{\gamma}(f') = D(ff'\gamma) - fD(ff'\gamma)$$

$$= D_{f}(f'\gamma) = (D_{f})^{\gamma}(f'), \text{ i.e. } (D_{f})^{\gamma} = (D^{\gamma})_{f} \quad . \tag{2.11}$$

By induction hypothesis, since $D_{f} \in D^{r-1}(\Gamma,\Gamma')$, then $(D_{f})^{\gamma} \in D^{r-1}(F,\Gamma')$, hence (2.11) proves that $(D^{\gamma})_{f} \in D^{r-1}(F,\Gamma')$, which shows that $D^{\gamma} \in D^{r}(F,\Gamma')$.

Definition

$D \in D^{1}(\Gamma,\Gamma')$ is a *homogeneous first order differential operator* if, for each $\psi \in \Gamma$, $D^{\psi} \in D^{1}(F,\Gamma')$ is a derivation of F into Γ'.

Now we turn to the description of $D^{2}(F,\Gamma')$. Given $f \in F$, by Theorem 2.2 there is a derivation: $F \to \Gamma'$ such that

$$D_{f}(f') = X_{f}(f') + D_{f}(1)f' \quad . \tag{2.12}$$

But,

$$D_{f}(f') = D(ff') - fD(f') \quad .$$

Hence,

$$D(ff') = X_{f}(f') + D_{f}(1)f' + fD(f') \quad . \tag{2.13}$$

Set $f' = 1$:

$$D(f) = D_{f}(1) + fD(1) \quad . \tag{2.14}$$

Thus,

$$D(ff') = X_{f}(f') + D_{f}(1)f' + fD(f') = X_{f'}(f) + D_{f'}(1)f + f'D(f) \quad .$$

Subtracting,

$$X_f(f') - X_{f'}(f) = f'(D_f(1) - D(f)) + f(D(f') - D_{f'}(1))$$

$$= \text{using (2.14)} \quad f'fD(1) - ff'D(1) = 0 \quad .$$

i.e.

$$X_f(f') = X_{f'}(f) \quad . \tag{2.15}$$

Theorem 2.5

$$X_{ff'} = fX_{f'} + f'X_f \ , \quad \text{for} \quad f,\ f' \in F \quad . \tag{2.16}$$

Proof.

$$X_{ff'}(f'') = \text{using (2.15)},\ X_{f''}(ff') = X_{f''}(f)f' + fX_{f''}(f')$$

$$= X_f(f'')f' + fX_{f'}(f'') = (f'X_f + fX_{f'})(f'') \quad .$$

This proves (2.16).

Remark. Let $V(F,\Gamma')$ denote the F-module of derivations of F into Γ'. Then, (2.16) says that the map $:f \to X_f:$ defined by D determines an element of $V(F,V(F,\Gamma'))$.

We can now leave as an exercise to the reader showing that the decomposition (2.12) characterizes second order differential operators. One can also proceed further to study higher order operators by the same methods.

3. ALGEBRAIC STUDY OF SCHWINGER TERMS

Consider the "Sugawara model" commutation relations, (1.6-1.8). The second term on the right hand side of (1.8) is, of course, called a "Schwinger term". We will now attempt an analysis, in the language of Section 2, of this particular sort of "Schwinger term".

Let Γ be an F-module. Suppose that $[\ ,\ ]$ is an R-bilinear map $:\Gamma \times \Gamma \to \Gamma:$ of the following form:

$$[\gamma_1,\gamma_2] = [\gamma_1,\gamma_2]_0 + \lambda D(\gamma_1,\gamma_2) \ , \quad \text{for} \quad \gamma_1,\gamma_2 \in \Gamma \tag{3.1}$$

where $[\ ,\ ]_0$ is an F-bilinear map $:\Gamma \times \Gamma \to \Gamma:$ which is a Lie algebra structure, and where D is a skew-symmetric, R-bilinear map $:\Gamma \times \Gamma \to \Gamma:$ that is a homogeneous first order differential operator. λ is a real parameter.

We will now investigate the validity of the Jacobi identity for [,] assuming that it is true for $[\ ,\ ]_0$. For $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ set:

$$T(\gamma_1,\gamma_2,\gamma_3) = [\gamma_1,[\gamma_2,\gamma_3]] - [[\gamma_1,\gamma_2],\gamma_3] - [\gamma_2,[\gamma_1,\gamma_3]] \tag{3.2}$$

$$= [\gamma_1,[\gamma_2,\gamma_3]] + \gamma_1 - [\gamma_2,[\gamma_1,\gamma_3]] + [\gamma_3,[\gamma_1,\gamma_2]] \quad . \tag{3.3}$$

Thus, (3.2) exhibits the relation of T to the "Jacobi identity", while (3.3) indicates how T is formed by permuting 1, 2, and 3 in the expression $[\gamma_1,[\gamma_2,\gamma_3]]$. Then, the following formula holds:

$$T = \frac{1}{2} \epsilon_{ijk}[\gamma_i,[\gamma_j,\gamma_k]] \quad . \tag{3.4}$$

We will now compute this explicitly, using (3.1).

$$[\gamma_1,[\gamma_2,\gamma_3]] = [\gamma_1,[\gamma_2,\gamma_3]_0 + \lambda D(\gamma_2,\gamma_3)]$$
$$= [\gamma_1,[\gamma_2,\gamma_3]_0]_0 + \lambda D(\gamma_1,[\gamma_2,\gamma_3]_0)$$
$$+ \lambda[\gamma_1,D(\gamma_2,\gamma_3)]_0 + \lambda^2 D(\gamma_1,D(\gamma_2,\gamma_3)) \quad . \tag{3.5}$$

Combining (3.4) and (3.5), together with the fact that the Jacobi identity is valid for $[\ ,\ ]_0$, gives the following formula:

$$T = \frac{1}{2} \lambda\, \epsilon_{ijk}[D(\gamma_i,[\gamma_j,\gamma_k]_0) + [\gamma_i,D(\gamma_j,\gamma_k)]_0 + \lambda D(\gamma_i,D(\gamma_j,\gamma_k))] \tag{3.6}$$

Then, if $T = 0$ for all λ, we have

$$\epsilon_{ijk}(D(\gamma_i,[\gamma_j,\gamma_k]_0) + [\gamma_i,D(\gamma_j,\gamma_k)]_0) = 0 \tag{3.7}$$

$$\epsilon_{ijk}D(\gamma_i,D(\gamma_j,\gamma_k)) = 0 \quad . \tag{3.8}$$

Condition (3.7) is a cocycle-type condition. (See [8] for an explanation of the relation between the "deformation" of Lie algebra structures and Lie algebra cohomology theory.) It is not too clear what is the "general" meaning of condition (3.8), although certain simple ways of satisfying it can be readily presented.

Let us attempt to solve relations (3.7-3.8) with a special Ansatz which may be thought of as a general case of the Sugawara conditions (1.6-1.8). Namely, let us suppose that there is a fixed element labeled "γ_0" of Γ such that:

$$[\Gamma,\gamma_0]_0 = 0 \quad . \tag{3.9}$$

Suppose also that there is a homogeneous 1-differential operator $d: \Gamma \times \Gamma \to F$ such that:

$$D(\gamma_1,\gamma_2) = d(\gamma_1,\gamma_2)\gamma_0 \ , \quad \text{for} \quad \gamma_1,\gamma_2 \in \Gamma \quad , \tag{3.10}$$

$$d(\gamma_1,\gamma_2) = -d(\gamma_2,\gamma_1) \tag{3.11}$$

$$d(\gamma_0,\gamma) = 0 \ , \quad \text{for} \quad \gamma \in \Gamma \quad . \tag{3.12}$$

Then, (3.12) guarantees that (3.8) is satisfied. (3.7) is the only condition that needs to be taken into account. Note that, in view of (3.9) and (3.11), (3.7) takes the following form:

$$d(\gamma_1,[\gamma_2,\gamma_3]_0) - d([\gamma_1,\gamma_2]_0,\gamma_3) - d(\gamma_2,[\gamma_1,\gamma_3]_0) = 0, \tag{3.13}$$

$$\text{for} \quad \gamma_1,\gamma_2,\gamma_3 \in \Gamma \quad .$$

4. THE SYMBOL OF DIFFERENTIAL OPERATORS ON VECTOR BUNDLES

We now aim to put the conditions found in Section 3 for the existence of "current algebras" on a slightly different foundation. Let M be a manifold. (See [14] for the notations and ideas of differential geometry to be used here.) Let F be the algebra of C^∞ real valued functions on M. As is well known, differential-geometric ideas can be described in two "languages", that of F-modules and that of vector bundles over M. It is important to be able to pass back and forth between them. The "symbol" of a differential operator expresses the operator-defined generally in the F-module language of Section 2 in terms of vector bundles.

Let $\pi: E \to M$ be a map between manifolds that defines E as a vector bundle over M. Let $\Gamma(E)$ denote the space of cross-section map: $M \to E$. Such cross-section maps can be added (because the fibers of π are vector spaces) and multiplied by functions in F, i.e. $\Gamma(E)$ is an F-module.

Suppose $D \in D^{r}(\Gamma(E),\Gamma(E'))$. Given a point $p \in M$, we will define the *symbol of* D *at* p, denoted by $\sigma(p,D)$: as an element of the fiber of a vector bundle defined over M, which depends on r.

For $r = 0$, proceed as follows. D is then an F-linear map: $\Gamma(E) \to \Gamma(E')$.

Lemma 4.1

If $\gamma \in \Gamma(E)$ vanishes at p, so does $D(\psi)$.

Proof. Suppose first that γ can be written as: $f\gamma_1$, where $\gamma_1 \in \Gamma(E)$, $f \in F$, and $f(p) = 0$. Then,

$$D(\gamma) = D(f\gamma_1) = fD(\gamma_1) \quad ,$$

hence

$$D(\gamma)(p) = f(p)D(\gamma_1)(p) = 0 \quad .$$

Using the local product structure for the vector bundle and a partition of unity for M, one sees that an arbitrary $\gamma \in \Gamma(E)$ that vanishes at p can be written as the sum of elements of the form $:f\gamma_1:$, hence the lemma is proved.

Let $E(p) = \pi^{-1}(p)$, $E'(p) = \pi^{1-1}(p)$ denote the fiber of the vector bundles over E. Then, the point-evaluation map defines R-linear map: $\Gamma(E) \to E(p)$, $\Gamma(E') \to E'(p)$. Lemma 4.1 shows that D (an element of $D^0(\Gamma(E), \Gamma(E'))$ passes to the quotient to define a linear map which we define as $\sigma(p,D)$ of $E(p) \to E'(p)$.

Now, suppose $r = 1$, and $D \in D^1(\Gamma(E), \Gamma(E'))$. For $f \in F$, define $D_f \in D^0(\Gamma(E),\Gamma(E'))$ as in Section 2. For $p \in M$, let M_p^* denote the vector space of cotangent vectors at p, i.e. M_p^* is the dual space to the tangent space M_p to M at p. Then, df(p), the value at p of the differential of f, is an element of M_p^*.

Lemma 4.2

If $f(p) = 0$ and $df(p) = 0$, then $\sigma(p,D_f) = 0$.

Proof. For $\gamma \in \Gamma(E)$, recall that

$$D_f(\gamma) = D(f\gamma) - fD(\gamma) \quad .$$

Thus, since $f(p) = 0$,

$$D_f(\gamma)(p) = D(f\gamma)(p) \quad .$$

As we have proved, the map $f \to D(f\gamma)$ is a first order differential operator on f. Hence, if also $df(p) = 0$, then all first order derivatives of f vanish at p, hence: $D_f(\gamma)(p) = 0$. From the definition of $\sigma(p,D_f)$, we see that it is zero.

Thus, let $\theta \in M_p^*$, $v \in E(p)$. Let $f \in F$ be a function which vanishes at p, such that:

$$df(p) = \theta \quad .$$

Thus, we see from Lemma 4.2 that $\sigma(p,D_f)(v) \in E'(p)$ only depends on θ. Let us denote this element as follows:

$$\sigma(p,D)(\theta,v) = \sigma(p,D_f)(v) \tag{4.1}$$

It is readily seen that (4.1) defines $\sigma(p,D)$ as bilinear map: $M_p^* \times E(p) \to E'(p)$. This map is the symbol of D at p.

One can continue inductively to define the symbol of an r-th order operator. It is a multilinear map,

$$\sigma(p,D) = M_p^* \circ \dots \circ M_p^* \times E(p) \to E'(p) \quad .$$

(See [7, 15].) Here, o denotes "symmetric tensor product". However, for our immediate purpose in discussing "Schwinger terms" that only involve first order derivatives of delta functions--it suffices to deal with the cases $r = 0$ or 1, hence we will restrict our attention to these cases.

5. THE SYMBOL ASSOCIATED WITH CURRENT ALGEBRAS

Suppose now that M is a manifold; that $\pi: E \to M$ is a vector bundle over M; that F = the algebra of C^∞, real valued functions on M; and that $\Gamma(E)$ is the F-module of C^∞ cross-sections of E. Suppose that [,] is an R-bilinear, first-order differential operator $:\Gamma(E) \times \Gamma(E) \to \Gamma(E):$ on $\Gamma(E)$ that makes $\Gamma(E)$ into a "current algebra". (Thus, in the situation suggested by quantum field theory, M will be R^3, which can be identified with a space-like hypersurface in R^4, the manifold of space-time.) Let us suppose that:

$$[\gamma_1,\gamma_2] = D_0(\gamma_1,\gamma_2) + \lambda D_1(\gamma_1,\gamma_2) \quad , \quad \text{for} \quad \gamma_1,\gamma_2 \in \Gamma(E) \quad , \tag{5.1}$$

where D_0 and D_1 are zero and first order homogeneous differential operators: $\Gamma(E) \times \Gamma(E) \to \Gamma(E)$, and where λ is a real parameter. (Notice that we are changing our notations slightly from those used in Section 3. To make the identification, change $[\gamma_1,\gamma_2]_0$ to $D_0(\gamma_1,\gamma_2)$, $D(\gamma_1,\gamma_2)$ to $D_1(\gamma_1,\gamma_2)$). Let us also suppose that $D_0(\gamma_1,\gamma_2)$ satisfies the Jacobi identity; i.e.,

$$D_0(\gamma_1,D_0(\gamma_2,\gamma_3)) = D_0(D_0(\gamma_1,\gamma_2),\gamma_3) + D_0(\gamma_2,D_0(\gamma_1,\gamma_3)) \tag{5.2}$$

$$D_0(\gamma_1,\gamma_2) = -D_0(\gamma_2,\gamma_1) \tag{5.3}$$

Now, for $p \in M$, the symbol $\sigma(p,D_0)$ is a bilinear map: $E(p) \times E(p) \to E(p)$. The conditions (5.2-5.3) pass to the quotient to define analogous conditions on the symbol. Namely, they express the fact that $\sigma(p,D_0)$ for each $p \in M$ defines a Lie algebra structure on the fiber $E(p)$, i.e. E is a "bundle of Lie algebras". Let us then denote the Lie algebra bracket defined on $E(p)$ by $\sigma(p,\pi_0)$ by the notation: $[\ ,\ ]_p$.

Now, to express the fact that (5.1) defines a Lie algebra structure on $\Gamma(E)$ for each λ, one must impose condition (3.7) and (3.8). The symbol at p of the differential operator D_1 may be defined as follows:

$$\sigma(p,D_1)(\theta,v_1,v_2) = D_1(f\gamma_1,\gamma_2)(p) \quad , \tag{5.4}$$

where $f \in F$ satisfies: $f = 0$, $df(p) = 0$; $v = \gamma_1(p)$, $v_2 = \gamma_2(p)$.

Suppose now that γ_1, γ_2, γ_3 are elements of $\Gamma(E)$, with:

$$\gamma_i(p) = v_i \quad .$$

Then (3.7) implies the following conditions:

$$\epsilon_{ijk}[D_1(f\gamma_i,D_0(f\gamma_j,\gamma_k)) + D_0(f\gamma_i,D_1(f\gamma_j,\gamma_k))] = 0 \quad . \tag{5.5}$$

In turn, this implies the following condition on the symbol:

$$\epsilon_{ijk}([v_i \sigma(p,D_1)(\theta,v_j,v_k)]_p - \sigma(p,D_1)(\theta,[v_j,v_k]_p,v_i) = 0 \quad . \tag{5.6}$$

In turn, (5.6) readily interpretable in terms of Lie algebra cohomology, namely, the following result holds.

Theorem 5.1

For each $\theta \in M_p^*$ consider the skew-symmetric bilinear map

$$\omega_\theta = (v_1,v_2) \to \sigma(p,D_1)(\theta,v_1,v_2) \tag{5.7}$$

of $E(p) \times E(p) \to E(p)$ as a 2-cocycle associated with the adjacent representation of the Lie algebra structure defined on $E(p)$ by the bracket $[\ ,\ ]_p$. Then, condition (5.6) expresses the fact that ω_θ is a 2-cocycle.

This result illustrates the general technique one may use. Now, let us turn to consideration of more special sets of current algebras, immediate generalization of the Sugawara model relation, (1.7-1.9).

6. MODELS WITH C-NUMBER SCHWINGER TERMS

Suppose now that $F = F(R^3)$, the C^∞, real valued functions $:x \to f(x):$ of a real 3-vector x. Let Γ be an F-module. Suppose that $\underset{\sim}{G}$ is a real subspace of Γ which has a real Lie algebra structure, denoted by $:[\ ,\ ]:$. Also suppose that there is an element, denoted by "1", of Γ, which is linearly independent from $\underset{\sim}{G}$. (Thus, the multiples $F1$ are the "c-number" in the title of this section.) Let us suppose that there is an algebra structure for Γ, whose bracket is also denoted by $[\ ,\ ]$ such that:

$$[fX,f'Y] = ff'[X,Y] + B_i(X,Y)(\partial_i(f)f' - \partial_i(f')f)1 \ , \quad \text{for } X, Y \in \underset{\sim}{G}, f,f' \in F \quad . \tag{6.1}$$

Here, the B_i are symmetric, bilinear maps: $\underset{\sim}{G} \times \underset{\sim}{G} \to R$. Again, notice that the Sugawara model relations, (1.7-1.9), are of this form. Our aim in this section is to investigate the conditions for Jacobi-identity type relations.

Thus, for $X, Y, Z \in G, f,f',f'' \in F$ set

$$T(X,Y,Z;\ f,f',f'') = [fX,[f'Y,f''Z]] - [f'Y,[fX,f''Z]] - [[fX,f'Y],f''Z] \quad . \tag{6.2}$$

Now,

$$[fX,[f'Y,f''Z]] = [fX,f'f''[Y,Z] + B_i(X,Y)(\partial_i(f')f'' - \partial_i(f''f')]$$

$$= ff'f''[X,[Y,Z]] + B_i(X,[Y,Z])(\partial_i(f)f'f'' - \partial_i(f'f'')f) \quad . \qquad (6.3)$$

Now, our goal in this section is not to derive the sort of condition considered in Section 5, but, a more general case that we can explain as follows.

Notice that T given by (6.2) is always a multiple of the element "1" of Γ. Now, we are ultimately interested in linear representations of the [,]-algebra structure on Γ, i.e. assignment of linear operators to elements of Γ in which the bracket [,] goes over into operator commutator. In order that this be possible, it is not essential that the Jacobi identity be satisfied, i.e., T be zero, in Γ, but that it be zero modulo a certain ideal zero. Now, for the sake of physical applications, it is desirable that all elements of the form $f1$, where f is a compact support function in F such that $:\int f(x)dx = 0:$, go over into the zero operator. Putting these remarks together, we see that it is desirable that T satisfy the following condition:

$$\int T(X,Y,Z;\ f,f',f'')(x)dx = 0 \ ,$$

$$\text{for } X,Y,Z \in G;\ f,f',f'' \text{ compact support functions} \quad . \qquad (6.4)$$

We shall call condition (6.4) the *up-to-a divergence Jacobi identity*. Presumably, the general symbol-type condition derived in Section 5 can be generalized to deal with this condition, but in this case it is just as easy to proceed directly; the general conditions will be investigated in a later publication.

In fact, notice from (6.3) that after integrating by parts

$$\int [fX,[f'Y,f''Z](x)dx = (\int ff'f''(x)dx)[X,[Y,Z]]$$

$$+ 2B_i(X,[Y,Z]) \int (\partial_i(f)f'f'')(x)dx$$

Hence,

$$\int T(X,Y,Z;\ f,f',f'')(x)dx = 2(B_i(X,[Y,Z]) \int \partial_i(f)f'f''dx$$

$$- 2B_i(Y,[X,Z]) \int \partial_i(f')ff''dx + 2B_i(Z,[X,Y]) \int \partial_i(f'')ff'dx$$

$$= \text{, after integrating by parts, } 2B_i(X,[Y,Z]) \int \partial_i(f)f'f''dx$$

$$+ 2B_i(Y,[X,Z]) \int (f'\partial_i(f)f'' + f'f\partial_i(f'')dx$$

$$+ 2B_i(Z,[X,Y]) \int \partial_i(f')f''dx \quad .$$

Thus, in order that (6.4) be satisfied, we must have the following relations.

$$B_i(X,[Y,Z]) + B_i(Y,[X,Z]) = 0 \ , \quad \text{for } X,Y,Z \in G \quad . \qquad (6.5)$$

Now, the skew-symmetry of the [,] bracket on Γ requires that B_i be a symmetric real valued form on $\mathcal{G} \times \mathcal{G}$. Thus, condition (6.5) requires that B_1, B_2, B_3 be symmetric bilinear forms on $\mathcal{G}$ that are invariant under the adjoint representation. For example, the Killing form on $\mathcal{G}$ is a candidate. More generally, it is known that each second order Casimir operator for $\mathcal{G}$ corresponds to such a form [13]. Thus, we see that the calculations of this section provide a general method for constructing one class of "current algebras" which satisfy the Jacobi identity up to a divergence. In fact, by specializing $\mathcal{G}$ and the form B_i suitably one obtains the Sugawara model relations, (1.7-1.9). (There the $\mathcal{G}$ is non-semisimple the direct sum of an abelian ideal and a subalgebra. It would perhaps be interesting to discuss the physical situations whose G itself is semisimple.)

Remark. In summary, we have provided in Sections 2-6 samples (without a definitive discussion) of the sort of work that must be done in order to classify "current algebras", from a purely algebraic point of view.

7. GENERAL REMARKS ABOUT DYNAMICS

What we have done so far is, a-priori, without great physical interest, since we as yet do not know enough data to make a Lorentz invariant theory. So far, we have been dealing with "currents" $\gamma_a(x)$ that are "functions" of a space point x. What is needed is some method for constructing objects $\gamma_a(x,t)$ that depend on space-time points in a Lorentz covariant manner.

Now, it is typical of the "current algebra" approach to physics that one approaches quantum field theory from the "Heisenberg picture" point of view. Thus, instead of regarding $\gamma_a(x,t)$ as "functions" of space time points, one ought to introduce test functions $F = F(R^3)$, as before, and objects of the following form:

$$\gamma_a(f,t) = \int \gamma_a(x,t) f(x) dx \quad .$$

Thus, if Γ denotes the Γ-module spanned by the $\gamma_a(f)$, one might expect to see "dynamics" defined by curves $t \to \gamma_a(f,t)$ in Γ, defined by differential equations, say of the form

$$\frac{\partial}{\partial t} \gamma_a(f,t) = [h, \gamma_a(f,t)] \quad , \tag{7.1}$$

where h is an element of Γ (the "Hamiltonian") and where [,] is an algebra structure on Γ of the "current algebra" type.

Unfortunately, this hope is too simple minded. In model situations, (say the Sugawara model) h is of the following formal form:

$$h = \int h_{ab}\gamma_a(x)\gamma_b(x)dx \quad . \tag{7.2}$$

Now, the bracket of something quadratic of the form (7.2) with $\gamma_a(f)$ goes "outside" of Γ.

In fact, what is required is a construction of the following type: Imbedded Γ as a submodule of an F-module Γ', and find an $h \in \Gamma'$ and a bracket [,] in Γ' so that the "dynamics" is given by (7.1).

In the next few sections we will sketch the construction of such a Γ' in a general situation suggested by the Sugawara model, namely, we will attempt to define "polynomial" objects like (7.2) in a consistent algebraic way.

8. POLYNOMIALS OF CURRENTS

To see what is involved mathematically in carrying out the construction of the F-module Γ' suggested in Section 7, consider the following Sugawara model type of commutation relations:

$$[v_a(x),v_b(y)] = c_{abc}v_c(x)\delta(x - y) - d_{abi}\partial_i^x\delta(x - y) \quad . \tag{8.1}$$

Introduce the symbols $:v_a(f):$ for $f \in F = F(R^3)$, as follows:

$$v_a(f) = \int v_a(x)f(x)dx \quad . \tag{8.2}$$

Let Γ be the F-module spanned by the $v_a(f)$. Then, the bracket in Γ is defined, consistently with (8.1) and (8.2), as follows:

$$[v_a(f),v_b(f')] = c_{abc}v_c(ff') + \frac{1}{2}\, d_{abi}(\partial_i(f)f' - \partial_i(f')f) \quad . \tag{8.3}$$

Now, introduce new objects of the following sort:

$$v_{ab}(f) = \int f(x)v_a(x)v_b(x)dx$$

$$v_{abc}(f) = \int f(x)v_a(x)v_b(x)v_c(x)dx \tag{8.4}$$

and so forth.

Also, introduce the "partial derivatives" $\partial_i v_a(x), \partial_{ij}v_a(x), \ldots,$ so that the following algebraic rules are satisfied:

$$(\partial_i v_a)(f) = \int \partial_i v_a(x)f(x)dx = -\int v_a(x)\partial_i f(x)dx = -v_a(\partial_i(f)) \tag{8.5}$$

$$(\partial_i\partial_j v_a)(f) = v_a(\partial_j\partial_i(f)) \tag{8.6}$$

and so forth.

Now,

$$[v_{ab}(f),v_c(y)] = \int f(x)[v_a(x)v_b(x),v_c(y)]dx = \int f(x)([v_a(x),v_c(y)]v_b(x)$$
$$+ v_a(x)[v_b(x),v_c(y)])dx \int f(x)([c_{acd}v_d(x)\delta(x-y) - d_{aci}\partial_i^x\delta(x-y)]v_b(x)$$
$$+ v_a(x)[c_{bcd}v_d(x)\delta(x-y) - d_{bci}\partial_i^x\delta(x-y)])dx = f(y)c_{acd}v_d(y)v_b(y)$$
$$+ d_{aci}(\partial_i(f)(y)v_b(y) + f(y)\partial_i v_b(y)) + f(y)c_{bcd}v_a(y)v_d(y)$$
$$+ d_{bci}(\partial_i(f)(y)v_a(y) + f(y)\partial_i v_a(y)) = \partial_i(f)(y)(d_{aci}v_b(y) + d_{bci}v_a(y))$$
$$+ f(y)(c_{acd}v_d(y)v_b(y) + c_{bcd}v_a(y)v_d(y) + d_{aci}\partial_i v_b(y) + d_{bci}\partial_i v_a(y)) \quad . \tag{8.7}$$

In particular, for $f,f' \in F$,

$$[v_{ab}(f),v_c(f')] = d_{aci}v_b(\partial_i(f)f') + d_{bci}v_a(\partial_i(f)f') + c_{acd}v_{db}(ff')$$
$$+ c_{bcd}v_{ad}(ff') + d_{aci}\partial_i(v_b)(ff') + d_{bci}(\partial_i v_a)(ff') \quad . \tag{8.8}$$

Now, introduce Γ' as the vector space spanned by all "polynomials" of the following form:

$$v_{a_1\ldots a_r}(f) = \int v_{a_1}(x)\ldots v_{a_r}(x)f(x)dx \quad . \tag{8.9}$$

One can calculate commutation relations of the following form:

$$[v_{a_1\ldots a_r}(f),v_{b_1\ldots b_s}(f')] \quad , \tag{8.10}$$

using the calculations that led into (8.8) as a pattern. Notice that again Γ' is an F-module (multiply $f \in F$ by $v_{a_1\ldots a_r}(f')$ to get $v_{a_1\ldots a_r}(ff')$, and the formula for the bracket (8.10) will be of the type that we have called "current algebra" bracket, i.e., will involve differential operator: $\Gamma' \times \Gamma' \to \Gamma'$. (Notice that Γ' is some sort of generalization "universal enveloping algebra" of a Lie algebra.)

Thus we have explained the algebraic background of the work of Sommerfield and Sugawara [18,21]. They showed that a Lorentz invariant dynamical theory could be obtained in which the energy-momentum tensor was a second degree polynomial in the currents. Of course, actually solving these equations is enormously difficult, with no kind of a procedure or approximation method available, and the whole theory is, as of right now, therefore useless from the view point of the practical physicist. However, there is an important point of principle involved. The Sugawara model--and others that one may construct using the generalized procedure sketched here--is a theory in which the dynamics is determined completely by the currents. If one believes that the "currents", and not the "fields", are the basic mathematical and/or physical objects involved in the interaction and classification of elementary particles, then a theory in which the equations of motion can be expressed strictly in terms of the currents is very attractive.

In the Sugawara model, these equations of motion have a very interesting classical analogue. Let G be a Lie group, whose Lie algebra is that described by the structure constants "c_{abc}" appearing in the current algebra commutation relations. Bardacki and Halpern, for special choices of G, and the author in general, have shown [3, 9] that the Lagrangian which gives rise in the simplest way (it is still unknown whether there are other Lagrangians which also do so) to the Sugawara model has as its classical externals the space of *harmonic maps:* $R^4 \to G$, in the sense of Eels and Sampson [5].

We will briefly explain what is involved here. Eels and Sampson define the concept of a harmonic map $\varphi: N \to M$ between two Riemannian manifolds. In this case, the system of differential equations defining φ is a system of elliptic partial differential equations--in general, non-linear--which, as the name indicates, generalize the concept of "harmonic function". (In fact, the harmonic map $\varphi: R^n \to R$, with the Euclidean metric on R^n and R, are the harmonic functions in the usual sense.)

Now, their definition makes perfectly good sense in the case either N or M or both are pseudo-Riemannian manifolds. For example, take $N = R^4$, space-time, with the Lorentz metric, and take $M = G$, a compact, semisimple Lie group, with the bi-invariant metric defined by the Killing form on G. Then, the differential equations defining the harmonic maps are identical with Sugawara's, and form a non-linear, hyperbolic system. Unfortunately, very little seems to be known about such systems. Perhaps their possible usefulness as equations for elementary particles will stimulate some relevant mathematics.

9. CURRENTS AS FUNCTIONS ON JET SPACES

Up to this point, all of our efforts have gone into explaining independently of quantum field theory the mathematical nature of currents. In fact, one of the most useful features of current algebra theory is the fact that it throws a new, more algebraic and geometric light on the more traditional aspects of quantum field theory. In this section, we will explain how currents arise in the context of classical field theory.

First we must explain briefly the differential geometric notion of a "jet" of a mapping. (See [12, 15] for more details.) Let E and M be manifolds, and let $\pi: E \to M$ be a mapping of E onto M. Let N be another manifold. The ordered set (E, M, π, N) is said to define a (local product) fiber space if each point p of M has a neighborhood U in which $\pi: \pi^{-1}(U) \to U$ looks like the

Cartesian projection map: $U \times N \to U$. Then, a "fiber space" is a "globalization" of a product space.

Let $\Gamma(E)$ denote the space of cross-section maps, i.e., $\psi \in \Gamma(E)$ is a map: $M \to E$ such that:

$$\pi\psi(p) = p \quad \text{for} \quad p \in M \quad ,$$

i.e., $\psi(p) \in E(p) = \pi^{-1}(p)$, the "fiber" of E over p.

Now, if E were the product $M \times N$, it should be clear that the elements of $\Gamma(E)$ can be written precisely in the form:

$$p \to (p, \psi'(p)) \quad ,$$

where ψ' is a map: $M \to N$. Then the notion of "cross-section" is a "globalization" of the idea of mapping between two spaces.

Suppose now that ψ, ψ' are two elements of $\Gamma(E)$, and p is a point of M. Let us say that ψ and ψ' *agree to the first order at* p *if:*

a) $\psi(p) = \psi'(p)$

b) In terms of a local product structure in a neighborhood U of p, with ψ, ψ' identified with maps: $U \to N$, the partial derivatives of ψ and ψ' of first order agree at p.

Definition. Consider the following equivalence relation on $M \times \Gamma(E)$: (p,ψ) is equivalent to (p',ψ') if and only if

a) $p = p'$

b) ψ and ψ' agree to the first order at p. (9.1)

Then, $J^1(E)$, the *manifold of first order jets of cross-sections*, is defined as the quotient of $M \times \Gamma(E)$ by the equivalence relation given by (9.1).

As shown in [11] and [12], the manifold $J^1(E)$ is the appropriate one for consideration of the calculus of variation problems underlying quantum field theory. For example, a "Lagrangian" is just a real-valued function: $J^1(E) \to R$.

If $\psi \in \Gamma(E)$, denote by $j^1(\psi)$ (its "one-jet") as a mapping: $M \to J^1(E)$ defined as follows:

$$j^1(\psi)(p) = \text{equivalence class to which the point } (p,\psi) \text{ belongs.} \quad (9.2)$$

Then, if $L: J^1(E) \to R$ is a "Lagrangian", if dx is a volume element form for M, if $\psi \in \Gamma(E)$, then:

$$L(\psi) = \int_M L(j^1(\psi)(x))dx \qquad (9.3)$$

is the value assigned by L to the cross-section ψ.

In order to establish the equivalence with the more usual formulas of field theory, we must introduce coordinate systems for M and E. Suppose that $M = R^4$. Let $x = (x_\mu)$, $0 \le \mu, \nu \le 3$, be Euclidean coordinates for R^4. Suppose also that (φ_a), $1 \le a, b \le n$, is a coordinate system for the fiber N.

We will define a coordinate system, that we will label $(x_\mu, \varphi_a, \varphi_{b\mu})$ for $J^1(E)$ in the following way. Suppose that (p,ψ) is an element of $M \times \Gamma(E)$. We will define the values of these functions on this point:

a) x_μ are the Euclidean coordinates of the point p.

b) φ_a are the φ-coordinates of the point $x(p)$.

c) $\varphi_{a\mu}$ are the derivatives $\frac{\partial\varphi_a(x)}{\partial x_\mu}$ of the function $x \to \varphi_a(x)$ which determine ψ locally as a map: $R^4 \to N$.

Thus, the Lagrangian L becomes a function $L(x,\varphi_a,\varphi_{a\mu})$ of the indicated variables. If $\psi \in \Gamma(E)$, with functions $x \to (\varphi_a(x))$ defining ψ locally, then (9.2) takes the following more classical form:

$$L(\psi) = \int L(x,\varphi_a(x),\ \partial_\mu\varphi_a(x))dx \quad . \tag{9.4}$$

Suppose we are given such a Lagrangian L. Define functions L_a, $L_{a\mu}$ on $J^1(E)$ as follows:

$$L_a = \frac{\partial L}{\partial \varphi_a}$$

$$L_{a\mu} = \frac{\partial L}{\partial \varphi_{a\mu}} \quad .$$

Then, a cross-section determined by functions $:x \to \varphi_a(x):$ is an *extremal* if it satisfies the following differential equations (called the *Euler-Lagrange equations)*:

$$\frac{\partial}{\partial x_\mu}(L_{a\mu}(x,\varphi(x),\partial\varphi(x)) = L_a(x,\varphi(x),\partial\varphi(x)) \quad . \tag{9.5}$$

We now proceed to show how "currents" may be defined. Suppose X is a vector field on the manifold E (see [14] for differential geometric notions, such as vector field) of the following form:

$$X = A_\mu(x)\frac{\partial}{\partial x_\mu} + A_a(x,\varphi)\frac{\partial}{\partial \varphi_a} \quad , \tag{9.6}$$

where A_μ, A_a are functions of the indicated variables. (Geometrically, vector fields of the form (9.6) generate one parameter groups of transformations of E that act on M and permute the fibers of E over this action; they may be called "fiber space automorphisms".)

We can now define a "prolonged" vector field X' on $J^1(E)$, by the following formula:

$$X' = A_\mu\frac{\partial}{\partial x_\mu} + A_a\frac{\partial}{\partial\varphi_a} + \left(\frac{\partial A_a}{\partial x_\mu} - \varphi_{a\nu}\frac{\partial A_\nu}{\partial x_\mu} + \frac{\partial A_a}{\partial \varphi_b}\varphi_{b\mu}\right)\frac{\partial}{\partial\varphi_{a\mu}} \quad . \tag{9.7}$$

This prolongation process is a Lie algebra homomorphism: $V(E) \to V(J^1(E))$, i.e.

$$[X,Y]' = [X'Y'] \quad , \tag{9.8}$$

if X, Y are vector fields on h of form (9.5).

Suppose now that $f \in F(M)$. Then,

$$(fX)' = fX' + \frac{\partial f}{\partial x_\mu}(A_a - \varphi_{a\nu}A_\nu)\frac{\partial}{\partial \varphi_{a\mu}} \quad . \tag{9.9}$$

Thus, if L is a Lagrangian,

$$(fX')(L) = fX'(L) + \frac{\partial f}{\partial x_\mu}(A_a - \varphi_{a\nu}A_\nu)L_{a\mu} \quad . \tag{9.10}$$

In particular, suppose that

$$A_\mu = 0;\; X'(L) = 0 \quad . \tag{9.11}$$

This means, geometrically, that if the one parameter group of automorphisms of E generated by X map the fibers of E into themselves, and the group is a one-parameter group of "internal symmetries" of the Lagrangian L, then

$$(fX)'(L) = \frac{\partial f}{\partial x_\mu}(A_a L_{a\mu}) \quad . \tag{9.12}$$

Now, $A_a L_{a\mu} = V_\mu^X$ is the very familiar formula in quantum field theory for the "vector current" generated by a one-parameter group of symmetries.

In general then, we might associate to each vector field X of form (9.7) the following set of functions on $J^1(E)$:

$$V_\mu^X = A_a L_{a\mu} - L_{a\mu}\varphi_{a\nu}A_\nu \quad . \tag{9.13}$$

This method of defining "currents" in classical field theories may be compared to the now-classical work of Belinfante and Rosenfeld [4, 17]. Now that we have seen that "currents" at the level of classical field theory may be interpreted as functions on the jet spaces, the road is open to use the current commutation relations of quantum field theory to define a "Poisson bracket" operation for functions on the jet spaces. However, we will not pursue this topic here. Instead, we will turn to the study of another related connection between "current algebras" and differential geometry.

10. REPRESENTATIONS OF GAUGE ALGEBRAS BY DIFFERENTIAL OPERATORS

Now we turn to the question of representing current algebras in a natural geometric way--as differential operators on manifolds. This corresponds, roughly, to finding their physical consequences as *classical* dynamical systems. The problem of realizing them irreducibly as operators on Hilbert space is related to their consequences in quantum mechanics, and is a much more difficult (and still unsolved) technical problem. (See the work of Araki, Streater and Wulfsohn [2, 19, 20].)

Now, part of our "grand design" is to see how "current algebras" arise in a natural geometric way. Indeed, I feel that this study will have interesting repercussions in "pure" differential geometry. (Of course, differential geometry used to be not unrelated to events in physics. However, there has been a period of introspection in the last twenty years, and now most of the active workers in this field know nothing of these roots.) Lie algebras first arose in mathematics, in the works of S. Lie, as Lie algebras of differential operators on finite dimensional manifolds. It is still an interesting, unsolved mathematical problem to classify the possible ways a given Lie algebra can so act. In the next few sections we will treat a fragment of this problem for the sorts of Lie algebras (or their generalizations, i.e., algebras satisfying the Jacobi identity up to a divergence) encountered in current algebra theory. In this section, we will treat the simplest case--where the "current algebra" contains no "Schwinger terms", hence is what might be called a "gauge algebra". Precisely, let us adopt the following definition.

Definition. Let F be a commutative, associative algebra over the real numbers, and let Γ be an F-module. A real Lie algebra structure [,] on Γ is said to define a *gauge algebra* if the bilinear map $(\gamma_1,\gamma_2) \to [\gamma_1,\gamma_2]$ of $\Gamma \times \Gamma \to \Gamma$ is also F-linear.

This concept is most useful when combined with the idea of a "free" F-module.

Definition. Let V be a real subspace of Γ, and let

$$\alpha\colon V \otimes F \to \Gamma$$

be the linear map constructed as follows:

$$\alpha(v)(f) = fv , \quad \text{for} \quad v \in V,\ f \in F \quad . \tag{10.1}$$

($\otimes$ denotes the tensor product defined with the real numbers as ground field.) Then, Γ is said to be a *free F-module* with *basis space* V if the map α defined by (10.1) is an isomorphism.

The modules which arise as "current algebras" in physical situations are usually also "free". If this is the case, and if V is a basis space, let us use the following notation; suggested by the physicists' notation:

$$v(f) = fv = \alpha(v \otimes f) , \quad \text{for} \quad f \in F,\ v \in V \quad . \tag{10.2}$$

Let $\underset{\sim}{G}$ be a real Lie algebra.

Definition. An F-module Γ is a *free gauge algebra based on the Lie algebra of charges* $\underset{\sim}{G}$ if the following conditions are satisfied:

a) Γ is a free F-module, with basis subspace V.

b) Γ is a gauge Lie algebra, in the sense defined above.

c) With respect to the Lie algebra bracket [,] defined by b), V is a Lie subalgebra of the real Lie algebra Γ.

d) $\underset{\sim}{G}$ is isomorphic, as a Lie algebra, to the Lie subalgebra V.

Now, let us suppose that $F = F(R^3)$. Let $\pi: M \to R^3$ be a map from a manifold M to R^3 that defines M as a fiber space over R^3. If $f \in F = F(R^3)$ is a function on the base space, R^3, then $f \to \pi^*(f)$ defines an imbedding of F or a subalgebra of F(M). In turn, this enables us to consider the tensor fields on M as F-modules. For example, if X is a vector field on M, i.e., an element of V(M), and $f \in F$, we denote by "fX" the product of the function $\pi^*(f)$ and the vector field X.

Now, suppose that Γ is a free gauge Lie algebra with the basis Lie subalgebra V. Thus, the following commutation rules are satisfied:

$$[v_1(f_1),v_2(f_2)] = [v_1,v_2](f_1f_2) \ , \quad \text{for } v_1,v_2 \in V;\ f_1,f_2 \in F \ . \tag{10.3}$$

We will now attempt to find a homomorphism h of the Lie algebra Γ-defined by the commutation rules (10.3) - into the Lie algebra V(M) of vector fields on the manifold M. In fact, we will restrict ourselves, at this point at least, to the search for h of the following form:

$$h(v(f)) = fX_v + \partial_i(f)X_v^i \ . \tag{10.4}$$

For $v \in V$, X_v and X_v^i are vector fields on M. The map $v \to X_v$ and X_v^i then define linear mappings: $V \to V(M)$. We will also suppose that

$$X_v(\pi^*(f)) = 0 = X_v^i(\pi^*(f)) \ , \quad \text{for } v \in V,\ f \in F(= F(R^3)) \ . \tag{10.5}$$

Comparing (10.3-10.5), we can readily write down the conditions that h be a Lie algebra homomorphism:

$$\begin{aligned} h([v_1(f_1),v_2(f_2)]) &= [h(v_1(f_1),h(v_2(f_2))] = [f_1X_{v_1} + \partial_i(f_1)X^i_{v_1}, f_2X_{v_2} \\ &+ \partial_i(f_2)X^i_{v_2}] = f_1f_2[X_{v_1},X_{v_2}] + \partial_i(f_1f_2[X^i_{v_1},X_{v_2}] + f_1\partial_i(f_2)[X_{v_1},X^i_{v_2}] \\ &+ \partial_i(f_1)\partial_j(f_2)[X^i_{v_1},X^j_{v_2}] = h([v_1,v_2](f_1f_2)) = f_1f_2X_{[v_1,v_2]} \\ &+ \partial_i(f_1f_2)X^i_{[v_1,v_2]} \end{aligned}$$

Thus, the conditions that h be a homomorphism read as follows:

$$[X_{v_1},X_{v_2}] = X_{[v_1,v_2]} \tag{10.6}$$

$$[X^i_{v_1}, X^i_{v_2}] = 0 \tag{10.7}$$

$$[X^i_{v_1}, X_{v_2}] = X^i_{[v_1,v_2]} \quad . \tag{10.8}$$

(Condition (10.8) can be analyzed further in terms of the cohomology of the Lie algebra V but we will not go into that here.)

In summary, we have presented in this section a geometric method for realizing gauge Lie algebras by means of differential operators. Of course, the method can be generalized considerably beyond what has been presented in this section. What we have done amounts to an illustrative example. Our main immediate goal is to lead into the work of the next section on "Schwinger terms".

11. Schwinger Terms for Gauge Lie Algebras

Continue with the notations of Section 10. Let Γ be a free gauge Lie algebra, with basis subalgebra V, i.e., the commutation relations for the Lie algebra structure on Γ take the form (10.3). Let M be, as in Section 10, a fiber space over R^3.

Now, let $D^1(M)$ denote the Lie algebra of first order inhomogeneous differential operators on M. Let us modify the definition (10.4) of h, to define the linear mapping $h'\colon \Gamma \to D^1(M)$, as follows:

$$h'(v(f)) = fX_v + \partial_i(f)X^i_v + fk_v \ , \quad \text{for } f \in F,\ v \in V \quad . \tag{11.1}$$

In (11.1), X_v, X^i_v are vector fields on M; k_v is a zero-th order differential operator on M, i.e., a function on M. Thus, $v \to k_v$ defines a linear mapping of $V \to F(M) = D^0(M)$.

Let us also suppose that conditions (10.5) are satisfied. (They mean, geometrically, that the vector fields X_v, X^i_v are tangent to the fibers of the map $\pi\colon M \to R^3$.) Then, for $v_1, v_2 \in V$; $f_1, f_2 \in F$,

$$\begin{aligned}[h'(v_1(f_1)), h'(v_2(f_2))] &= f_1 f_2 [X_{v_1}, X_{v_2}] + \partial_i(f_1) f_2 [X^i_{v_1}, X_{v_2}] \\ &+ f_1 \partial_i(f_2)[X_{v_1}, X^i_{v_2}] + \partial_i(f_1)\partial_j(f_2)[X^i_{v_1}, X^j_{v_2}] - f_1(f_2 X_{v_2}(k_{v_1}) \\ &+ \partial_i(f_2) X^i_{v_2}(k_{v_1})) \quad .\end{aligned} \tag{11.2}$$

Let us suppose, as in Section 10, that V is a Lie algebra. Thus, using (10.3), Γ can be made into a Lie algebra, with a bracket denoted by [,]. Let Γ' be the direct sum of Γ and F itself. Define a "new" bracket for Γ', denoted by [,]', by the following formula:

$$[v_1(f_1),v_2(f_2)]' = [v_1,v_2](f_1f_2)$$
$$+ \beta_i(v_1,v_2)\partial_i(f_1)f_2 - \beta_i(v_2,v_1)f_1\partial_i(f_2) \quad . \qquad (11.3)$$

(β_i are bilinear maps: $V \times V \to R$. Then,

$$h'([v_1(f_1),v_2(f_2)]') = f_1f_2X_{[v_1,v_2]} + \partial_i(f_1f_2)X^i_{[v_1,v_2]}$$
$$+ \beta_i(v_1,v_2)\partial_i(f_1)f_2 - \beta_i(v_2,v_1)f_1\partial_i(f_2) \quad . \qquad (11.4)$$

Let us now equate (11.2) and (11.4). This imposes the following conditions:

$$[X_{v_1},X_{v_2}] = X_{[v_1,v_2]} \qquad (11.5)$$

$$[X^i_{v_1},X^j_{v_2}] = 0 \qquad (11.6)$$

$$[X^i_{v_1},X_{v_2}] = X^i_{[v_1,v_2]} \qquad (11.7)$$

$$X_{v_1}(k_{v_2}) = X_{v_2}(k_{v_1}) \qquad (11.8)$$

$$X^i_{v_1}(k_{v_2}) = \beta_i(v_1,v_2) \qquad (11.9)$$

What we have done now is to find the conditions, namely (11.5-11.9), that the set of first order differential operators, of the form (11.1), satisfy the commutation relations whose "abstract" structure relations are given by (11.3). Now we turn to the study of more specific structures of this sort, which arise in the study of the current algebras of quantum field theory.

12. THE CURRENT ALGEBRAS OF QUANTUM FIELD THEORY

Let us now change notations slightly. Choose the following range of indices, together with the corresponding summation conventions

$$1 \leq a,\ b \leq m;\ 1 \leq 1,\ j \leq 3;\ 0 \leq \mu,\ \nu \leq 3 \quad .$$

Let $x = (x_i)$, $y = (y_i)$ denote elements of R^3. Let $F = F(R^3)$ be the commutative, associative algebra of real-valued functions on R^3.

Consider objects that are labeled as follows:

$$v^a_\mu(x)$$

Typically, they are "currents" associated with a Lie algebra of symmetries of a physical system. One aspect of the "current algebra" approach to quantum field theory is an attempt to construct Lie algebras from these objects, and investigate how these abstract Lie algebras are realized in terms of physical systems. In this final section of this paper, I will rework some of the ideas in a previous paper of mine [10], in the algebraic language developed here.

First of all, for the currents constructed from "Noether's theorem" (essentially equivalent to the material presented in Section 9), using the most common sort of Lagrangians, the "time" components of the current satisfy the following commutation relations:

$$[v_0^a(x), v_0^b(y)] = c_{abc} v_0^c(x)\delta(x - y) \quad . \tag{12.1}$$

Here, the "c_{abc}" are structure constants of a semisimple Lie algebra $\underset{\sim}{G}$.

Second, postulate the following time-space commutation relations:

$$[v_0^a(x), v_j^b(y)] = c_{abc} v_i^c(x)\delta(x - y) - \partial_j^x (v_{ij}^{ab}(x)\delta(x - y)) \quad . \tag{12.2}$$

In (12.2) the $v_{ij}^{ab}(x,y)$ are objects that are model dependent.

Let us put the commutation relations (12.1-12.2) into "module" form. Introduce

$$v_0^a(f) = \int v_0^a(x) f(x) dx$$

$$v_i^a(f) = \int v_i^a(x) f(x) dx$$

$$v_{ij}^{ab}(f_1, f_2) = \int v_{ij}(x) f_1(x) f_2(x) dx dy. \tag{12.3}$$

Then, (12.1) - (12.2) take the following form:

$$[v_0^a(f_1), v_0^b(f_2)] = c_{abc} v_0^c(f_1 f_2) \tag{12.4}$$

$$[v_0^a(f_1), v_j^b(f_2)] = c_{abc} v_i^c(f_1 f_2) + v_{ij}^{ab}(\partial_j(f_1), f_2) \quad . \tag{12.5}$$

Let us now try to find realizations of the commutation relations (12.4-12.5). Let Γ be an F-module. We will construct a mapping of the objects $v_0^a(f), v_1^a(f)$ into the space $D_0(\Gamma,\Gamma)$ of F-linear mappings: $\Gamma \to \Gamma$.

Set:

$$\rho(v_0^a(f)) = fA^a + \partial_i(f) A_j^a \quad , \tag{12.6}$$

where A_a, A_a^j are operators in $D_0(\Gamma,\Gamma)$. Thus, following the pattern described in Section 10, it is readily verified that the following conditions are equivalent to (12.4):

$$[A^a, A^b] = c_{abc} A^c \quad , \tag{12.7}$$

$$[A_i^a, A_j^b] = 0 \quad , \tag{12.8}$$

$$[A_i^a, A^b] = c_{abc} A_j^c \quad . \tag{12.9}$$

Now, let us attempt to satisfy (12.5) by means of the following assignment:

$$\rho(v_i^a(f)) = f B_i^a \quad , \tag{12.10}$$

with $B_i^a \in D_0(\Gamma,\Gamma)$. Then,

$$[\rho(v_0^a(f_1)), \rho(v_i^b(f_2))] = [f_1 A^a + \partial_j(f_1) A_j^a, f_2 B_i^b]$$

$$= f_1 f_2 [A^a, B_i^b] + \partial_j(f_1) f_2 [A_j^a, B_i^b] \quad .$$

Then, we see that ρ will be a representation of the commutation relations (12.5) provided that:

$$[A^a, B_i^b] = c_{abc} B_i^c \tag{12.11}$$

$$v_{ij}^{ab}(f_1, f_2) = \partial_j(f_1) f_2 [A_j^a, B_i^b] \quad . \tag{12.12}$$

To obtain a model having common features of the "Sugawara model", one can further require that the operators $[A_j^a, B_i^b]$ commute with the operators A^a, B_i^b. A method for an explicit realization of these operators in terms of differential operators has been presented in [10], to which we refer for further details. The next step in this program would be to search for more general (possibly even the most general) realizations of this sort, a task we will attempt in volume III of [12].

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