OPTIMUM DESIGN OF STRUCTURES THROUGH VARIATIONAL PRINCIPLES

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1. INTRODUCTION

The application of the calculus of variations to the design of structures for minimum volume leads to necessary conditions for the structural volume to be stationary, and a local or global minimum is not guaranteed. However, if an appropriate variational principle applies for the class of structures under consideration, design criteria can be established which lead to structures of minimum volume. A direct design procedure was first given by Michell [1] for framed structures composed of a material of limited strength. For perfectly-plastic structures, direct design procedures were introduced by Drucker and Shield [2, 3, 4] and here the upper bound theorem of limit analysis provided the appropriate variational principle. For elastic structures, variational principles can provide direct design methods for design for a given stiffness, for a given buckling load or for given fundamental frequency of vibration (see Prager and Taylor [5] and Shield and Prager [6]). The major part of this paper surveys the direct design procedures which have been developed through the use of variational principles.

Section 2 describes the procedures for minimum-volume design of structures of perfectly-plastic materials which are required to carry a given set of loads. Section 3 discusses uniform strength designs in which the structural material is required to be stressed within a certain range under a given system of loads. The stress range may be chosen to ensure that the stresses remain in the elastic range, for example, or to ensure that an appreciable amount of creep will not occur. Section 4 treats elastic design for a given stiffness in order to illustrate the design procedures for elastic structures. Minimum-volume framed structures of material of limited strength in tension and compression are considered in Section 5. The Michell design method fails when kinematic constraints are present (except when the tensile and compressive strengths are equal) but an alternative approach [7] does not have this limitation. An example is given to show that minimum-volume frames are not necessarily unique, and some new plane structures of the Michell type are described, including the layout for pure bending.

This paper is not intended to provide a comprehensive survey of the literature on optimum design. The reader will find additional references in [7, 8, 9, 10].

2. PLASTIC DESIGN OF STRUCTURES

In this section we discuss the optimum design of structures composed of perfectly-plastic materials. A restricted formulation of the problem is indicated in Figure 1. It has the advantage of ensuring that the structure will consist of the usual structural elements, such as frames, plates and shells, and the limitations also increase the chances of determining the optimum solution. We suppose that the structure is to have a prescribed middle surface A. The loading is prescribed and is distributed over A and its boundary. At supports either the components of displacement and rotation are prescribed to be zero or the corresponding components of edge traction and moment are given. For a solid shell, problem (i), the structure is formed by placing a certain thickness h of a given material at points of the middle surface. For a sandwich type structure, problem (ii), we suppose that the shell has a core of prescribed thickness H. The core carries shear force only, and bending moments and force resultants are carried by membrane stresses in two thin identical face sheets of thickness h. In both cases we wish to design the shell, that is choose h, so that the shell is just at collapse under the given loading and is optimum for a given criterion. Here we design so that the volume

$$V = \int h \, dA$$

is minimized but the methods extend readily to minimization of the functional

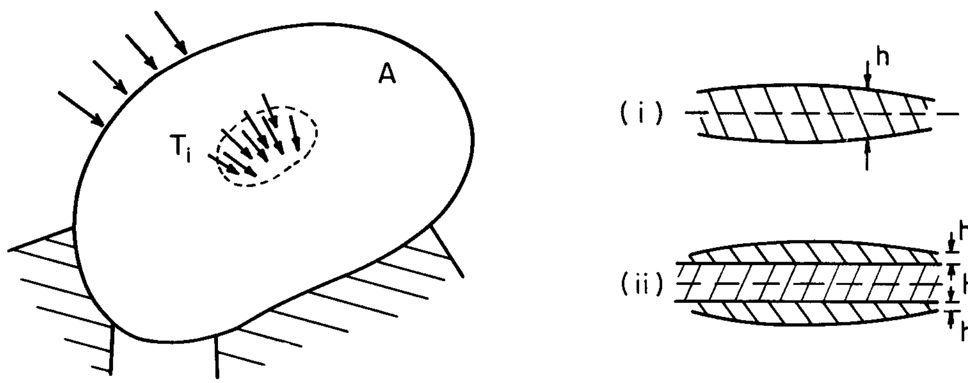

Figure 1. Solid and sandwich shells

$$\int h f(X)\, dA,$$

where f (X) is a non-negative function of position over A. The extension allows for minimum-weight design when different materials are to be used in different parts of A, and also admits design for minimum moment of inertia about an axis.

The generalized stresses, such as bending moments and stress resultants across a section, will be denoted by $Q_1, Q_2, .., Q_N$ or simply Q_n. Yielding can occur in the shell when the stresses Q_n lie on the yield surface

$$F(Q_n; h) = 0 \tag{1}$$

in N-dimensional stress space. For stress states Q_n represented by points on the yield surface, purely plastic strain rates are possible; and we use q_n ($n = 1, 2, .., N$) to denote the generalized strain rates, such as rates of curvature change and extension of the middle surface. The vector representing q_n is normal to the yield surface at regular points, while at singular points the vector lies between adjacent normals. For a convex yield surface, the plastic rate D_A of dissipation of energy per unit area of the middle surface is then uniquely determined by the values of q_n,

$$D_A = D_A(q_n; h) = Q_1 q_1 + Q_2 q_2 + .. + Q_N q_N. \tag{2}$$

For the solid shell, D_A is quadratic in h in general; while for the sandwich shell, D_A is directly proportional to h. Shear forces have little influence on yielding, even for highly localized loading [11], and they are not included in the generalized stresses Q_n.

The theorems of limit analysis [12, 13] can be used to provide information about the volume of a design which can carry the loads. The appropriate theorems are the following:

Lower-bound theorem. If the applied loads can be carried by an equilibrium distribution of moments and force resultants Q_n in the shell which are at or below yield, the loading is at or below the collapse loading.

Upper-bound theorem. If the applied loads are such that a deformation of the shell can be found for which the rate at which the applied loads do work exceeds the rate of internal energy dissipation, the loading is above collapse.

The lower-bound theorem can be used to determine upper bounds on the volume of the minimum-volume design for given loads [3]. If a stress distribution Q_n over A is in equilibrium with the applied loads, we can choose the thickness h_s so that the yield condition (1) is satisfied everywhere on A. Since the design h_s is then a permissible design, the minimum volume V_m must be less than or equal to V_s,

$$V_m \le V_s = \int h_s \, dA.$$

The upper-bound theorem can be used to provide lower bounds for V_m, as in [3], but the theorem also leads to a direct design procedure [4]. For a shell h_s which is at or below collapse under the applied loads and for any kinematically admissible velocity field u_i $(i = 1, 2, 3)$ we have

$$\int D_A (q_n; h_s) \, dA - \int T_i u_i \, dA \ge 0, \tag{3}$$

for otherwise the use of the upper-bound theorem with the deformation u_i would predict that the loading exceeds collapse. In (3), T_i are the components of the applied loads, the repeated index i implies summation over 1, 2, 3, and the integral of $T_i u_i$ is to include the rate of work at the edge of the shell. Inequality (3) is a variational principle for the permissible design h_s; equality holds in (3) only when u_i is a collapse mode for h_s. The application of this principle to the determination of a minimum-volume design is much more direct than the use of the calculus of variations. Suppose that a design h_c for a solid shell, problem (i), is just at collapse under the loads in a collapse mode u_i, and suppose that a neighboring design $h_s = h_c + \delta h$ is also a permissible design. If we neglect second order terms, the dissipation rate for the shell h_s in the deformation mode u_i is

$$D_A(q_n; h_c) + \delta h \frac{\partial D_A}{\partial h},$$

assuming that D_A is continuously differentiable in h. Applying the variational principle (3) to the design h_s, we conclude that

$$\int \delta h \frac{\partial D_A}{\partial h} \, dA \geq 0$$

because equality holds in (3) for the design h_c. It now follows that if the design h_c is such that

$$\frac{\partial}{\partial h} D_A(q_n; h) = \text{constant} \tag{4}$$

over A then

$$\int \delta h \, dA \geq 0, \tag{5}$$

so that the design h_c provides a relative minimum for the volume of permissible designs. Thus, assuming that the neglect of the second-order terms is permissible, the variational principle leads to a direct design procedure. Mroz [14] has given an example in which the application of (4) leads only to a stationary value for the volume.

The case when D_A is directly proportional to h, as for the ideal sandwich shell of problem (ii), is simpler and a stronger result is possible. We again suppose that the design h_c is at collapse in a deformation mode u_i, and we use the mode u_i in the variational principle (3) for another permissible design h_s. We obtain

$$\int D_A(q_n; h_s) \, dA \geq \int T_i u_i \, dA = \int D_A(q_n; h_c) \, dA.$$

Since

$$D_A(q_n; h_s) = D_A(q_n; h_c) \frac{h_s}{h_c},$$

we can conclude that if the design h_c is such that

$$\frac{D_A(q_n; h)}{h} = \text{constant} \tag{6}$$

over A, then

$$\int h_s \, dA \geq \int h_c \, dA.$$

Not only does the condition (6) lead to an absolute minimum for the design volume but the condition (6) is much easier to use than condition (4) because (6) does not involve the design thickness directly.

In order to illustrate the use of these design methods we consider the minimum-volume design of a circular plate with a built-in edge under uniform pressure loading on its upper face (for problems involving other symmetric and non-symmetric pressure distributions see [15, 16]). When the Tresca yield condition is assumed, the yield condition on the radial bending moment M and the circumferential bending moment N is the familiar hexagon in (M, N) space,

$$\max \left(|M| \, , \, |N| \, , \, |M - N| \right) = M_o,$$

in which $M_o = \sigma_o H h$ for the sandwich plate and $M_o = \frac{1}{4} \sigma_o h^2$ for the solid plate. The curvature rates κ, λ in the radial and circumferential directions are derived from the downward deflection rate w of the middle surface through

$$\kappa = -\frac{d^2 w}{dr^2}, \qquad \lambda = -\frac{1}{r} \frac{dw}{dr},$$

where r measures distance from the center.

For the sandwich plate, the design criterion (6) does not involve the design thickness h, and it is readily found that (6) can only be satisfied for a finite range of r when either

$$\text{(i)} \quad M = N = \pm M_o, \qquad \text{or} \qquad \text{(ii)} \quad M = \pm M_o, \quad N = 0.$$

For these regimes, condition (6) reduces to

$$\text{(i)} \quad \kappa + \lambda = \pm \alpha, \qquad \text{or} \qquad \text{(ii)} \quad \kappa = \pm \alpha,$$

where α is a constant when the core thickness H is constant. The curvature rates κ, λ must have the same sign in regime (i) while in regime (ii) they are of opposite sign and $|\kappa| \geq |\lambda|$. For a plate with a built-in edge, regime (i) with the positive sign will apply in a central region $r \leq a$ and regime (ii) with the negative sign will apply in the remaining portion $a \leq r \leq R$. At the built-in edge w and dw/dr are zero and dw/dr is zero at the center. In order to have w and dw/dr continuous at the junction $r = a$, it is found that we

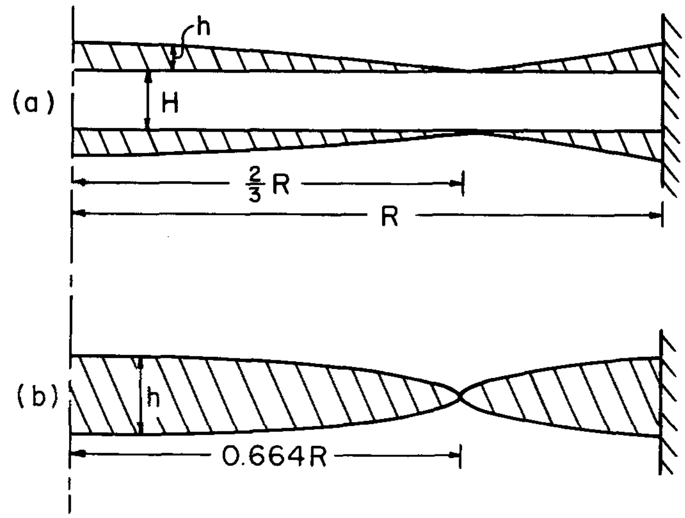

Figure 2. Minimum-volume designs for built-in circular plate under uniform pressure (a) sandwich plate (b) solid plate

must have $a = 2R/3$, and at this radius M must vanish because κ changes sign. With $M = 0$ at $r = 2R/3$, the moment distribution in the two regions of the plate, and hence the thickness $h = |M|/\sigma_0 H$, can now be found from equilibrium. The design is indicated in Figure 2 (a).

For the solid plate, we again assume that M and N have the fully plastic value M_0 for for $r \leq a$ and that N is zero for $a \leq r \leq R$, with $M = 0$ at $r = a$. From equilibrium the moment distribution and therefore the thickness h(r),

$$h = \left\{4\ |M|/\sigma_0\right\}^{\frac{1}{2}},$$

can be found for a general value of a. In order to satisfy the design criterion (4) we must have

$$\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr} = -\frac{\alpha}{h} \quad \text{and} \quad \frac{d^2 w}{dr^2} = \frac{\alpha}{h}$$

in the inner and outer regions, respectively. The continuity of dw/dr at $r = a$ leads to (see [15])

$$\int_0^a \frac{r\,dr}{h(r)} = a \int_a^R \frac{dr}{h(r)},$$

and this equation serves to determine the junction radius a. For uniform pressure loading $a = 0.664 R$ and the design is as indicated in Figure 2 (b).

Minimum-volume design for other one-dimensional situations, such as symmetrically loaded circular cylindrical shells of sandwich type [4], is also relatively straight-forward, but design problems which are two-dimensional can be much more difficult to treat. So far designs for non-circular plates with built-in edges have only been obtained by an inverse method [17].

The design procedure can be modified to include body forces (such as weight) which act only when material is present (see [4]). Also the procedure has been extended to the design of multi-purpose structures which are to support different systems of loads at different times [18], and to the quasi-static design of structures under moving loads [19].

3. UNIFORM STRENGTH DESIGNS

The methods for plastic design have been extended [7] to materials which are not perfectly plastic, so that design limitations other than plastic collapse are involved. For example, we may use a work-hardening material and in order to minimize the possibility of fracture we may wish to design the shell so that it remains elastic everywhere. For another material, we may wish to keep the stresses below the level at which an appreciable amount of creep will occur. In both cases there will be a limiting surface in stress space to restrict the stress states in a section of the shell. We shall say that a design is a uniform strength design for the given loading if the stresses everywhere in the shell are on the limiting surface. We seek the uniform strength design which has least volume.

Figure 3 indicates a limiting surface in generalized stress space. We suppose that the surface is given by

$$L\,(Q_n;\,h) = 0, \tag{7}$$

where L is a known function, and we assume that the surface is convex. In purely elastic design, for example, the limiting surface is determined by those values Q_n for which the yield limit is reached in the outer fibers of the shell.

Consider an infinitesimal virtual deformation of the shell defined by middle surface displacements v_i and associated generalized strains e_n. We shall say that a virtual defor-

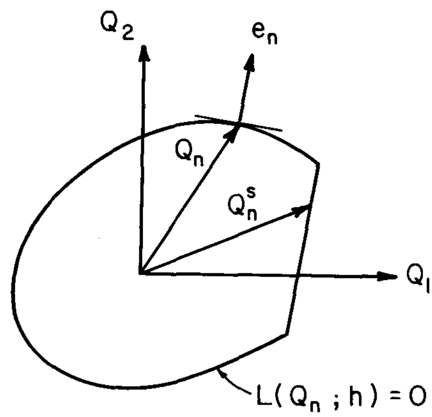

Figure 3. Limiting surface in generalized stress space

mation with strains e_n is compatible with a limiting stress state Q_n if the vector representing e_n is normal to the limiting surface at the stress point Q_n, Figure 3. At a singular point of the limiting surface the strain vectors representing compatible deformations lie in the fan bounded by adjacent normals. For a given convex limiting surface, the virtual work W_A in a compatible virtual deformation is then uniquely determined by the virtual strains e_n,

$$W_A = Q_n e_n = W_A (e_m; h),$$

in which the repeated index n implies summation over 1, 2, .., N. Moreover, for any other stress state Q_n^s inside or on the limiting surface we have

$$Q_n^s e_n \leq W_A (e_m; h) \tag{8}$$

with equality only if e_n are also compatible with Q_n^s.

The approach to minimum-volume design for uniform strength is similar to that for plastic design, and we shall treat the case when W_A is directly proportional to the design thickness h. If the shell h_c has limiting stresses under the loads and if there is an admissible compatible virtual deformation v_i, then by virtual work we have

$$\int T_i v_i \, dA = \int W_A (e_n; h_c) \, dA. \tag{9}$$

For any other shell h_s with stresses Q_n^s at or below the limiting values we can use (8) and virtual work to deduce that

$$\int T_i v_i \, dA = \int Q_n^s e_n \, dA \le \int W_A (e_m; h_s) \, dA,$$

where v_i is any admissible virtual displacement. We therefore have the variational principle

$$\int W_A (e_n; h_s) \, dA - \int T_i v_i \, dA \ge 0 \qquad (10)$$

for permissible designs and equality holds only when h_s is a uniform strength design and v_i is compatible with the stresses Q_n^s in the shell. We now use the compatible virtual deformation v_i for the design h_c in the variational principle (10) and with (9) we derive

$$\int W_A (e_n; h_s) \, dA \ge \int W_A (e_n; h_c) \, dA.$$

Because W_A is proportional to h, we see that if the design h_c is such that

$$\frac{W_A (e_n; h)}{h} = \text{constant}$$

over A, then the volume of h_c will be an absolute minimum for all permissible designs.

For the solid shell the design criterion is

$$\frac{\partial}{\partial h} W_A (e_n; h) = \text{constant}$$

over the shell.

Uniform strength designs have been discussed by Save [20].

4. ELASTIC DESIGN FOR GIVEN STIFFNESS

Direct design methods can be developed in the same way for other problems of optimum design provided that a suitable variational principle holds for the structure under investigation. This can be the case in the minimum-volume design of an elastic structure which is to have a given stiffness under a given set of loads (or, equivalently, elastic design for maximum stiffness with a given volume of material). Other examples are minimum-volume design for a given buckling load or for a given fundamental frequency of vibration. Techniques for design problems such as these that have been developed in a unified way by Prager and Taylor [5]. Here we outline the procedure in the case of elastic design for a given stiffness.

For an elastic shell there is a strain-energy function E_A, per unit area of the middle surface, which is uniquely determined by the generalized strains q_n derived from middle surface displacements u_i. The strain energy also depends on the design thickness h so that we write it as $E_A(q_n; h)$. The potential energy U is defined as

$$U\{u^*; h\} = \int E_A(q_n^*; h)\, dA - \int T_i u_i^*\, dA,$$

where the integral of $T_i u_i$ represents all the virtual work of the prescribed loads including the edge loading and where u_i^* is a displacement field which satisfies any imposed displacement conditions. When E_A is a positive definite quadratic function of the strains, the Principle of Minimum Potential Energy holds. The principle states that the potential energy U is minimized by the actual displacements u_i produced by the loads,

$$U\{u^*; h\} \geq U\{u; h\}.$$

We now define the compliance of the shell for the given loads to be twice the total strain-energy of the shell and we note that

$$2\int E_A(q_n; h)\, dA = \int T_i u_i\, dA.$$

For two designs h and h_s with the same compliance, we have

$$\int E_A(q_n; h_s)\, dA \geq \int E_A(q_n^s; h_s)\, dA = \int E_A(q_n; h)\, dA, \qquad (11)$$

where q_n^s are the strains for the design h_s. The inequality in (11) follows from the Principle of Minimum Potential Energy applied to the design h_s. When E_A is directly proportional to h, we see from (11) that in designing for a given compliance, the design with E_A/h constant will have least volume. For other types of shells the procedure would be to design so that $\partial E_A/\partial h$ is constant over the shell, and the design would provide a relative minimum for the volume of permissible designs.

As a simple example, suppose we have an elastic beam of length 2ℓ which is built-in at both ends and which has a transverse point load P at the center. We wish to design the beam so that the central deflection does not exceed δ and such that the beam has minimum volume. For a beam of the sandwich type, minimizing the volume is the same as minimizing the integral of the bending stiffness over the beam. If two beams with stiffnesses s and $\overline{s}$ have the same central deflection δ under the load, they have the same compliance $P\delta$ and in the same way that (11) was derived we can use the Principle of Minimum Potential Energy to get

$$\int \bar{s}\kappa^2 \, dx \geq \int s\kappa^2 \, dx,$$

where κ is the curvature of the design s under the load P and x measures distance from one end. We now see that the design s will have least volume if $|\kappa|$ is constant. In order to satisfy the constraints at the ends, the deflection with constant $|\kappa|$ must have inflection points at the quarter points $x = \ell/2, \; 3\ell/2$. Since the moment $M = s\kappa$ must vanish at the quarter points where κ changes sign, the moment distribution is now statically determinate and $M(x)$ and therefore $s(x)$ can be found.

The design procedure obtained from the Principle of Minimum Potential Energy applies for design with given compliance. However, the design criterion does not always coincide with the compliance. Thus if we have a distributed load over the built-in beam and we wish to limit the central deflection as before, the compliance will not be known in advance. Similarly, if we have an off-center point load P at the section $x = x_o$ and we wish to limit the maximum deflection of the beam, the compliance is $P u_o$, where u_o is the deflection at $x = x_o$ and is not necessarily the maximum deflection. These design problems can be approached by using a variational principle of a different type called the Principle of Stationary Mutual Potential Energy [6]. Let u_i and $\bar{u}_i$ be two middle surface displacement fields for a design of thickness h and let q_n, Q_n and $\bar{q}_n$, $\bar{Q}_n$, respectively be the associated generalized strains and stresses. We define the <u>mutual strain energy</u> through

$$E_A(q_n, \bar{q}_n; h) = \sum_1^N Q_n \bar{q}_n = \sum_1^N \bar{Q}_n q_n.$$

For two different sets T_i and $\bar{T}_i$ of applied loads, the <u>mutual potential energy</u> U_M is defined as

$$U_M\{u^*, \bar{u}^*; h\} = \int E_A(q_n^*, \bar{q}_n^*; h)\, dA - \int T_i \bar{u}_i^* \, dA - \int \bar{T}_i u_i^* \, dA,$$

where u_i^*, $\bar{u}_i^*$ are kinematically admissible displacement fields.

If u_i and $\bar{u}_i$ are the actual displacements that the loads T_i and $\bar{T}_i$, respectively, would induce in the shell, then

$$U_M\{u, \bar{u}; h\} = -\int T_i \bar{u}_i \, dA = -\int \bar{T}_i u_i \, dA. \qquad (12)$$

With the use of the Principle of Virtual Work, it can now be shown (see [6] for details) that

$$U_M\{u^*, \bar{u}^*; h\} - U_M\{u, \bar{u}; h\} = \int E_A(q_n^* - q_n, \bar{q}_n^* - \bar{q}_n; h)\, dA. \tag{13}$$

If we apply (13) when u_i^* and $\bar{u}_i^*$ are neighboring displacements to the actual displacements u_i and $\bar{u}_i$, the right-hand side will be zero to first order. Thus $U_M\{u^*, \bar{u}^*; h\}$ is stationary at the values $u_i^* = u_i$, $\bar{u}_i^* = \bar{u}_i$, and this is the Principle of Stationary Mutual Potential Energy.

Suppose we wish to design a structure so that the transverse deflection at a particular point X_o of the mid-surface is of amount δ under the loads T_i. We take the second system of loads $\bar{T}_i$ to be a single unit concentrated load $\bar{P}$ acting normal to the middle surface at the point X_o. From (12) we then see that the value of $U_M\{u, \bar{u}; h\}$ is $-\bar{P}\delta$, so that designs which satisfy the design criterion will have the same value for $U_M\{u, \bar{u}; h\}$. We can therefore use the Principle of Stationary Mutual Potential Energy in the same way as the Principle of Minimum Potential Energy was used in design for a given compliance. In this way we find that the design such that

$$\frac{\partial}{\partial h} E_A(q_n, \bar{q}_n; h) = \text{constant}$$

over the shell will provide a stationary value for the volume for designs which have transverse deflection of amount δ at the point X_o.

Applications to the minimum-volume design of beams for given deflections (or rotations) are described in [6]. Suppose we wish to design a beam of sandwich type and we require the deflection at the section $x = x_o$ to be of amount δ under a certain system of loads. Let s and s^* be the bending stiffnesses of two designs that satisfy the constraint on the deflection at x_o, and let u, u^* and $\bar{u}$, $\bar{u}^*$ be the corresponding deflections of these designs under the given loads and under a unit concentrated load $\bar{P}$ at x_o. From (12) we have

$$U_M\{u, \bar{u}; s\} = U_M\{u^*, \bar{u}^*; s^*\} = -\bar{P}\delta,$$

where we have identified the bending stiffnesses s and s^* with the design thicknesses h and h^*, as we may do for sandwich beams. The deflections u, $\bar{u}$ are kinematically admissible for the design s^* and if we apply (13) to this design we get

$$U_M\{u, \bar{u}; s^*\} - U_M\{u^*, \bar{u}^*; s^*\} = \int s^*(\kappa^* - \kappa)(\bar{\kappa}^* - \bar{\kappa})\, dx, \tag{14}$$

where κ, $\bar{\kappa}$, ... are the curvatures associated with the deflections u, $\bar{u}$, ... If we replace

$U_M\{u^*, \bar{u}^*; s^*\}$ by $U_M\{u, \bar{u}; s\}$ in (14) and use the definition of U_M we find that

$$\int (s^* - s)\, \kappa\, \bar{\kappa}\, dx = \int s^* (\kappa^* - \kappa)\, (\bar{\kappa}^* - \bar{\kappa})\, dx. \tag{15}$$

When s^* is a neighboring design to s, the right-hand side of (15) is zero to first-order and we see that

$$\kappa\, \bar{\kappa} = \text{constant} = c^2 \tag{16}$$

is a sufficient condition for the design s to provide a stationary value for the volume $\int s\, dx$. If $M = s\,\kappa$ and $\bar{M} = s\,\bar{\kappa}$ are the bending moments for the optimum design s under the two systems of loads then

$$s = (M\bar{M})^{\frac{1}{2}}/(\kappa\,\bar{\kappa})^{\frac{1}{2}} = \frac{1}{c}\,(M\bar{M})^{\frac{1}{2}}.$$

The constant c can be determined from $U\{u, \bar{u}; s\} = -\bar{P}\delta$ and we finally arrive at

$$s = \frac{(M\bar{M})^{\frac{1}{2}}}{\bar{P}\delta} \int (M\bar{M})^{\frac{1}{2}}\, dx. \tag{17}$$

For a statically determinate beam, the moment distributions M, $\bar{M}$ can be determined directly so that the optimum design (17) is readily found without calculation of deflections. Moreover, for a statically determinate beam it can be shown [6] that the design satisfying (16) actually furnishes an absolute minimum for the design volume. In this case the moment distributions M and $\bar{M}$ are independent of the stiffnesses, so that

$$M = s\,\kappa = s^*\,\kappa^*, \qquad \bar{M} = s\,\bar{\kappa} = s^*\,\bar{\kappa}^*.$$

These equations imply that

$$\kappa^* - \kappa = -\frac{(s^* - s)\,\kappa}{s^*}, \qquad \bar{\kappa}^* - \bar{\kappa} = -\frac{(s^* - s)\,\bar{\kappa}}{s^*}.$$

Substituting in (15) we obtain, with (16),

$$\int (s^* - s)\, dx = \int \frac{(s^* - s)^2}{s^*}\, dx \geq 0,$$

which shows that the design s satisfying (16) provides an absolute minimum for the design volume.

For a statically indeterminate beam, an extra step is required in order to arrive at a design that provides an absolute minimum for the volume. Consider, for example, a beam of length 2ℓ which is built-in at both ends and is loaded by a uniform pressure p along its length. We wish to restrict the deflection at the center $x = \ell$ to be of amount δ. For a beam built-in at both ends, $u''(x)$ must change sign at least twice for otherwise no deflection is possible; thus $M(x)$ will have at least two zeros. Assuming a symmetrical design, we suppose that the bending moment $M(x)$ is zero at $x = \ell \pm b$. If we now consider only designs for which the stiffness vanishes at $x = \ell \pm b$, we have a statically determinate beam and we can determine the design (17) which has least volume in this class of designs. We can now choose b so that the volume will have the least value for all possible designs, and this value of b is found to be $\ell/2.01$.

When the loading is not symmetric, the maximum deflection may be off center. Suppose, for example, that we have a simply supported beam of length 2ℓ under a system of loads which produces a bending moment $M(x)$. We wish to limit the maximum deflection to an amount δ. Let $\overline{M}(x)$ be the bending moment distribution caused by a unit point load $\overline{P}$ at $x = b$. The design (17) will then be optimum for a deflection of amount δ at $x = b$. We can ensure that the section $x = b$ will have the greatest deflection if we choose b so that $u'(x)$ is zero at $x = b$. In order to determine b, we note that if $u'(b) = 0$ and $u = 0$ at the ends, then

$$u(b) = -\int_0^b \int_y^b u''(x)\,dx\,dy = -\int_b^{2\ell} \int_b^y u''(x)\,dx\,dy,$$

and this implies that

$$\int_0^b x\kappa\,dx = \int_b^{2\ell} (2\ell - x)\,\kappa\,dx. \tag{18}$$

Because $\kappa = M/s = c\,(M/\overline{M})^{\frac{1}{2}}$, we can write (18) as

$$\int_0^b x\,(M/\overline{M})^{\frac{1}{2}}\,dx = \int_0^{2\ell} (2\ell - x)\,(M/\overline{M})^{\frac{1}{2}}\,dx,$$

and this equation serves to determine b. To give an example, when the beam is loaded by a point load P at $x = a$, the value of b varies from ℓ to 1.11ℓ as a varies from ℓ to 2ℓ.

The procedures described here for elastic design can be extended to design with two or more constraints on deflection or rotation under a single system of loads [6] and to the design of multi-purpose structures [6, 21].

5. MICHELL STRUCTURES

In formulating the problem of optimum design in Section 2, we assumed that the type of the structure and the layout, that is the middle surface A, were specified. A less restrictive formulation merely specifies the region in which the given material can be placed and leaves the type and layout of the structure to be determined. In 1904 Michell published his paper [1] on the minimum-volume design of framed structures. He specified that the structure should consist of tie-bars in tension and struts in compression, but the layout of the structure was not specified. The material to be used allows a maximum tensile stress σ_t and a maximum compressive stress σ_c, and for a design which carries the prescribed loads, the minimum volume allowable is

$$V = \Sigma \ell_t f_t/\sigma_t + \Sigma \ell_c f_c/\sigma_c. \tag{19}$$

Here f_t is the tension in any tie-bar of length ℓ_t and f_c is the thrust in any strut of length ℓ_c. Michell showed that a framed structure will be of minimum volume if there is a virtual small deformation of the space such that each tie-bar suffers an extensional strain of amount e and each strut suffers a compressive strain of amount e and no linear element of space suffers a strain numerically greater than e, where e is a constant. Note that the actual deformation of the minimum-volume frame under the loads involves extensional and compressive strains of amounts σ_t/E and σ_c/E, respectively, along the frame elements, where E is Young's modulus.

In the proof of his results, Michell used a theorem due to Maxwell. By imposing a uniform dilatation on the whole of space, Maxwell showed that for all structures under the same system of applied loads

$$\Sigma \ell_t f_t - \Sigma \ell_c f_c = \text{constant}.$$

(The constant is $\Sigma \underline{F} \cdot \underline{r}$, where $\underline{F}$ is an applied load at a point with position vector $\underline{r}$.) However, Maxwell's theorem does not apply to structures with kinematic constraints imposed by support conditions because the reactions at the supports can vary with the structure. An exception is a structure with one fixed point but in this case the reaction at the support is determined by overall equilibrium. Because Maxwell's theorem is essential to Michell's proof when $\sigma_t \neq \sigma_c$, the design procedure of Michell will not be valid in general when kinematic constraints are imposed. This limitation on the use of Michell's method does not appear to have been mentioned explicitly in the literature.

An alternative approach, which does not have the limitation of the Michell method, has been given by Shield [7]. The procedure is to design a frame compatible with a virtual small deformation in which the principal strains are of magnitude e/σ_t if extensional and of magnitude e/σ_c if compressive, the directions of frame elements coinciding with the principal directions of strain as before. The virtual deformation must satisfy any imposed kinematic constraints. The proof that the procedure leads to a minimum-volume frame is straightforward and it makes direct use of the Principle of Virtual Work as in the method of Section 3 for uniform strength designs. The proof has been repeated by Hemp [22] and by Hegemier and Prager [23] for the case $\sigma_t = \sigma_c$ (when the Michell method and the alternative method become identical).

Michell [1] supplied some examples of minimum-volume framed structures and other examples are given in [22, 24, 25, 26]. Cox [27] has shown that a Michell structure has greater stiffness under the loads than any other structure which is stressed to the limiting values σ_t and σ_c. More recently, Hegemier and Prager [23] have shown that an elastic frame with a specified stiffness (i.e. compliance) has least volume when it has the layout of a Michell structure, and this holds also for frames designed for a given stiffness in stationary creep or for a given fundamental frequency of vibration. In the following we give an example to show that minimum-volume frames are not necessarily unique, and we describe some new additions to the list of Michell structures.

The diagram at the top of Figure 4 indicates the layout given by Michell [1] for a single force applied at the midpoint C of the line AB and balanced by equal parallel forces at A and B. The struts AD, EB and the curved bar DE carry a uniform compressive force and a quadrantal fan of tie-bars from C to DE maintains the equilibrium of the curved bar. The layout is symmetrical about AB with tie-bars replacing struts and vice-versa. The virtual deformation with principal strains $\pm e$ associated with the layout can be adjusted so that the

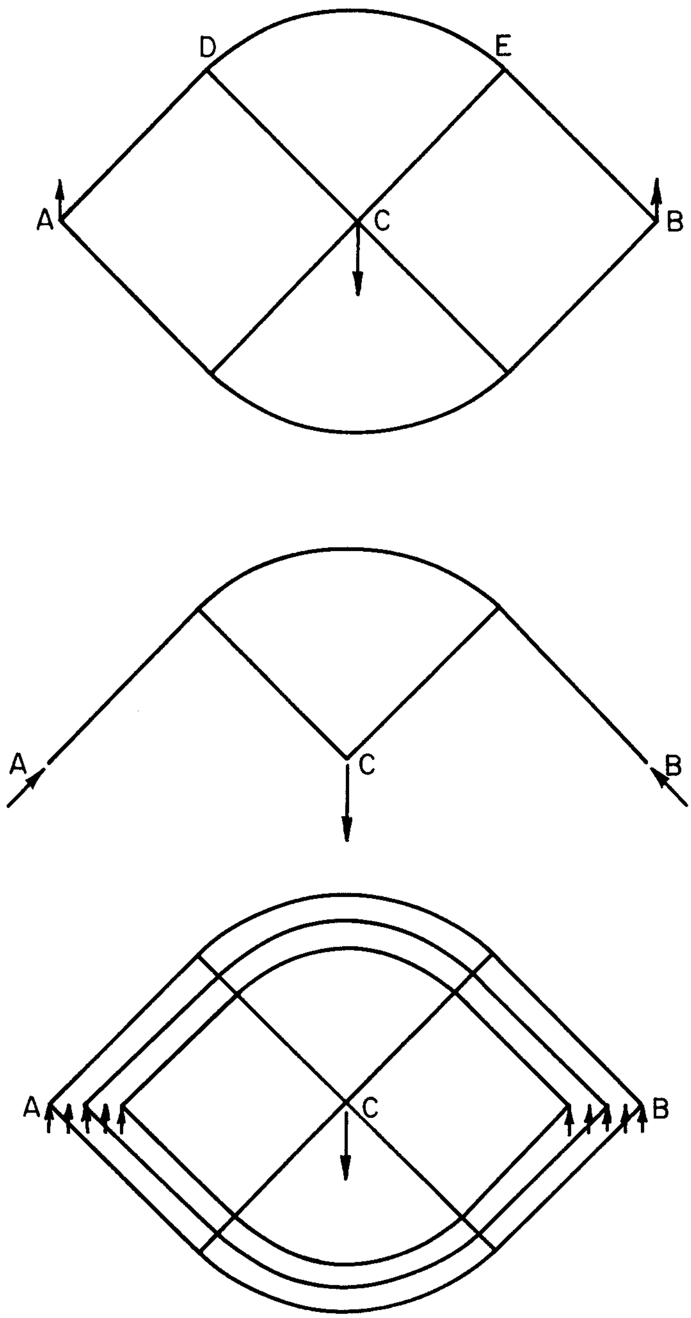

Figure 4. Load at C supported at A, B

displacement is zero at points A and B. If we assume $\sigma_t = \sigma_c$, we can use this virtual deformation for the case when we have the same force at C, but now A and B are fixed points of support. The optimum structure has the same volume as the structure with specified parallel forces at A, B, but the optimum design is not unique. For example, the load at C can be carried by a frame entirely above AB as indicated in the middle diagram of Figure 4. An infinity of optimum designs results from arbitrarily assigning a fraction of the load at C to be carried by a structure above the line AB and the remainder by a structure below the line AB. We note that if we had specified that the load at C be carried by a beam with centerline AB and built-in at A and B, the optimum design would have bending moments at A and B. The Michell structure has no moments at the fixed points A, B.

The minimum-volume design indicated at the bottom of Figure 4 uses the same virtual deformation with principal strains $\pm$ e, but now it is specified that distributed loads at A and B balance the load at C.

Figure 5 shows the optimum layout for pure bending. A bending moment at the point A is to be transmitted to the point B by a framed structure of minimum volume, composed of a material of limited strength (or the structure has an assigned bending stiffness). In the circular regions around the points A and B, the tie-bars and struts follow logarithmic spirals. The spiral regions are connected by a strut GH in compression at the top and a tie-bar carrying the same force at the bottom of the structure. The associated virtual deformation with principal strains $\pm$ e is, apart from a rigid displacement, purely circumferential in the circular regions. The regions between the larger circles and the straight-line boundaries (such as GC, CH) of the upper and lower quadrants which meet at C move as rigid bodies. In the quadrants meeting at C, the principal strain directions are vertical and horizontal, and the quadrants deform like a plastic hinge in a beam in pure bending. The total volume of material required is

$$M \left[1 + 2 \ln \frac{a}{\sqrt{2}\, r_o}\right] \left[\frac{1}{\sigma_t} + \frac{1}{\sigma_c}\right] .$$

Here M is the moment applied at A and B, a is the length of AC or AB and r_o is the radius of small circles at A and B over which the forces equivalent to the moments M are distributed.

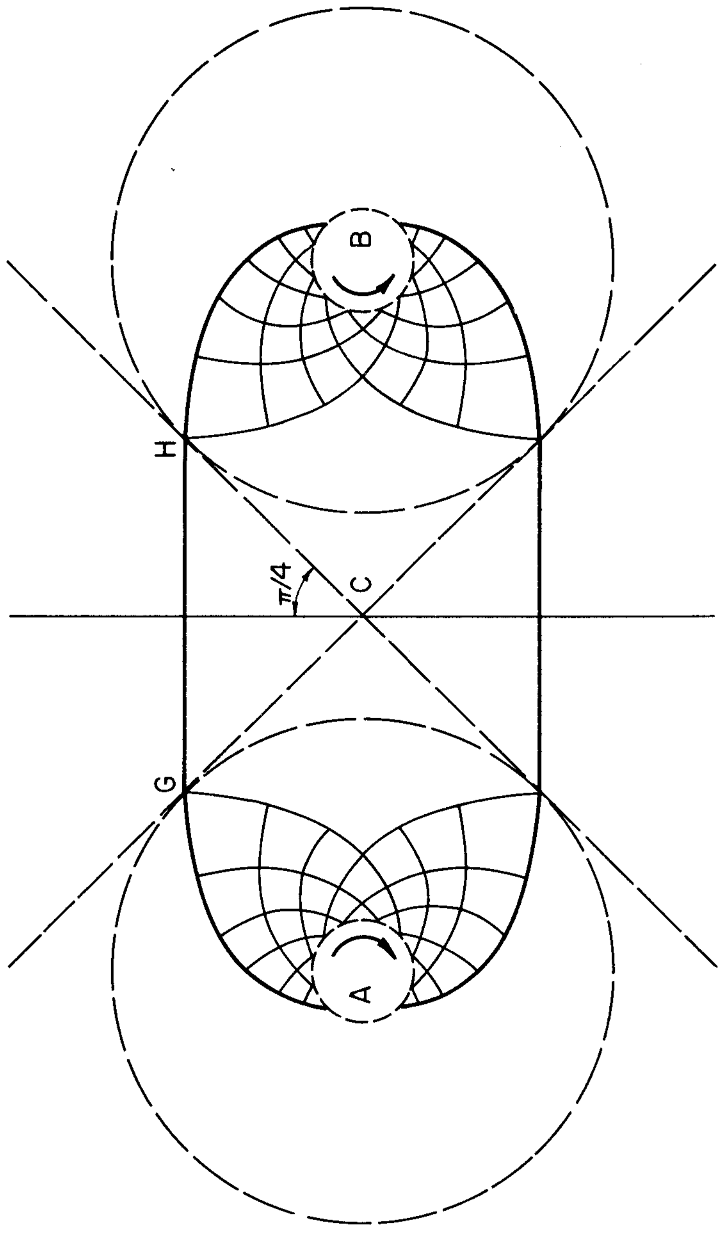

Figure 5. Layout for pure bending

Figure 6 indicates the optimum layout when a downward force P is added at the central point C and upward parallel forces P/2 are added at the points A and B. The moment M applied at A and B and the force P are related to the angle 2α of the fan regions through

$$4M/Pa = \cot\alpha - 1.$$

As the ratio P/M increases, the angle α tends to $\pi/4$ and the structure approaches the Michell structure for three parallel forces. It may be noted that the moments at A, B are of opposite sign to those that would be developed at the ends of a built-in beam by a downward central load. The optimum layout for the case of reversed moments at A and B remains to be determined. In the particular case when there is no moment across the central section, that is the case $M = Pa/2$, the optimum layout is as shown in Figure 7. In the associated virtual deformation, the space outside the circular regions does not move while inside the circular regions the displacement is purely circumferential.

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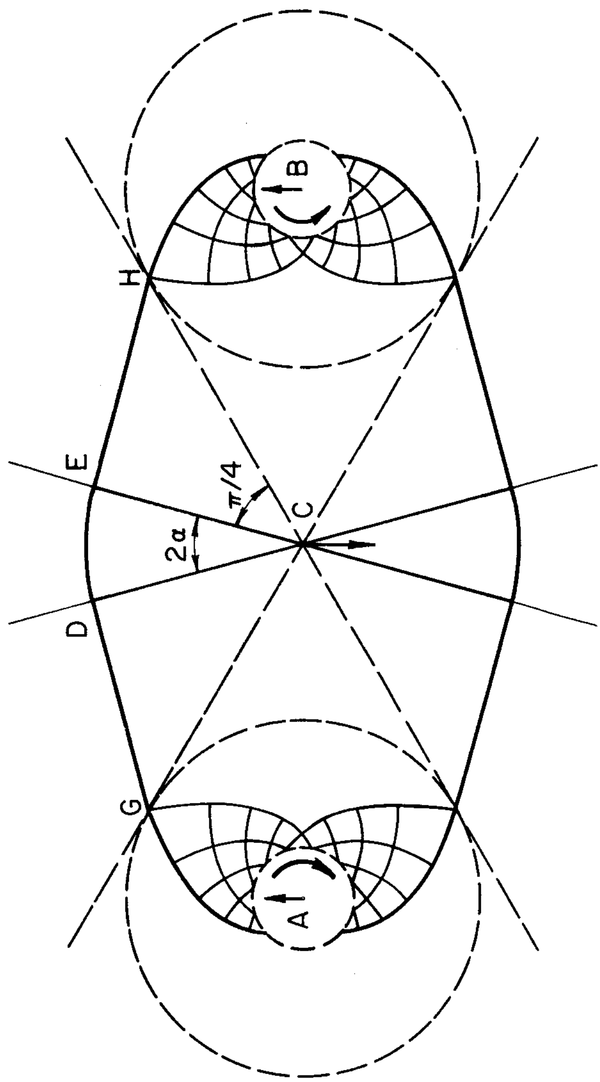

Figure 6. Bending with central load

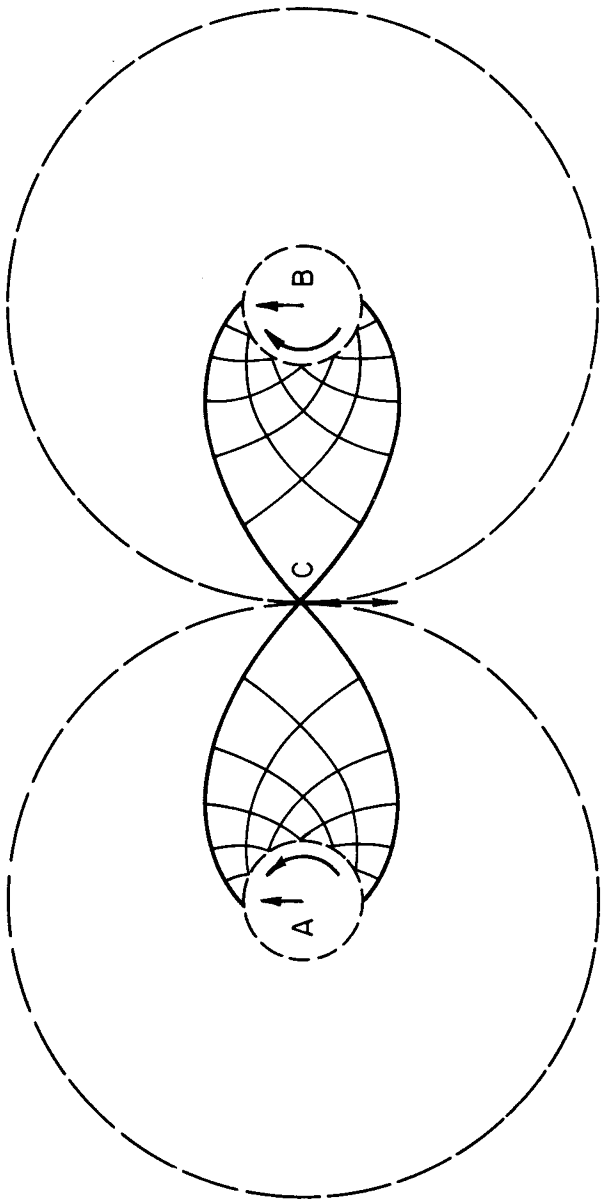

Figure 7. Bending with central load and zero central moment

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