OPTIMIZATION PROBLEMS IN HYDROFOIL PROPULSION*

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This paper attempts to apply the principle of control theory to investigate the possibility of extracting flow energy from a fluid medium by a flexible hydrofoil moving through a gravity wave in water, or by an airfoil in gust. The present optimization consideration has led to the finding that although the flexible hydrofoil may have an infinite number of degrees of freedom, the optimum shape problem is nevertheless a finite-dimensional one. The optimum shape sought here is the one which minimizes the required power subject to the constraint of fixed thrust. A primary step towards the solution is to reduce the problem to one of minimizing a finite quadratic form; after this reduction the solution is determined by the method of variational calculation of parameters. It is found that energy extraction is impossible if the incident flow is uniform, and may be possible when the primary flow contains a wave component having a longitudinal distribution of the velocity component normal to both the mean direction of flight and the wing span. When such waves of sufficiently large amplitude are present, not only flow energy but also a net mechanical power can be extracted from the surrounding flow.

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* This paper includes further extension to that which was originally presented at the Symposium.

1. Introduction

Some previous observations on fish swimming and bird flight seem to suggest that some species may have learned, through experience, to acquire the key to high performance by executing the optimum movement that may be of great interest to control theory related to fluid mechanics. An especially intriguing aspect of the optimization problem concerns with the possibility of extracting energy from surrounding flow by an oscillating lifting surface (such as the fish body and fins, bird wings, and artificial wings like airfoil and hydrofoil) and its associated effect on the control of motion.

This general problem has been explored to various degrees of generality. Based on the approximation of potential flow with small amplitude, it has been found by Lighthill (1960) for slender bodies, and by Wu (1961) for two-dimensional plates, that if the basic flow is uniform, energy is always imparted by an oscillating wing to the surrounding fluid, and an extraneous mechanical work must therefore be continuously supplied to maintain the motion. Even though it is impossible in this case to extract energy from the flow field, the highest possible hydromechanical efficiency that can be attained by a wing, subject to delivering a given forward thrust, can be very high, as found by Wu (1971 b, c) for the two-dimensional plate and a slender lifting surface.

As was subsequently pointed out by Wu (1972), the situation becomes drastically different when the basic flow is no longer uniform, but contains a wave component, such as gravity waves in water, or wavy gust in air. The contention that the wave energy stored in a fluid medium can be utilized to assist propulsion has been suggested by intuitive observations. Sea gulls and pelicans have been observed to skim ocean waves over a long distance without making noticeable flapping motions (save some gentle twisting) of their wings. In an extensive study of the migrating salmon, Osborne (1960) found that the increased flow rate in a swollen river did not slow the salmon down (for known biochemical energy expended during the travel) by that much a margin as would be predicted by the law of resistance in proportion to the square of their velocity relative to the flowing water. Several possible explanations were conjectured by Osborne, including the prospect that the flow energy associated with the eddies in river could be converted to generate thrust. To explore this possibility Wu (1972) introduced an energy consideration to an earlier study of Weinblum (1954) on the problem of heaving and pitching of a rigid hydrofoil in regular water waves. It was found that the greatest possible rate of energy extraction is provided by the optimum mode of heaving and pitching. When waves of sufficiently large amplitude are present, not only flow energy but also a net mechanical power can be extracted from the wave field.

In the present study this problem is further generalized by allowing the hydrofoil to be flexible so as to admit an infinitely many degrees of freedom (of small amplitudes). This general problem merits study for several reasons. First, it is of a theoretical interest to find out how much improvement in the hydromechanical efficiency and energy extraction can be gained by admitting the additional degrees of freedom. Second, the results of the present study of energy transfer between an oscillating body and surrounding stream can be useful to the development of control theory for hydrofoil ships and to the analysis of flutter phenomena. In the case of flutter in a uniform stream, it is usually assumed that the engine maintains the constant forward speed regardless of the flutter-created inertial drag. In a wavy stream, however, the flutter may create a propulsive thrust, which may amplify further instability and a self-excited flutter may develop. Some of these aspects have already been observed by Küssner (1935) and Garrick (1936, 1957); this paper is aimed at the general case of propulsive energy balance.

Further, from the standpoint of development of control theory, the present problem also merits study in its own right since it presents some new features and difficulties that apparently do not confirm with the known classical cases. A brief description can be given as follows. Section 2 presents the general (linearized) theory for a two-dimensional hydrofoil oscillating in waves, which is applied in Section 3 to the general case of a flexible plate wing. In Section 4 the problem of optimum motion is formulated as to find a hydrofoil profile that minimizes the energy loss C_E subject to a constrained thrust coefficient C_T. It is shown that although the flexible hydrofoil may have infinitely many degrees of freedom, C_E and C_T can be reduced to quadratic forms of finite dimensions. After this crucial step the optimization problem reduces to one defined on a three-dimensional vector space ($\zeta_1, \zeta_2, \zeta_3$). With this drastic reduction it is possible to show that an optimal solution does not exist unless appropriate bounds are imposed on the independent variables ζ_n's. Under this condition the optimal solution is determined and compared with the previous special cases. It is felt that the present method of solution is still heuristic, to some extent, for much of the intuitive physical picture was relied on for guidance. It is with the hope to stimulate further development of the general theory for this class of control problem that the present study is presented before this Symposium.

2. Two-dimensional Hydrofoil Oscillating in Waves

With specific applications in view we consider the basic flow to be a sinusoidal gravity wave of small amplitude in water of finite depth, H, in which a two-dimensional hydrofoil of chord 2ℓ moves horizontally with velocity U while submerged at a mean depth h_1 underneath the free surface. In terms of the body coordinate system (x, y), the wave profile of the basic flow (see Fig. 1) may be written as

$$y = h_1 + \mathrm{Re}\,[\, a\, e^{i(\omega_o t - kx)} \,] \quad , \tag{1}$$

the wave amplitude, a, being assumed small such that $ka \ll 1$. The corresponding velocity $(U + u_o,\ v_o)$ of the wave field, by classical theory, is

$$u_o - iv_o = A_* \cos [\, k(x + iy + ih_2) - \omega_o t \,] \quad , \tag{2}$$

where $h_2 = H - h_1$ is the distance from the bottom and ω_o is the encounter frequency

$$\omega_o = \omega_* \pm kU \quad , \tag{3}$$

$$\omega_*^2 = gk \tanh kH \quad , \qquad A_*/a = (2gk/\sinh 2kH)^{\frac{1}{2}} \quad , \tag{4}$$

where g is the gravitational acceleration and in (3), the + sign is for heading sea, and - sign for following the waves. In particular, the y-component wave velocity at the x-axis (which coincides with the mean position of the hydrofoil), denoted by $v_o(x, 0, t) \equiv V_o(x, t)$, is

$$V_o(x, t) = iA_o\, e^{i(\omega_o t - kx)} \quad , \qquad A_o = A_* \sinh kh_2 \quad . \tag{5}$$

Here and henceforth, the real part of a complex expression will be understood for physical interpretation.

Since the problem of central interest at hand is to determine the effect of a waving stream on the propulsive performance of a hydrofoil in unsteady motion, we shall further assume, for simplicity, that the hydrofoil is located sufficiently far from both the free water surface and solid bottom so as to curtail the complicated (but only secondary) corrections due to these boundary effects. This condition would be nearly satisfied if the hydrofoil is at a distance more than two chords away from each of these boundaries, that is for the chord $2\ell < \frac{1}{2} \max\{h_1, h_2\}$, this estimate being inferred by the known results of the steady flow case (see Wu, 1954) which is assumed to remain valid in the unsteady case. As an additional simplifying assumption, the ratio $\epsilon = A_o/U$ of the magnitude of the orbital wave velocity to the mean free stream velocity is taken to be small so that the x-component

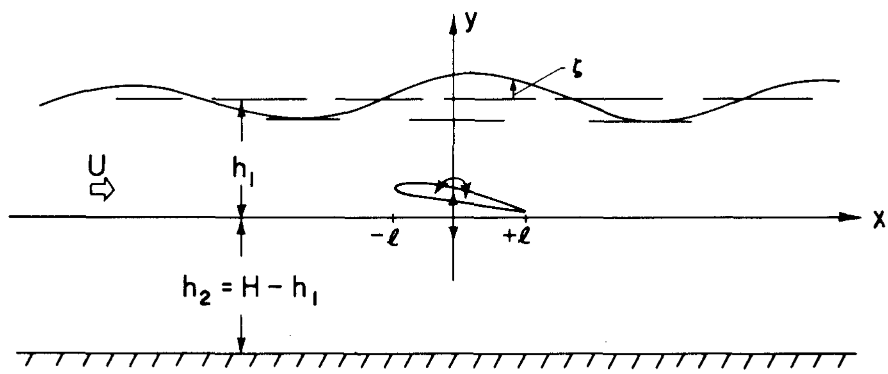

FIGURE 1

orbital velocity, u_o, may be neglected in comparison with U in formulating the present linear theory. Although the simple water wave is chosen as a concrete example, it makes little difference to the subsequent discussion if other kinds of wavy streams are considered as long as the transverse velocity of the basic flow can be represented by equation (5). For arbitrary V(x, t) the result can be obtained by the Fourier synthesis of this fundamental case.

The hydrofoil (or airfoil) is assumed to be thin, though sufficiently rounded at the leading edge to keep the flow from being separated there. The foil-thickness effect is then only secondary and will be further disregarded in this study. For brevity, the semi-chord, ℓ, of the hydrofoil will be normalized to unity as the reference length. The unsteady motion of the hydrofoil assumes the fundamental form

$$y = h(x,t) = \hat{h}(x)\, e^{i\omega t} \qquad (-1 < x < 1) , \qquad (6)$$

where the circular frequency ω is arbitrary, and $\hat{h}$ may be a complex function of x (with respect to $i = \sqrt{-1}$ in the time factor). With the resultant flow velocity denoted by $(U + u_o + u_1,\ v_o + v_1)$, the linearized boundary condition that the flow be always tangential to the moving body surface requires

$$v_1^{\pm}(x,t) \equiv V_1(x,t) = V(x,t) - V_o(x,t) \qquad (|x| < 1) \qquad (7a)$$

$$V(x,t) = D\, h(x,t) , \qquad D \equiv \partial/\partial t + U\, \partial/\partial x , \qquad (7b)$$

where $V_o(x,t)$ is given by equation (5), and $v_1^{\pm}(x,t)$ signifies $v_1(x, 0\pm, t)$. Like in the uniform stream case, it is convenient to use the acceleration potential, defined by

$$\phi = (p_\infty - p)/\rho = \phi_0 + \phi_1 , \qquad (8)$$

as a new dependent variable, particularly since it is continuous throughout the fluid. It is related to the velocity by

$$Du_o = \partial\phi_o/\partial x , \quad Dv_o = \partial\phi_o/\partial y - g ; \quad Du_1 = \partial\phi_1/\partial x , \quad Dv_1 = \partial\phi_1/\partial y, \quad (9)$$

on linearized theory. The component ϕ_o, which gives the pressure distribution in the primary wave field, can be readily obtained by integration of the first two equations in (9); it gives no hydrodynamic force (except a bouyancy) or moment on the hydrofoil since it is continuous across the plate. The effect of waving stream on the hydrodynamic performance comes with calculations on ϕ_1, explicitly through the term $V_o(x,t)$ in condition (7). The problem of ϕ_1 is specified by the boundary conditions

$$\partial\phi_1^{\pm}/\partial y = D\, V_1(x,t) \qquad (|x| < 1,\ y = 0\pm), \tag{10a}$$

$$\phi_1^{\pm} = 0 \qquad (|x| > 1,\ y = 0\pm), \tag{10b}$$

together with the Kutta condition that $\phi_1^{\pm} = 0$ at $x = 1$, and that ϕ_1 vanishes at infinity. Condition (10a) follows from substituting (7a) into the last equation of (9); and (10b) is a consequence of the pressure being continuous in the flow and the fact that ϕ_1 is odd in y.

The solution to this mixed-type boundary problem of ϕ_1 is known (see Wu, 1971a); in particular, the value of $\phi_1(x, 0\pm, t)$ at the plate is given by

$$\phi_1^{\pm}(x,t) = \pm\frac{U}{2}a_o\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}} \pm \frac{1}{\pi}\int_{-1}^{1}\left(\frac{1-x^2}{1-\xi^2}\right)^{\frac{1}{2}}\frac{\psi_1(\xi,t)}{\xi - x}\,d\xi \tag{11}$$

$$\psi_1(x,t) = -D\int_{-1}^{x} V_1(\xi,t)\,d\xi\ , \tag{12}$$

$$a_o = [\, b_1 - (b_o + b_1)\,\Theta(\sigma)] - [\, b_1' - (b_o' + b_1')\Theta(\sigma_o)]\ , \tag{13}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} V(\cos\theta, t)\cos n\theta\, d\theta \qquad (x = \cos\theta,\ \ n = 0, 1, 2, \ldots)\ , \tag{14}$$

$$b_n' = \frac{2}{\pi}\int_0^{\pi} V_o(\cos\theta, t)\cos n\theta\, d\theta = 2A_o(-i)^{n-1} J_n(k)\, e^{i\omega_o t}\ , \tag{15}$$

$$\Theta(\sigma) = \mathfrak{F}(\sigma) + i\,\mathfrak{G}(\sigma)\ , \qquad \sigma \equiv \omega\ell/U\ , \qquad \sigma_o \equiv \omega_o\ell/U\ , \tag{16}$$

$$\sigma_o = (\delta\kappa)^{\frac{1}{2}} \pm \kappa\ , \qquad \delta \equiv (g\ell/U^2)\tanh kH\ , \qquad \kappa \equiv k\ell\ . \tag{17}$$

Here the integral in equation (11) assumes its Cauchy principal value; $J_n(k)$ is the Bessel function of the first kind; $\Theta(\sigma)$ is the Theodorsen function, $\mathfrak{F}$ and $\mathfrak{G}$ being its real and imaginary parts (for a tabulation of Θ, see Luke and Dengler, 1951); σ is the reduced frequency of the body motion, σ_o the wave reduced frequency, both being based on the half-chord ℓ. The function $\sigma_o(\kappa)$ in equation (17) is the non-dimensional form of equation (3). We shall write κ as k since $\ell = 1$.

The differential lift distribution along the chord is clearly

$$\mathcal{L}(x,t) = p^-(x,t) - p^+(x,t) = 2\rho\phi_1^{+}(x,t) \qquad (|x| < 1)\ . \tag{18}$$

The integral representation of the lift L and the moment M (about the mid-chord, positive in the nose-up sense) are

$$L = \int_{-1}^{1}\mathcal{L}(x,t)\,dx\ , \tag{19}$$

$$M = -\int_{-1}^{1} \mathcal{L}(x,t)\, x\, dx \quad . \tag{20}$$

Also, the formulas for calculating the thrust, T, the power required for maintaining the motion, P, and the kinetic energy imparted to the fluid, E, remain the same as in the uniform stream case (Wu, 1971a),

$$T = \int_{-1}^{1} \mathcal{L}(x,t)\, h_x\, dx + S \quad , \qquad S = \tfrac{1}{2}\, \pi\rho(\mathrm{Re}\ a_o)^2 \quad , \tag{21}$$

$$P = -\int_{-1}^{1} \mathcal{L}(x,t)\, h_t\, dx \quad , \tag{22}$$

$$E = -\int_{-1}^{1} \mathcal{L}(x,t)(h_t + Uh_x)\, dx - SU \quad , \tag{23}$$

where the subscripts designate partial differentiation, S represents the leading-edge suction, which now includes the contribution from the wave component. From the above expressions we note that the energy balance can be expressed as

$$P = TU + E \quad , \tag{24}$$

which is formally the same as in the uniform stream case. However, unlike the case of uniform free stream, the time average of E here is no longer always positive, and we shall see that energy can be extracted from the waves when E becomes negative.

3. Flexible Plate Wing

We shall consider the general case when the motion is periodic in time, as prescribed by (6), with arbitrary amplitude function $\hat{h}(\cdot)$, such as any one that can be performed by a completely flexible plate. Substituting the differential lift $\mathcal{L}(x,t)$ given by (11) - (18) in (19) and (20), we obtain the total lift L, and the moment M, as

$$L = \pi\rho U \{a_o - (b_1 - b_1') - \tfrac{1}{2} i\sigma(b_o - b_2) + \tfrac{1}{2} i\sigma_o(b_o' - b_2')\} \quad , \tag{25}$$

$$M = \tfrac{1}{2}\pi\rho U \{a_o + (b_2 - b_2') + \tfrac{1}{4} i\sigma(b_1 - b_3) - \tfrac{1}{4} i\sigma_o(b_1' - b_3')\} \quad . \tag{26}$$

The corresponding results for the thrust T, power P, and energy loss E can be obtained by substituting (11) - (18) in (21) - (23) and by following the same procedure as that used for the uniform stream case by Wu (1971a, in deriving his equations 44 - 46); the intermediate manipulation again can be considerably simplified by making use of the relationship

$$\int_{-1}^{1} f'(x)\,dx \,⨍_{-1}^{1} \left(\frac{1-x^2}{1-\xi^2}\right)^{\frac{1}{2}} \frac{g(\xi)\,d\xi}{\xi - x} = \int_{-1}^{1} g'(x)\,dx \,⨍_{-1}^{1} \left(\frac{1-x^2}{1-\xi^2}\right)^{\frac{1}{2}} \frac{f(\xi)\,d\xi}{\xi - x} , \qquad (27)$$

where f(x), g(x) are two arbitrary functions, provided they and their derivatives f'(x), g'(x) are continuous in $-1 \leqslant x \leqslant 1$. The mean values of thrust $\overline{T}$, power $\overline{P}$, and energy loss $\overline{E}$ can be deduced by averaging T, P, and E over a long time period. Two different cases arise according as $\omega = \omega_o$ or $\omega \neq \omega_o$.

(i) When $\omega = \omega_o$, that is when the wing oscillates at the wave encounter frequency, the two motions are correlated. In this case we obtain $\overline{T}$ as

$$\overline{T} = \frac{\pi}{4}\rho \,\mathrm{Re}\{(a_o + b_o - \dot{\beta}_o)(a_o^* - b_1^* + \dot{\beta}_1^*) - b_o'\dot{\beta}_1^* - b_1'\dot{\beta}_o^* + \dot{\beta}_o\dot{\beta}_1^* + 2I\} , \qquad (28a)$$

$$I = \frac{2}{\pi^2}\int_{-1}^{1} ⨍_{-1}^{1} \left(\frac{1-x^2}{1-\xi^2}\right)^{\frac{1}{2}} \frac{V(x,t)\,V_o^*(\xi,t)}{\xi - x}\,dx\,d\xi , \qquad (28b)$$

where the superscript * denotes the complex conjugate, $\dot{\beta}_n \equiv d\beta_n(t)/dt$, and

$$\beta_n(t) = \frac{2}{\pi}\int_0^{\pi} h(x,t)\cos n\theta \, d\theta \qquad (x = \cos\theta, \;\; n = 0, 1, 2, \ldots) , \qquad (28c)$$

hence $\dot{\beta}_n = i\omega\beta_n$ when h is given by (6). The mean power $\overline{P}$ can be shown, after some manipulation, to have the following expression

$$\overline{P} = \frac{\pi}{4}\rho U \,\mathrm{Re}\{(a_o + b_o - b_o')\dot{\beta}_1^* + (b_1 - b_1' - a_o)\dot{\beta}_o^* + 2I_1\} , \qquad (29a)$$

$$I_1 = -\frac{2}{U\pi^2}\int_{-1}^{1} ⨍_{-1}^{1} \left(\frac{1-x^2}{1-\xi^2}\right)^{\frac{1}{2}} \frac{\partial V_o^*(x,t)/\partial t}{\xi - x} \left(\int_{-1}^{\xi} V(\eta,t)\,d\eta\right) dx\,d\xi . \qquad (29b)$$

By substituting in (29b) the relationship

$$\partial V_o(x,t)/\partial t = -(\omega/k)\,\partial V_o/\partial x ,$$

(see (5), with $\omega = \omega_o$ for the present case), and applying the formula (27), it immediately follows that

$$I_1 = (\sigma/k)\,I , \qquad (29c)$$

where I is given by (28b). Whence, by (24), $\overline{E} = \overline{P} - U\overline{T}$, or

$$\overline{E} = \frac{\pi}{4}\rho U \,\mathrm{Re}\{(a_o + b_o)(b_1^* - a_o^*) + 2(\sigma/k - 1)I\} . \qquad (30)$$

We note that, upon substituting (13) in (30), the first term on the right-hand side of (30) involves only the first two Fourier coefficients of V, in the particular combination of $(b_o + b_1)$. However, the second term in (30) with I, which results from the interaction between the wave action and body motion, involves all the Fourier coefficients of V since the integral I has the following Fourier-Bessel expansion

$$I = A_o e^{-i\omega t} \sum_{n=1}^{\infty} (i)^{n+1} J_n(k) (b_{n+1} - b_{n-1}) . \tag{31}$$

We further note that the expressions for $\overline{T}$ and $\overline{P}$ involve, in addition to the b_n's, also the first two Fourier coefficients, β_o and β_1, of h(x, t).

To facilitate the subsequent consideration of the optimum shape problem, it is useful to recast the above expressions for $\overline{T}$, $\overline{P}$, and $\overline{E}$ in terms of certain inner products. Let $\mathcal{H}$ denote a subset of the complex Hilbert space $L_2[-1,1]$

$$\mathcal{H} \equiv \left\{ f \in L_2[-1,1] \ : \ \frac{2}{\pi} \int_{-1}^{1} |f(x)|^2 (1 - x^2)^{-\frac{1}{2}} dx < \infty \right\} \tag{32a}$$

and let the inner product between f(•) and g(•) on $\mathcal{H}$ be defined by

$$< f, g > \equiv \frac{2}{\pi} \int_{-1}^{1} f(x) g^*(x) (1 - x^2)^{-\frac{1}{2}} dx = < g, f >^* , \quad (f, g \in \mathcal{H}) \tag{32b}$$

where the weighting function $(1 - x^2)^{-\frac{1}{2}}$ is introduced in order to convert the Fourier coefficients into the inner product form. Any two functions f, g in $\mathcal{H}$ will be said to be orthogonal on $\mathcal{H}$ if $< f, g > = 0$.

Substituting (13) - (16) and (28c) in (29) - (30), we obtain the mean coefficients of thrust, power, and energy loss, defined by $(C_P, C_E, C_T) = (\overline{P}, \overline{E}, \overline{T}U)/(\frac{1}{4}\pi\rho U^3 \ell)$, in terms of the inner products as

$$C_P = \text{Re} \{ -i\sigma[< v, f_1 > - 2\epsilon (J_1 + iJ_o)] < g_1, \hat{h} > + 2\epsilon (\sigma/k) < v, g_2 > \} , \tag{33}$$

$$C_E = \text{Re} \{ B(\sigma)|< v, f_1 >|^2 + 2\epsilon (1 - 2\Theta)(W_1 + iW_2) < v, f_1 > + 2\epsilon (\sigma/k - 1) < v, g_2 > - 4\epsilon^2 W^2 \} , \tag{34}$$

where

$$v(x) = e^{-i\omega t} V(x, t)/U , \quad \hat{h}(x) = e^{-i\omega t} h(x, t) , \tag{35a}$$

$$f_1(x) = 1 + x , \quad g_1(x) = (1 - \Theta) x + \Theta , \quad \epsilon = A_o/U , \tag{35b}$$

$$\Theta(\sigma) = \mathcal{F}(\sigma) + i \mathcal{G}(\sigma) , \quad B(\sigma) = \mathcal{F} - (\mathcal{F}^2 + \mathcal{G}^2) , \tag{35c}$$

$$W_1 - i W_2 = J_1(k)[1 - \Theta(\sigma)] - iJ_o(k)\Theta(\sigma) , \quad W^2 = W_1^2 + W_2^2 , \tag{35d}$$

$$g_2(x) = \frac{i}{\pi}(1 - x^2) \oint_{-1}^{1} \frac{e^{-ik\xi}\, d\xi}{(1-\xi^2)^{\frac{1}{2}}(\xi - x)} \quad . \tag{35e}$$

In the above, as well as in the sequel, the argument k of the Bessel functions $J_n(k)$ will be understood unless otherwise designated. The mean thrust coefficient is simply (the coefficient form of (24))

$$C_T = C_P - C_E \quad . \tag{36}$$

Another flow quantity of interest is the mean leading-edge-suction coefficient, $C_S = \overline{S}/\frac{1}{4}\pi\rho U^2\ell$. From (21), (13), (14) we obtain

$$C_S = |\langle v, f_1 \rangle \Theta - \langle v, f_o \rangle + 2\epsilon\,(W_1 - iW_2)|^2 \quad , \tag{37a}$$

where $\qquad f_o(x) = x \qquad (-1 \leqslant x \leqslant 1) \quad .$

As suggested by Lighthill (1969, 1970), the ratio C_S/C_T provides a measure of the relative strength of the leading-edge suction; moderate and large values of C_S/C_T (as compared to unity) suggest a tendency that the flow would separate, or stall, near the leading edge (such a category of separated flow would be quite different from the completely wetted flow as assumed here).

(ii) $\omega \neq \omega_o$ --- In this case the mean product of $\exp(i\omega t)$ and $\exp(\pm i\omega_o t)$ vanishes as the body motion and wave action become uncorrelated. Consequently the terms which are linear in ϵ in (33) and (34) drop out of the expressions for C_P and C_E; further, W^2 in (34) then assumes its value at σ_o. The corresponding C_S likewise becomes

$$C_S = |\langle v, f_1 \rangle \Theta - \langle v, f_o \rangle|^2 + 4\epsilon^2 W^2(\sigma_o, k) \quad . \tag{38}$$

The result of this case therefore reduces virtually to the case of uniform stream except for the additional term $(-4\epsilon^2 W^2)$ in the expression for C_E and $(4\epsilon^2 W^2)$ in C_S. These added terms indicate that energy is invariably being supplied by the primary wave, through the mechanism of generating a greater leading-edge suction, at no expense of C_P. It thus follows that for C_P fixed, C_T becomes greater and C_E smaller (hence higher efficiency) with increasing wave action (greater ϵW). The energy gain in this case, however, is always accompanied by an appreciable increase in the leading-edge suction, suggesting an easier leading-edge stall. When the suction is required to remain reasonably small, the optimum motion and the corresponding improvement of efficiency are not significantly different from the uniform stream case which has been discussed earlier by Wu

(1971b). For this reason this second case will not be further pursued here.

4. The Optimum Motion ($\omega = \omega_0$)

The present problem of optimum motion is formulated especially to analyze the interaction between the body motion and wave action; it can be stated as follows:

Given a reduced frequency $\sigma > 0$ (hence also the wave number k, see (3)) and a thrust coefficient $C_{T,0} > 0$, find a velocity profile v, or a hydrofoil profile $\hat{h}$ in the set $\mathcal{H}$ (defined by (32a)) such that C_E is minimized subject to the constraint

$$C_T = C_{T,0} > 0 \quad , \tag{39}$$

assuming that the wing oscillates at the wave encounter frequency.

It is desirable to choose C_T (rather than C_P or C_E) to be a constrained quantity since a constant thrust is required to overcome the (nearly constant) viscous drag if the uniform forward motion is to be maintained. No additional constraints are imposed here on the total lift L and moment M for balancing the rectilinear and angular recoils of the flexible plate (see Wu, 1971a, Eqs. (56a,b)); this choice is made for two reasons. First, when a body structure consists of components other than the flexible plate, the recoil consideration must take the motion of the entire body into account. Second, even when the wing alone comprises a self-propelling body in its entirety, there will still be other degrees of freedom left to be used to satisfy the recoil conditions, if desired, as we shall see later.

In choosing the independent functionals for the optimization calculation, we note that only two of C_P, C_E, C_T are independent since they are related by (36). There are great advantages in the choice of C_P and C_E as the independent functionals of v and $\hat{h}$ because C_E, in particular, does not involve $\hat{h}$, and C_P is also simpler in expression than C_T. In the expression (34) for C_E, the first term on the right-hand side is the same as in the uniform-stream case (see Wu, 1971b, Eq. (13)); it is always non-negative since $B(\sigma) > 0$ for $\sigma > 0$. The second and third terms, which are bilinear in ϵ and v, represent the body-wave interaction. The last term, which is proportional to ϵ^2, is solely due to the wave action. This result actually proves the statement that extraction of energy from the surrounding flow by an oscillating flexible wing is impossible if the incident flow is uniform. In the presence of a primary wave, with appropriate v and increasing wave parameter ϵ, the last three terms in (34) may become negative and numerically so large as to reduce C_E at first, and C_P eventually, to negative values, as will be seen later. The case of $C_P < 0$ signifies the operation in which a mechanical power is received by the body, instead of being consumed by it, through a favorable extraction of the wave energy. In spite of these possibilities, we shall still continue to use the Froude efficiency

$$\eta = C_T/C_P = C_{T,0}/C_P = (1 + C_E/C_{T,0})^{-1} \tag{40a}$$

as a measure of the hydromechanical performance. Aside from its usual significance for $0 < \eta < 1$, now we may have new generalized interpretations as follows:

$$\text{(i)}\ \eta > 1 \quad \text{for} \quad C_E < 0,\ C_P > 0; \tag{40b}$$

$$\text{(ii)}\ \eta < 0 \quad \text{for} \quad C_E < C_P < 0. \tag{40c}$$

Another step of primary importance is to choose the independent function for the optimization calculation. Although either v or h may serve as an independent function (since they are related by a differential equation (7b)), the advantage of taking v is clear, as was noted by Wu (1971b, section 6) in discussing the optimum shape of a flexible plate oscillating in a uniform stream. As another reason, we note that in the present formulation, an inner product of h with a given $f(\cdot)$ can be converted into an equivalent one involving v, whereas the converse is generally impossible.

Accepting v as the independent function, we proceed to recast the inner product $< g_1, \hat{h} >$ in (33) in terms of v. By (35a) and (7b), $\hat{h}$ and v are related by

$$(d/dx + i\sigma)\, \hat{h}(x) = v(x) \qquad (|x| < 1)\ , \tag{41a}$$

which has the general integral as

$$\hat{h}(x) = \int_{-1}^{x} e^{-i\sigma(x-\xi)}\, v(\xi)\, d\xi + \hat{h}_{-1} e^{-i\sigma(x+1)}\ , \tag{41b}$$

where $\hat{h}_{-1}$ is an arbitrary integration constant. Substituting (41b) and (35b) in $< g_1, \hat{h} >$, and integrating by parts, we obtain

$$< g_1, \hat{h} > = < g_3, v > + C_1 - iC_2\ , \tag{42a}$$

where
$$g_3(x) = (1 - x^2)^{\frac{1}{2}} \int_{x}^{1} e^{-i\sigma(x-\xi)}\, (1 - \xi^2)^{-\frac{1}{2}}\, g_1(\xi)\, d\xi\ , \tag{42b}$$

$$C_1 - iC_2 = 2i\hat{h}^{*}_{-1}\, e^{i\sigma} [\, J_1(\sigma)(1 - \Theta) - i\Theta J_o(\sigma)]\ . \tag{42c}$$

Consequently (33) becomes

$$C_P = \mathrm{Re}\, \{ -i\sigma[< v, f_1 > - 2\epsilon\, (J_1 + iJ_o)] \,[< g_3, v > + C_1 - iC_2] + 2\epsilon\, (\sigma/k) < v, g_2 > \}\ . \tag{33'}$$

Now the expression for C_P in (33)' and C_E in (34) are both expressed in terms of v and contain only three inner products: $<v, f_1>$, $<v, g_2>$, and $<g_3, v>$.

Since f_1, g_2, g_3 are not mutually orthogonal on $\mathcal{H}$, we next construct a set of three orthogonal functions, f_1, f_2, f_3 say (there being no need here to normalize them), by the Schmidt scheme:

$$f_1 = 1 + x \qquad [<f_1, f_1> = 3] \tag{43a}$$

$$g_2 = a_1 f_1 + f_2 \ , \tag{43b}$$

$$g_3 = a_2 f_1 + a_3 f_2 + f_3 \ , \tag{43c}$$

such that

$$<f_i, f_k> = 0 \qquad (i \neq k) \ . \tag{44}$$

The coefficients a_n are determined by the orthogonality condition (44) as

$$a_1 = <g_2, f_1>/<f_1, f_1> = \tfrac{1}{3} <g_2, f_1> = \tfrac{1}{3} [2J_1(k) - iJ_2(k)] \ , \tag{45a}$$

$$a_2 = <g_3, f_1>/<f_1, f_1> = \frac{2}{3\sigma^2} \{ \Theta[1 + i\sigma - e^{i\sigma} J_o(\sigma)] + i(1 - \Theta)[\tfrac{\sigma}{2} - e^{i\sigma} J_1(\sigma)] \}, \tag{45b}$$

$$a_3 <f_2, f_2> = <g_3, f_2> = <g_3, g_2> - a_1^* <g_3, f_1> = <g_3, g_2> - 3a_1^* a_2 \ . \tag{45c}$$

By separate calculations,

$$<f_2, f_2> = <g_2, g_2> - a_1 <f_1, g_2> - a_1^* <g_2, f_1> + a_1 a_1^* <f_1, f_1> = <g_2, g_2> - 3a_1 a_1^* \ ,$$

$$<g_2, g_2> = \frac{2}{\pi^3} \int_{-1}^{1} (1 - x^2)^{3/2} dx \int_{-1}^{1} \frac{e^{-ik\xi} d\xi}{(1-\xi^2)^{\frac{1}{2}} (\xi - x)} \int_{-1}^{1} \frac{e^{ik\eta} d\eta}{(1-\eta^2)^{\frac{1}{2}} (\eta - x)}$$

$$= 1 - J_o^2(k) + 2J_1^2(k) - 2J_o(k) J_2(k) \ ,$$

which can be shown by successive interchange of the order of integration and by making use of the Poincaré-Bertrand formula, and hence

$$<f_2, f_2> = 1 - J_o^2(k) + \tfrac{2}{3} J_1^2(k) - 2J_o(k) J_2(k) - \tfrac{1}{3} J_2^2(k) \ . \tag{45d}$$

Finally,

$$(g_3, g_2) = - \frac{2i}{\pi^2} \int_{-1}^{1} (1 - x^2)\, dx \int_{x}^{1} e^{-i\sigma(x-\eta)} \frac{g_1(\eta)\, d\eta}{(1-\eta^2)^{\frac{1}{2}}} \int_{-1}^{1} \frac{e^{ik\xi} d\xi}{(1-\xi^2)^{\frac{1}{2}} (\xi - x)}$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m} [N_m(\sigma) - (i)^m N_o(\sigma)] J_n(k) \{ (-1)^n [J_{m-n-2}(\sigma) + 2J_{m-n}(\sigma)$$

$$+ J_{m-n+2}(\sigma)] - [J_{m+n-2}(\sigma) + 2J_{m+n}(\sigma) + J_{m+n+2}(\sigma)] \} \ , \tag{45e}$$

where

$$N_n(\sigma) = -[1-\Theta(\sigma)]\,J_n'(\sigma) - i\Theta(\sigma)J_n(\sigma) \qquad (n=0,1,2,\ldots\ldots). \qquad (45f)$$

The above result can be shown by using the series expansion of g_2 and g_3 as

$$g_2(x) = 2i\sin\theta\sum_{n=1}^{\infty}(-i)^n J_n(k)\sin n\theta \qquad (x=\cos\theta)\ ,$$

$$g_3(x) = 2ie^{-i\sigma\cos\theta}\sin\theta\sum_{n=1}^{\infty}(i)^n[\,N_n(\sigma)-(i)^n N_o(\sigma)]\ \frac{\sin n\theta}{n} \qquad (x=\cos\theta)\ .$$

This completes the determination of a_3, hence also the orthogonalization.

It is now evident that v can be expressed as

$$v(x) = \sum_{n=1}^{3} B_n f_n(x) + v_\perp(x)\ , \qquad (-1\leqslant x\leqslant 1)\ , \qquad (46)$$

where B_n's are complex coefficients and $v_\perp$ is any function belonging to the orthogonal complement of the subspace spanned by $\{f_1, f_2, f_3\}$, that is, $\langle f_n, v_\perp\rangle = 0$ for $n=1,2,3$. For convenience of the subsequent computations, we introduce the real parameters ξ_n's by

$$\langle v, f_n\rangle = B_n\langle f_n, f_n\rangle \equiv \xi_{2n-1} + i\xi_{2n} \qquad (n=1,\ 2)\ , \qquad (47a)$$

$$\langle v, f_3\rangle = B_3\langle f_3, f_3\rangle \equiv \xi_5 + i\xi_6 - (C_1 + iC_2)\ , \qquad (47b)$$

where $C_1 + iC_2$ is given by (42c). From (43), (46) and (47), we have

$$\langle v, g_2\rangle = a_1^*\langle v, f_1\rangle + \langle v, f_2\rangle = a_1^*(\xi_1 + i\xi_2) + (\xi_3 + i\xi_4)\ , \qquad (48a)$$

$$\langle v, g_3\rangle = a_2^*(\xi_1 + i\xi_2) + a_3^*(\xi_3 + i\xi_4) + (\xi_5 + i\xi_6) - (C_1 + iC_2)\ . \qquad (48b)$$

Substitution of (47), (48) in (33)' and (34) yields

$$C_P = \sigma\{A_2(\sigma)(\xi_1^2 + \xi_2^2) + A_3(\xi_2\xi_3 - \xi_1\xi_4) + A_4(\xi_1\xi_3 + \xi_2\xi_4) + (\xi_2\xi_5 - \xi_1\xi_6) + 2\epsilon[\sum_{j=1}^{4} P_j\xi_j - J_o(k)\xi_5 + J_1(k)\xi_6]\}\ , \qquad (49)$$

$$C_E = B(\sigma)(\xi_1^2 + \xi_2^2) + 2\epsilon(Q_1\xi_1 + Q_2\xi_2 + Q_3\xi_3) - 4\epsilon^2 W^2\ , \qquad (50)$$

where

$$a_2 = A_1(\sigma) + iA_2(\sigma)\ , \qquad a_3 = A_3(\sigma, k) + iA_4(\sigma, k)\ , \qquad (51a)$$

$$P_1 = -A_1J_o(k) - A_2J_1(k) + \frac{2}{3k}J_1(k)\ , \quad P_2 = A_1J_1(k) - A_2J_o(k) - J_2(k)/3k\ , \qquad (51b)$$

$$P_3 = -A_3 J_o(k) - A_4 J_1(k) + 1/k \;, \qquad P_4 = A_3 J_1(k) - A_4 J_o(k) \;, \tag{51c}$$

$$Q_1 = (1 - 2\mathfrak{F})\, W_1 + 2\mathfrak{G} W_2 + \tfrac{2}{3}\left(\frac{\sigma}{k} - 1\right) J_1 \;, \tag{51d}$$

$$Q_2 = 2\mathfrak{G} W_1 - (1 - 2\mathfrak{F})\, W_2 - \tfrac{1}{3}\left(\frac{\sigma}{k} - 1\right) J_2 \;, \qquad Q_3 = (\sigma/k) - 1 \;, \tag{51e}$$

The other coefficients appeared here have been given in (35), (45).

Equations (49), (50) show that C_P depends on only six real parameters $\{\xi_1, \xi_2, \ldots \xi_6\}$, and C_E depends on only three parameters $\{\xi_1, \xi_2, \xi_3\}$, while both C_P and C_E, hence also C_T, are independent of $v_\perp(x)$. (Note that the orthogonal complement of the subspace spanned by $\{f_1, f_2, f_3\}$ is infinite dimensional.) Thus, it is clear that the optimization problem posed earlier now reduces to one defined on a finite-dimensional vector space.

Before we proceed with our discussion from this approach, further simplification of the expressions for C_P and C_E can be gained if we first eliminate the terms linear in ξ_1 and ξ_2 in (49), (50) and then reduce the number of quadratic terms in (49) by the following transformation

$$\zeta_1 + i\zeta_2 = \frac{1}{A}(A_4 - iA_3)\left[\xi_1 + i\xi_2 + \frac{\epsilon}{B}(Q_1 + iQ_2)\right] \;, \quad A = (A_3^2 + A_4^2)^{\frac{1}{2}} \;, \tag{52a}$$

$$\zeta_3 + i\zeta_4 = (\xi_3 + i\xi_4) + (A_3 + iA_4)(\zeta_5 + i\zeta_6)/A^2 \;, \tag{52b}$$

$$\zeta_5 + i\zeta_6 = (\xi_5 + i\xi_6) + (C_5 + iC_6) \;, \tag{52c}$$

where

$$C_5 = 2\epsilon\,(P_2 - A_2 Q_2/B) \;, \qquad C_6 = -2\epsilon\,(P_1 - A_2 Q_1/B) \,. \tag{52d}$$

Then (49) and (50) reduce to

$$C_P/\sigma = A_2(\zeta_1^2 + \zeta_2^2) + A(\zeta_1\zeta_3 + \zeta_2\zeta_4) + \epsilon\,(A_5\zeta_3 + A_6\zeta_4) + \epsilon\, A_o \tag{53}$$

$$C_E = B(\zeta_1^2 + \zeta_2^2) + 2\epsilon\, Q_3\zeta_3 - \epsilon\, Q_o \;, \tag{54}$$

where

$$A_5 = 2P_3 - (A_3Q_2 + A_4Q_1)/B \;, \qquad A_6 = 2P_4 - (A_4Q_2 - A_3Q_1)/B \;, \tag{55a}$$

$$\begin{aligned} A_o = {} & -2\left[J_o + (P_3A_3 + P_4A_4)/A^2\right]\zeta_5 + 2\left[J_1 + (P_3A_4 - P_4A_3)/A^2\right]\zeta_6 \\ & + 4\epsilon\left[J_oP_2 + J_1P_1 - A_2(J_oQ_2 + J_1Q_1)/B\right] - \epsilon\, A_2(Q_1^2 + Q_2^2)/B^2 \;, \end{aligned} \tag{55b}$$

$$Q_o = 2Q_3(A_3\zeta_5 - A_4\zeta_6)/A^2 + 4\epsilon\, W^2 + \epsilon\,(Q_1^2 + Q_2^2)/B \,. \tag{55c}$$

Thus in the above reduced form, C_P depends quadratically on $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$, C_E

depends quadratically on $\{\zeta_1, \zeta_2\}$ but is independent of ζ_4, while both C_P and C_E depend linearly on $\{\zeta_5, \zeta_6\}$. When the primary wave is absent (i.e. $\epsilon = 0$), equations (53) and (54), or equivalently (49) and (50), reduce to the case of a flexible plate in uniform flow treated earlier by Wu (1971b, see his equations (79) and (80), which involve also six independent parameters). The present result of C_P and C_E is also very similar to that for a flat plate oscillating in waves discussed by Wu (1972, see his equations (50), (51) for the four independent parameters proper to that problem). Like those simpler cases investigated previously, we note that in the three-dimensional Euclidean space $(\zeta_1, \zeta_2, \zeta_3)$ (i.e. with $\zeta_4, \zeta_5, \zeta_6$ held fixed), the C_E= const. surfaces are paraboloids of revolution with its generating axis lying along the ζ_3-axis, while C_P= const. surfaces are oblique hyperboloids, whose cross-sections with ζ_3= const. planes, if real, are circles.

The optimization problem posed earlier can now be reformulated as follows: Let R_6 denote the six-dimensional Euclidean space of ordered six-tuples $\vec{\zeta} \equiv (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6)$ of real numbers; and let Ω be a subset of R_6 defined by

$$\Omega \equiv \{ \vec{\zeta} \in R_6 : \quad C_T(\vec{\zeta}) = C_P(\vec{\zeta}) - C_E(\vec{\zeta}) = C_{T,o} > 0 \} . \tag{56}$$

The optimization problem is to find a vector $\vec{\zeta}^o \in \Omega$ such that $C_E(\vec{\zeta}^o) \leqslant C_E(\vec{\zeta})$ for all $\vec{\zeta} \in \Omega$.

From the known geometric properties of constant C_P and C_E surfaces, and hence also of $C_T = C_{T,o} > 0$ surface, it follows that Ω is an unbounded set in R_6. Consequently, it is possible that the optimization problem may not have a solution. It suffices to demonstrate two such cases. As the first, consider a sequence of points $\{\vec{\zeta}^k\}$ in the set $S_1 \subset \Omega$ defined by

$$S_1 = \{ \vec{\zeta} \in \Omega : \quad \sigma A \zeta_1 + \epsilon (\sigma A_5 - 2Q_3) = 0 \} , \tag{57}$$

such that $Q_3 \zeta_3^k \to -\infty$ as $k \to \infty$. It is readily shown that in the set S_1, C_T depends on $(\zeta_2, \zeta_4, \zeta_5, \zeta_6)$ while C_E depends on $(\zeta_2, \zeta_3, \zeta_5, \zeta_6)$; S_1 is therefore nonempty. But, since ζ_j^k's are all constant for j=2, 4, 5, 6 and for all k, we immediately see from (54) that the sequence of values $C_E(\vec{\zeta}^k) \to -\infty$, as $k \to \infty$, implying the non-existence of an optimal solution.

As the second example, consider another sequence of points $\{\vec{\zeta}^\ell\}$ in the set $S_2 \subset \Omega$ defined by

$$S_2 = \{ \vec{\zeta} \in \Omega : \quad \sigma A_o(\zeta_5, \zeta_6) + Q_o(\zeta_5, \zeta_6) = 0 \} , \tag{58}$$

such that $Q_o(\zeta_5^\ell, \zeta_6^\ell) \to \infty$ as $\ell \to \infty$. It is also easily seen that in the set S_2, $C_T = C_T(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$, $C_E = C_E(\zeta_1, \zeta_2, \zeta_3, \zeta_5)$ and consequently $C_E(\vec{\zeta}^\ell) \to -\infty$ while C_T remains unchanged as $\ell \to \infty$, implying again the nonexistence of an optimal solution.

To ensure the existence of an optimal solution that is physically meaningful, we shall minimize C_E over a subset $\widetilde{\Omega}$ of Ω which is closed and bounded, i.e.

$$\widetilde{\Omega} \equiv \{ \vec{\zeta} \in \widetilde{R}_6 \ : \quad C_T(\vec{\zeta}) = C_P(\vec{\zeta}) - C_E(\vec{\zeta}) = C_{T,o} > 0 \} , \tag{59a}$$

where $\widetilde{R}_6$ denote a bounded subset of R_6 such that

$$\sum_{n=1}^{6} \zeta_n^2 \leqslant M < \infty . \tag{59b}$$

The new optimization problem is to find a vector $\vec{\zeta}^{\,o} \in \widetilde{\Omega}$ such that $C_E(\vec{\zeta}^{\,o}) \leqslant C_E(\vec{\zeta})$ for all $\vec{\zeta} \in \widetilde{\Omega}$. Evidently, this optimization problem has a solution since C_E is continuous on the closed bounded set $\widetilde{\Omega}$.

In what follows, we shall consider the particular case where ζ_4, ζ_5 and ζ_6 are treated as free parameters so that the optimization problem reduces to a three-dimensional one. Moreover, the constant M in (59b) is adjusted so that the optimum solution can be determined from the points in $\widetilde{\Omega}$ at which (grad C_P) is proportional to (grad C_E). Thus, we set

$$\partial\,(C_P - \lambda' \sigma C_E) / \partial \zeta_j = 0 \ , \qquad j=1,2,3, \tag{60}$$

where λ' is a Lagrange multiplier, giving

$$\zeta_1 = \lambda A \zeta_3 \ , \tag{61a}$$

$$\zeta_2 = \lambda A \zeta_4 \ , \tag{61b}$$

$$\zeta_1 = (\epsilon / AB)(2A_2 Q_3 - A_5 B) + (\epsilon\, Q_3 / AB)\, \lambda^{-1} \ , \tag{61c}$$

where λ is related to λ' by $\lambda^{-1} = 2(B\lambda' - A_2)$. From the three equations (61a-c) we can determine the variables $(\zeta_1, \zeta_2, \zeta_3)$, which are subject to variation, in terms of $(\zeta_4, \zeta_5, \zeta_6, \lambda)$. Finally, the Lagrange multiplier λ can be determined in terms of ζ_4, ζ_5, ζ_6, $C_{T,o}$ and ϵ by invoking condition (39). This line of approach indicates that the extremal solution will involve $(\zeta_4, \zeta_5, \zeta_6, C_{T,o}, \epsilon)$ as free parameters. It is more desirable, however, to adopt

$$\zeta_o = (\zeta_3^2 + \zeta_4^2)^{\frac{1}{2}} \tag{62}$$

rather than ζ_4 as a free parameter since this replacement will facilitate computation as well as comparison with the earlier results for the uniform stream case (Wu, 1971b) and those for the rigid plate in waves (Wu, 1972). Thus, we first eliminate ζ_1, ζ_2 in (53), (54), and (61a-c), next we apply condition (39), giving

$$A^2 [\, T_2 \lambda^2 + \sigma \lambda] + \bar{\epsilon} [\, (\sigma A_5 - 2Q_3) z_3 + \sigma A_6 z_4] = \bar{C}_{T,o} - \bar{\epsilon}\, \Sigma_o \ , \tag{63}$$

$$z_3 = (\bar{\epsilon}/A^2B)(2A_2Q_3 - A_5B)\frac{1}{\lambda} + (\bar{\epsilon}Q_3/A^2B)\frac{1}{\lambda^2} \quad , \tag{64}$$

$$z_4 = \pm(1 - z_3^2)^{\frac{1}{2}} \quad , \tag{65}$$

where

$$T_2 = \sigma A_2 - B \quad , \qquad z_j = \zeta_j/\zeta_o \quad (j = 1, 2, \ldots 6) \quad , \tag{66a}$$

$$\bar{C}_{T,o} = C_{T,o}/\zeta_o^2 \quad , \quad \bar{\epsilon} = \epsilon/\zeta_o \quad , \quad \Sigma_o = (\sigma A_o + Q_o)/\zeta_o \quad . \tag{66b}$$

Equation (65) follows from the definition of z_3, z_4 and ζ_o as given by (62) and (66a), there being two branches of z_4 for given z_3, with $|z_3| \leqslant 1$.

The three equations (63)-(65) involve three unknowns, z_3, z_4, λ, and three parameters, namely $\bar{C}_{T,o}$ — the "proportional loading factor", $\bar{\epsilon}$ — the "proportional wave factor", and Σ_o — the "complementary mode factor" which includes the contribution from the mode $\zeta_5 + i\zeta_6$ and that from the waves.

As for the actual calculation of the Lagrange multiplier λ, we note that if equations (64), (65) are substituted for z_3, z_4 in equation (63), then, upon perfect squaring, there results an eighth degree algebraic equation for λ, of which the real solutions (appearing always in even numbers) are of interest. This equation seems to be too difficult for analytical solutions; resort was then made to numerical methods. The method which proved to be successful is as follows. Since the physically meaningful solutions also require z_3, z_4 to be both real, equation (65) suggests that z_3 can be used effectively for parametric computation of λ, with the obvious advantage of having a bounded range $-1 \leqslant z_3 \leqslant 1$. In this parametric form, both equations (63) and (64) are quadratic equations in λ, each giving two solutions of λ in closed form, from which the real solutions of λ satisfying both equations were determined by Newton's method, using z_3 as a parameter. The number of solutions depend on the values of σ, $\bar{C}_{T,o}$, $\bar{\epsilon}$ and Σ_o ; on occasions as many as eight real solutions were obtained, and there are cases in which two real solutions are very close to each other. In all the cases tried the two real solutions providing the highest and lowest efficiencies were taken as the desired optimal solutions.

The numerical results of η_{max}, as shown in Figs 2 - 5 for a few representative cases, exhibit the following salient features of the optimum solution. For $\bar{C}_{T,o}$ and $\bar{\epsilon}$ both as small as 10^{-3} and with $z_5 = z_6 = 0$, the maximum efficiency is already $\eta_{max} = 1.0$ for the reduced frequency $\sigma > 10^{-2}$. The corresponding results of the maximum efficiency for a rigid hydrofoil in a uniform stream ($\bar{\epsilon} = 0$) and in regular waves ($\bar{\epsilon} > 0$) are reproduced (see Wu 1971b, 1972) in Fig. 3 over a similar range of $\bar{C}_{T,o}$, although the $\bar{C}_{T,o}$ and $\bar{\epsilon}$ in those two cases are defined with reference to the heaving amplitude at mid-chord, whereas the definition of $\bar{C}_{T,o}$ and $\bar{\epsilon}$ in the present case (by referring to ζ_o, see (66), whose physical significance is not quite so simple) is slightly different. With this qualification, a comparison between Fig. 2 and 3 shows that η_{max} is further improved by the flexible over the rigid foil, in the frequency range of interest. When $\overline{C}_{T,o}$ is kept at 10^{-3} and $\overline{\epsilon}$ alone is increased (by having a stronger wave) to 10^{-2}, η_{max} becomes greater than 1, corresponding to the operation in which energy is extracted from the surrounding waves, but a power (somewhat smaller than before) is still required for maintaining the hydrofoil motion. When $\overline{\epsilon}$ is as large as 0.1, we see that η_{max} becomes negative, indicating that both energy and power are supplied by the exterior wave field. At this high level of $\overline{\epsilon}$, η_{max} becomes more negative as $\overline{C}_{T,o}$ is increased to 10^{-2}. This trend implies that more energy and power can be extracted from stronger waves at higher loadings. This is a quite remarkable feature since this trend is reversed from that at smaller $\overline{\epsilon}$ (or weaker waves, see the curves with $\overline{\epsilon} = 10^{-2}$).

To summarize the case of $z_5 = z_6 = 0$, we note that the overall features of η_{max} of the flexible and rigid hydrofoils are very similar, the difference, after the two definitions of $\overline{C}_{T,o}$ and $\overline{\epsilon}$ are properly reconciled, being rather small. Since the salient features of the optimum motion, including the variation of the leading-edge suction C_S, feathering of the hydrofoil to the trajectory, etc., have been thoroughly explored for the rigid plate case (see Wu, 1972), these features will not be further pursued here for the flexible plate. However, it must be stressed that the additional degrees of freedom provided by z_5 and z_6 for flexible plates can alter the η_{max} of a flexible plate. As indicated by Figs. 4 and 5 for two typical examples, η_{max} for the basic case of $z_5 = z_6 = 0$ can be increased by suitable choice of z_5 and z_6 (within the set $\widetilde{R}_6$ of (59)). The trend of the influence by z_5 and z_6 on the value of η_{max} can be seen clearly through the linear dependence of Σ_o on z_5 and z_6 (see (63) - (66)). Obviously, η_{max} will remain unchanged when the set of parameters ($\overline{C}_{T,o}$, $\overline{\epsilon}$, z_5, z_6) is replaced by ($\overline{C}'_{T,o}$, $\overline{\epsilon}$, 0, 0), where

$$\overline{C}'_{T,o} = \overline{C}_{T,o} - \overline{\epsilon}\,\Sigma_o(z_5, z_6) \quad . \tag{67}$$

This explains the opposite trend of η_{max} from its basic value at $z_5 = z_6 = 0$ when a set of nonvanishing z_5 and z_6 is reversed in sign. By making use of this property, or equivalently the simple formula (67), the utility and interpretation of Fig. 2 is thereby greatly extended.

Acknowledgments

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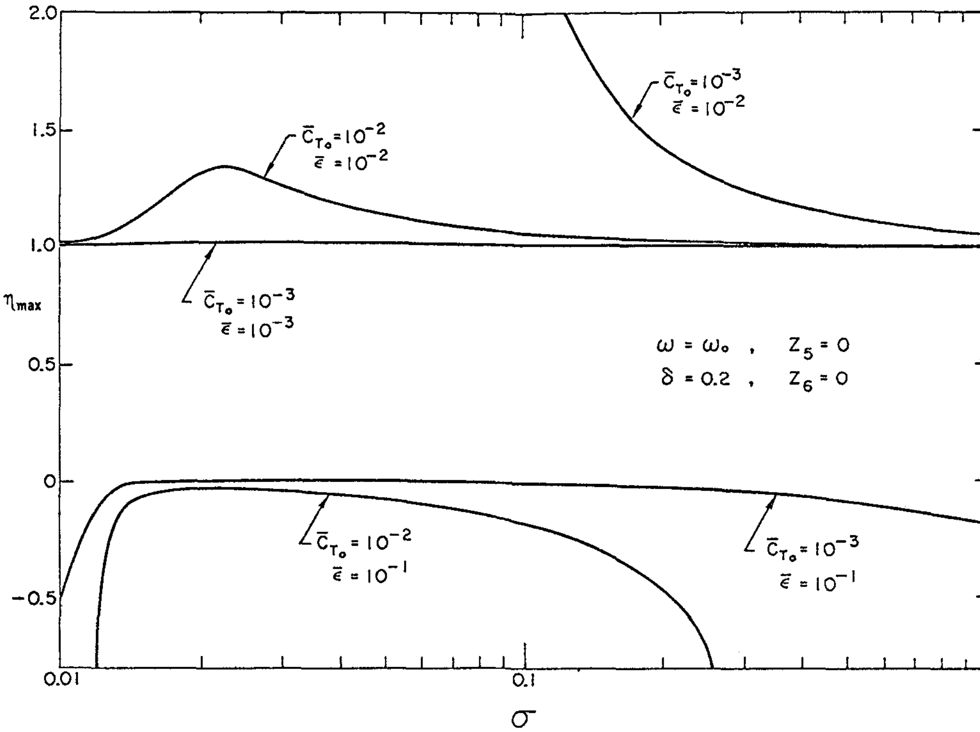
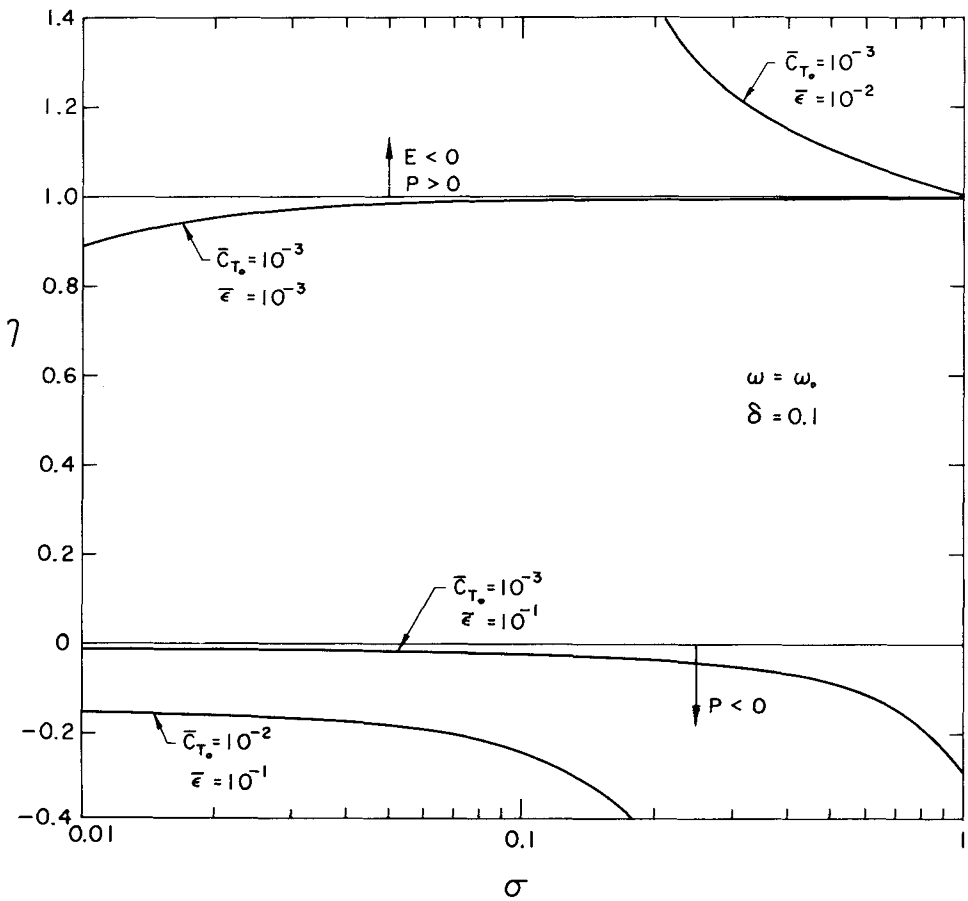

2.0
1.5
1.0
0.5
0
-0.5
η_{max}
0.01
0.1
1
σ
$\bar{C}_{T_o} = 10^{-3}$
$\bar{\epsilon} = 10^{-2}$
$\bar{C}_{T_o} = 10^{-2}$
$\bar{\epsilon} = 10^{-2}$
$\bar{C}_{T_o} = 10^{-3}$
$\bar{\epsilon} = 10^{-3}$
$\omega = \omega_o$, $Z_5 = 0$
$\delta = 0.2$, $Z_6 = 0$
$\bar{C}_{T_o} = 10^{-2}$
$\bar{\epsilon} = 10^{-1}$
$\bar{C}_{T_o} = 10^{-3}$
$\bar{\epsilon} = 10^{-1}$

FIGURE 2



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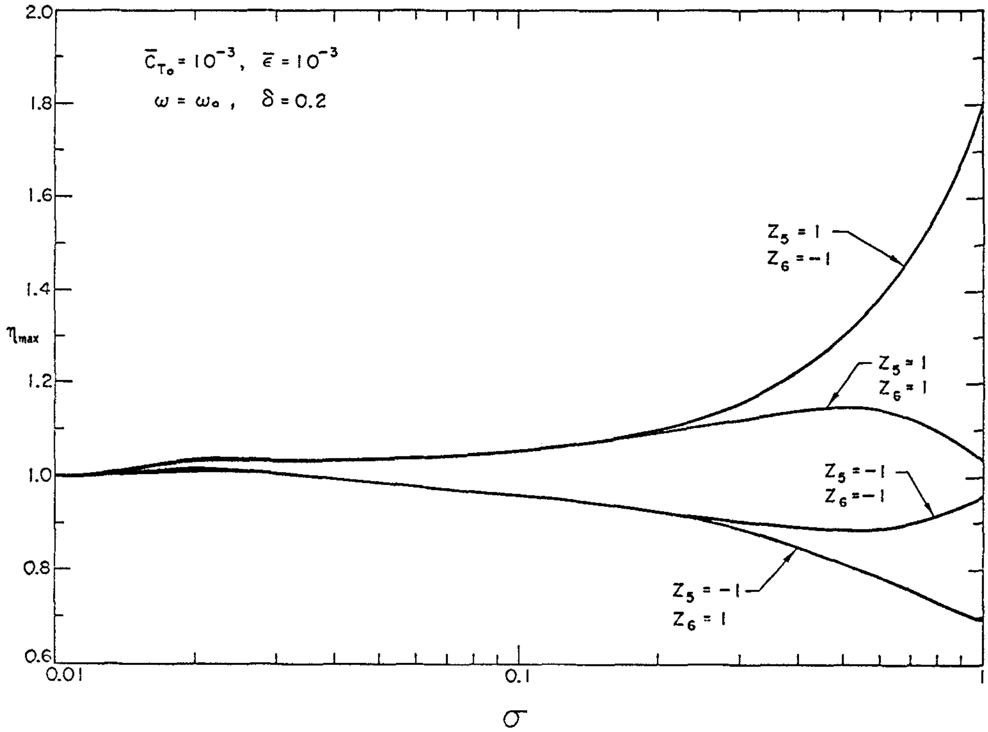
1.2
1.0
0.8
0.6
0.4
0.2
0
−0.2
−0.4
η
0.01
0.1
1
σ
$\bar{C}_{T_0} = 10^{-3}$
$\bar{\epsilon} = 10^{-2}$
E < 0
P > 0
$\bar{C}_{T_0} = 10^{-3}$
$\bar{\epsilon} = 10^{-3}$
$\omega = \omega_0$
$\delta = 0.1$
$\bar{C}_{T_0} = 10^{-3}$
$\bar{\epsilon} = 10^{-1}$
P < 0
$\bar{C}_{T_0} = 10^{-2}$
$\bar{\epsilon} = 10^{-1}$

FIGURE 3



$\bar{C}_{T_o} = 10^{-3}$, $\bar{\epsilon} = 10^{-3}$
$\omega = \omega_o$, $\delta = 0.2$
η_{max}
σ
$Z_5 = 1$
$Z_6 = -1$
$Z_5 = 1$
$Z_6 = 1$
$Z_5 = -1$
$Z_6 = -1$
$Z_5 = -1$
$Z_6 = 1$
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.01
0.1
1

FIGURE 4

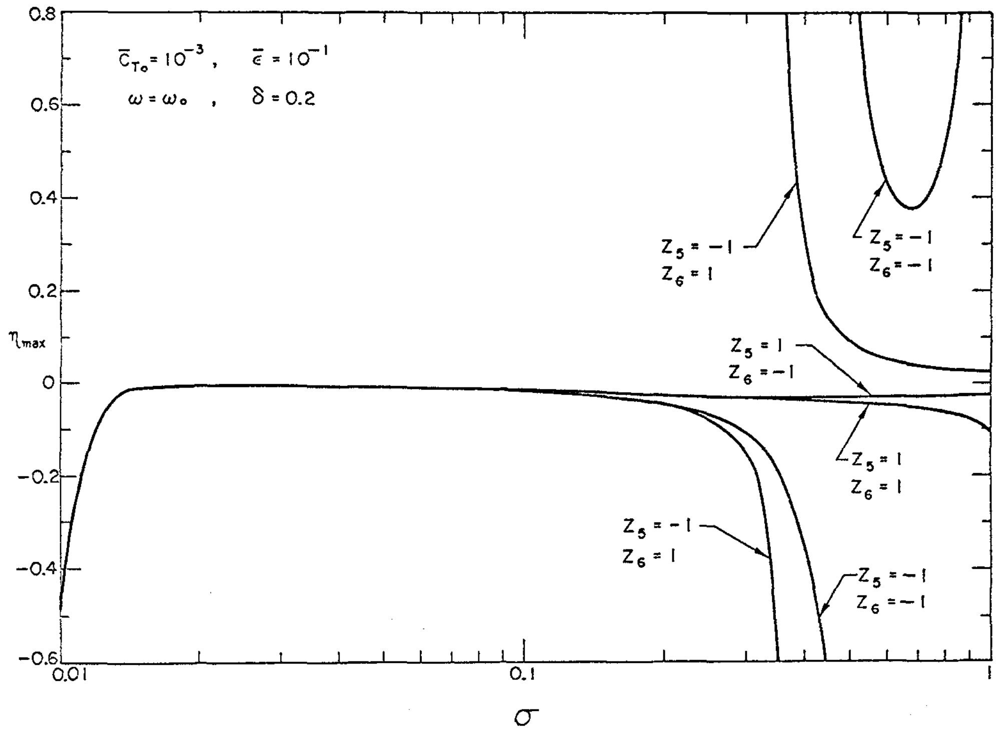

$\bar{C}_{T_0} = 10^{-3}$, $\bar{\epsilon} = 10^{-1}$
$\omega = \omega_0$, $\delta = 0.2$
η_{max}
σ
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0.01
0.1
1
$Z_5 = -1$ $Z_6 = 1$
$Z_5 = -1$ $Z_6 = -1$
$Z_5 = 1$ $Z_6 = -1$
$Z_5 = 1$ $Z_6 = 1$
$Z_5 = -1$ $Z_6 = 1$
$Z_5 = -1$ $Z_6 = -1$

FIGURE 5

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