STABILITY THEORY FOR GENERAL DYNAMICAL SYSTEMS AND SOME APPLICATIONS

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In this lecture, it will be attempted to describe some recent results in the stability of general dynamical systems from the viewpoint of Liapunov theory, and to illustrate and motivate them through examples.

The subject of stability theory is an extremely broad one. During the last ten years a large number of developments have taken place in this area; it could be said that, beside Liapunov theory and the classical theory for finite dimensional systems, the new branch of the functional analytic approach has sprung during this period.

This lecture is purposefully limited to a description of the Liapunov approach, and a rather limited one at that. To attempt more would surely lead to failure; furthermore, the author has worked much more extensively in this area than in the other ones. Fortunately, some recent expositions of a very readable nature are available [4, 13] to those interested in the functional analytic approach; furthermore, at the foundations of the functional analytic approach is the concept of a dynamical system, the central mathematical concept of this lecture.

Before embarking on the subject, the author feels obligated to apologize to the numerous workers in this area which he will neither reference nor acknowledge contributions from. This, it is hoped, is permissible, given the tutorial nature of the lecture. Interested readers will find the appropriate references in the few papers quoted.

The development of stability theory, from the time of Poincare and Liapunov, was considered a branch of ordinary differential equations and of mechanics; interest in the stability theory of partial differential equations and functional

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differential equations is rather more recent, especially in the engineering literature. The fundamental idea behind the concept of a dynamical system is to try to generalize to broader classes of evolutions the results known for ordinary differential equations.

1. Dynamical Systems and Some Examples

Let R^n denote an n-dimensional vector space with norm $|\cdot|$, R^+ denote the interval $[0,\infty)$ and $\mathscr{B}$ a Banach space with $\|\varphi\|$ the norm of an element in $\mathscr{B}$. Then [9, 11]

Definition 1.1. A dynamical system in a Banach space $\mathscr{B}$ is a function $u: R^+ \times \mathscr{B} \to \mathscr{B}$ such that

(i) u is continuous

(ii) $u(0,\varphi) = \varphi$

(iii) $u(t+\tau,\varphi) = u(t,u(\tau,\varphi))$ for every $t,\tau \geq 0$, every φ in $\mathscr{B}$.

Hence, a dynamical system has some continuity properties, the second condition states that at $t = 0$ the dynamical system is the identity map and, finally, that it has the semigroup property. It will be noted that the definition implies that a dynamical system is autonomous and that the map of the dynamical system is defined only forward in time. Except for this last restriction, it represents, at a slightly more abstract level, precisely the properties associated with the solutions of ordinary differential equations of autonomous type. Associated with the dynamical system we have

Definition 1.2. The positive orbit $0^+(\varphi)$ through φ is the set of elements in $\mathscr{B}$ defined by $0^+(\varphi) = \bigcup_{t\geq 0} u(t,\varphi)$.

Let us give some examples of dynamical systems.

Example 1. Ordinary differential equations. Consider the equation

$$\dot{x} = f(x), \tag{1.1}$$

where $f\colon R^n \to R^n$, is continuous for every ξ in R^n, and the solution $u(t,\xi)$, $u(0,\xi) = \xi$ exists for all $t \geq 0$, is unique and depends continuously upon t,ξ. Uniqueness of the solution implies $u(t+\tau,\xi) = u(t,u(\tau,\xi))$ for all t,τ and therefore for all $t,\tau \geq 0$. Clearly u is a dynamical system on R^n.

Example 2. Functional differential equations of the retarded type. Let $C = C([-r,0],R^n)$, $r \geq 0$, be the space of continuous functions from $[-r,0]$ to R^n with the uniform convergence topology. For any continuous function x defined on $[-r,s)$, $s > 0$ and any $0 \leq t < s$ let x_t be the function in C defined by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$. Then the functional differential equation

$$\dot{x}(t) = f(x_t), \tag{1.2}$$

with $f\colon C \to R^n$ continuous and locally lipschitz will have a solution $x = x(\varphi)(t)$ defined and continuous on $[-r,s)$, $s > 0$ and $x_0 = \varphi$, the initial value for every φ in C. With $u(t,\varphi) = x_t(\varphi)$, since local existence, uniqueness and continuous dependence is easily proved [12] then u is a dynamical system on C.

It should be noted that this functional differential equation (simplest example $\dot{x}(t) = \alpha x(t-1)$) only defines solutions forward in time, hence the dynamical system definition.

Example 3. Functional differential equations of the neutral type. Let $C = C([-r,0),R^n)$ with the same norm as above and, with the same notation consider the equation

$$\frac{d}{dt}(Dx_t) = f(x_t) \tag{1.3}$$

where D is a difference operator defined by

$$D\varphi = \varphi(0) - \sum_{k=1}^{N} A_k \varphi(-\tau_k) \tag{1.4}$$

where A_k are $n \times n$ constant matrices and $0 < \tau_k \leq r$ with τ_j/τ_n rational. This is a special case of a functional differential equation of the neutral type (for example, $\dot{x}(t) + d\dot{x}(t-1) + bx(t) + cx(t-1) = 0$ is such an equation). If f is continuous and locally lipschitzian in C then it is possible to show that with any initial value φ in C the solution $u(t,\varphi) = x(\varphi)$ exists, is continuous in t and φ and is unique [10]. If solutions exist for all $t \geq 0$, then $u(t,\varphi)$ defines a dynamical system on C. Note again that the solution can be defined only forward in time.

Example 4. Parabolic partial differential equations. In this case, consider the heat equation, with boundary and initial conditions

$$\begin{aligned} &u_t = u_{xx},\ 0 \leq x \leq \pi,\ t > 0 \\ &u(0,t) = u(\pi,t) = 0,\ t \geq 0 \\ &u(x,0) = \Phi(x) \end{aligned} \tag{1.5}$$

Consider the space X of functions $\Phi: [0,\pi] \to R$ continuously differentiable on $[0,\pi]$ with $\Phi(0) = \Phi(\pi) = 0$ and with norm $\|\Phi\|_1 = \sup\{|\Phi'(x)|: 0 \leq x \leq \pi\}$. Then it is well-known that $u(t,\Phi) = u(t,x;\Phi(x))$ exists for all $t \geq 0$ is unique and depends continuously on t, Φ, in the norm of X. Hence, we have a dynamical system in the Banach space X. Note that, once again, the solutions are not defined backward in time - as is well-known the backward solution will not be in X.

Example 5. Consider the equation

$$\begin{aligned} &v_{tt} = v_{xx} + f(v,v_t,v_x) \quad 0 \leq x \leq 1,\ t \geq 0 \\ &v(0,x) = \varphi(x), v_t(0,x) = \psi(x) \\ &v(t,0) = 0,\ \ v(t,1) = 0,\ \ t \geq 0 \end{aligned} \tag{1.6}$$

where f is analytic in its variables in the whole space. Let W_2^k the space of functions with all generalized derivatives of order less than an equal to k square integrable in [0,1] with norm $\|\varphi\|^2_{W_2^k} = \int_0^1 [\varphi^2 + (\varphi^1)^2 + \cdots + (\varphi^{(k)})^2]dx$, where $\varphi^{(j)}$ is the j th generalized derivative of φ. Then [19], it is known that (1.6) has a unique generalized solution $v(t,x,\varphi,\psi)$ on $-\eta \leq t \leq \eta$ for every φ in W_2^k and any ψ in W_2^{k-1} and that the pair $[v(t,x;\varphi,\psi), v_t(t,x;\varphi,\psi)]$ belongs to $W_2^k \times W_2^{k-1}$ and is continuous in t,φ,ψ. Hence, if it is assumed that such a solution exists for all $t \geq 0$, then $u(t,\Phi) = [(v(t,x;\varphi,\psi), v_t(t,x;\varphi,\psi)]$, is a dynamical system on $W_2^k \times W_2^{k-1}$ for any $k \geq 1$.

The purpose of these examples has been to illustrate the generality of the concept of dynamical systems. We shall return to some specific applications of a physical nature later.

2. Some Stability Theorems

Let us now state, for our general dynamical system, the fundamental theorems which we wish to exploit for the determination of stability results. For this purpose, let

Definition 2.1. Let a dynamical system $u(t,\varphi)$ be defined in the Banach space $\mathscr{B}$. If $u(t,\psi) = \psi$ for all $t \geq 0$, then ψ is an equilibrium solution of the dynamical system.

Definition 2.2. The equilibrium solution $\varphi = 0$ of $u(t,\varphi)$ is stable, if for every $\epsilon > 0$ there exists a $\delta(\epsilon)$ such that $\|\psi\| \leq \delta$ implies $\|u(t,\psi)\| \leq \epsilon$ for all $t \geq 0$. The equilibrium $\varphi = 0$ is asymptotically stable if it is stable and there exists a γ such that $\|\psi\| \leq \gamma$ implies $\lim_{t\to\infty} u(t,\psi) \to 0$ (in the norm in $\mathscr{B}$).

Definition 2.3. A set M in $\mathscr{B}$ is a positively invariant set of the dynamical system u if for each Φ in M, $O^+(\Phi) \subset M$. It is invariant if for each Φ in M there exists a function $U(s,\Phi)$, $U(0,\Phi) = \Phi$ defined and in M for $-\infty < s < \infty$ and such that $u(t,U(s,\Phi)) = U(t+s,\Phi)$ for all $t \geq 0$.

Definitions 2.1 and 2.2 are the natural generalization of the familiar ones. The first part of Definition 2.3 is well-known; the second part of the definition simply uses the device of extending the dynamical system backward, if possible, since the dynamical system is not defined backward. Note that the function U must exist only for those Φ in M.

Let us now define, in the manner of [9, 11]

Definition 2.4. If u is a dynamical system on $\mathscr{B}$ and V is a continuous scalar function on $\mathscr{B}$, define

$$\dot{V}(\Phi) = \overline{\lim_{t \to 0}} \frac{1}{t} [V(u(t,\Phi)) - V(\Phi)].$$

V is said to be a Liapunov functional on a set G in $\mathscr{B}$ is V is continuous on $\overline{G}$ and if $\dot{V}(\Phi) \leq 0$ for every Φ in G. Furthermore, let $S = \{\Phi \text{ in } \overline{G} | \dot{V}(\Phi) = 0\}$ and let M be the largest invariant set in S for the dynamical system u.

Then it is possible to prove [9]

Theorem 2.1. Suppose u is a dynamical system on $\mathscr{B}$. If V is a Liapunov functional on G and the orbit $O^+(\varphi)$ belongs to G then $u(t,\varphi) \to S$ as $t \to \infty$. Furthermore, if $O^+(\varphi)$ belongs to a compact set of $\mathscr{B}$ then $u(t,\varphi) \to M$, and M is nonempty, compact and invariant.

This is one of the most general stability theorems available. Note that first of all, we always require the orbit to remain in G; secondly, that compactness of the orbit allows much more to be said about the set of points approached if S contains more than one element.

In the next examples, we attempt to illustrate the application of this general theorem. Note that the elements needed are:

(i) a dynamical system

(ii) a set $G \subset \mathscr{B}$

(iii) a Liapunov functional on G and, finally, perhaps

(iv) compactness of the orbits

3. A Problem of Nonexistence of Oscillations

Consider the network shown in Figure 1. In this circuit the section between 0 and 1 is a lossless transmission line with specific capacitance C_s and specific inductance L_s. The current i and the voltage v of this line are functions of ξ and t and satisfy the equations

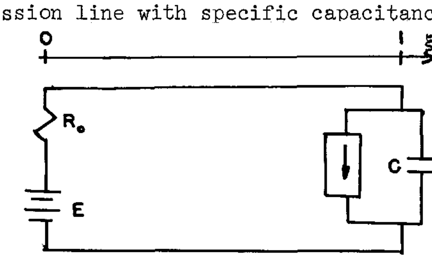

Figure 1

$$L_s \frac{\partial i}{\partial t} = - \frac{\partial v}{\partial \xi}, \qquad -C_s \frac{\partial v}{\partial t} = \frac{\partial i}{\partial \xi} \qquad 0 < \xi < 1, \quad t > 0. \tag{3.1}$$

The circuits at the ends of the line give rise to the boundary conditions

$$E = v_0 + R_0 i_0, \qquad C \frac{dv_1}{dt} + f(v_1) = i_1, \quad t > 0, \tag{3.2}$$

where $v_0(t) = v(0,t)$, $v_1(t) = v(1,t)$, $i_0(t) = i(0,t)$ and $i_1(t) = i(1,t)$. The function f which renders the problem nonlinear is pictured in Figure 2 and represents the general characteristic on an Esaki diode.

There has been considerable recent interest in circuits of this type, generally called flip-flops, particularly regarding the existence and nonexistence of oscillations. Moser [16], Brayton [2] and Brayton and Miranker [3] have considered increasingly sophisticated mathematical models for the study of such circuits, from lumped models to the present one. The equilibrium states of (3.1), (3.2) are given by

$$E = v_1 + R_0 i_1, \qquad (3.3)$$
$$i_1 = f(v_1),$$

and, as illustrated in Figure 2, we shall consider only the case of a unique equilibrium point, say (v^*, i^*). Translating the equilibrium state to the origin and denoting the new variables by the same notation yields

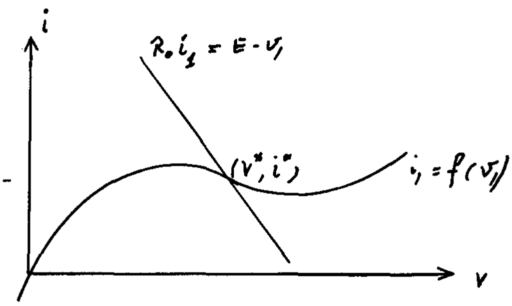

Figure 2

$$L_s \frac{\partial i}{\partial t} = - \frac{\partial v}{\partial \xi}, \quad 0 = v_0 + R_0 i_0, \qquad (3.4)$$
$$-C_s \frac{\partial v}{\partial t} = \frac{\partial i}{\partial \xi}, \quad C \frac{dv_1}{dt} + g(v_1) = i_1,$$

with $g(v_1) = f(v_1 + v^*) - f(v^*)$, which is assumed continuously differentiable and globally lipschitzian.

The behavior of the solutions of (3.4) is far from obvious. What is desired is to determine conditions on the parameters that guarantee the global asymptotic stability of the solution; because of the nature of the circuit, the lossless transmission line, it is suspected that periodic oscillations are possible.

To study this problem with some mathematical care it is necessary to have an existence theorem which suggests the appropriate space in which the problem should be viewed; for this purpose it is fairly simple to prove [17]:

<u>Theorem 3.1</u>. For the system (3.7), let the initial conditions $i(\xi,0) = \hat{i}(\xi)$ and $v(\xi,0) = \hat{v}(\xi)$ belong to $C^1[0,1]$ and satisfy the consistency conditions

(i) $0 = -\hat{v}(0) - R_0\hat{i}(0)$

(ii) $0 = L_s\hat{i}'(0) + R_0C_s\hat{v}'(0)$,

(iii) $\frac{C}{C_s}\hat{i}'(1) = -\hat{i}(1) + f(\hat{v}(1))$,

then there exists a unique solution $v(\xi,t)$, $i(\xi,t)$ in $C^1[0,1] \times C^1[0,\infty)$. Furthermore, this solution has the representation

$$v(\xi,t) = \frac{1}{2}[\phi(\xi-\sigma t) + \psi(\xi+\sigma t)],$$
$$i(\xi,t) = \frac{1}{2z}[\phi(\xi-\sigma t) - \psi(\xi+\sigma t)], \tag{3.5}$$

with $\sigma = \frac{1}{(L_sC_s)^{1/2}}$, $z = \left(\frac{L_s}{C_s}\right)^{1/2}$.

This theorem yields a representation for the solutions which is very suggestive; through the use of this representation it is possible to reduce this problem to a more tractable one. Indeed, introducing (3.5) into (3.4), the wave equation is automatically satisfied and the boundary conditions become

$$v_1(t) + zi_1(t) = -\psi_1\left(t - \frac{2}{\sigma}\right)\left(\frac{z-R_0}{z+R_0}\right),$$
$$v_1(t) - zi_1(t) = \psi_1(t), \tag{3.6}$$
$$c\frac{dv_1}{dt} + g(v_1) = i_1.$$

Eliminating i_1 and ψ_1 then yields the neutral functional differential equation

$$C\frac{d}{dt}[v_1(t) + kv_1(t-r)] = -\frac{v_1(t)}{z} + \frac{k}{z}v_1(t-r) - g(v_1(t)) - kg(v_1(t-r)), \tag{3.7}$$

where $r = \frac{2}{\sigma}$ and $k = \frac{R_0 - z}{R_0 + z}$. It is also simple to see that the given initial data $\hat{i}(\xi)$, $\hat{v}(\xi)$ in $C^1[0,1]$ completely determines the initial data $v_1 \in C^1[-r,0]$ for (1.7). Furthermore, it is not difficult to see that since $|k| < 1$ if

$\lim_{t\to\infty} v_1(t) = 0$, then $\lim_{t\to\infty} i(\xi,t) = 0$ and $\lim_{t\to\infty} v(\xi,t) = 0$ uniformly in ξ and that therefore oscillations will not exist.

The problem has then been reduced to the determination of conditions for the global asymptotic stability (3.7), which is rewritten for convenience of later computations as

$$\frac{d}{dt}\,[Dv_{1t}] = -[\frac{1}{Cz} + \frac{g(v_1(t))}{Cv_1(t)}]v_1(t) + [\frac{k}{Cz} - \frac{k}{C}\frac{g(v_1(t-r))}{v_1(t-r)}]v_1(t-r), \qquad (3.8)$$

where $D\varphi = \varphi(0) + k\varphi(-r)$, $x_t(\theta) = x(t+\theta)$ with $-r \leq \theta \leq 0$. Cruz and Hale [10] have developed existence, uniqueness and continuous dependence results for this type of neutral functional differential equation.

Indeed, it should be noted that this is a functional differential equation of the neutral type of the type described in Example 3. Within this context and considering the application of the first part of Theorem 2.1 leads to

Theorem 3.2. If the D operator is a stable one and V is a Liapunov functional on $G = G_\rho = \{\varphi \in C\colon V(\varphi) < \rho\}$. Then, if $\dot{V}(\varphi) \leq -\omega(|D\varphi|) \leq 0$ with $\omega(s) > 0$ for $S > 0$, with ω continuous, then every solution of (1.3) approaches zero as $t \to \infty$.

The result is precisely the one expected as a generalization of the usual theorems for ordinary differential equations. Now, through the use of this theorem it is not too difficult to obtain some stability results for our problem. Indeed, it is possible to prove [17].

Theorem 3.3. If g satisfies the sector criterion

$$\sup_\sigma \left(\frac{g(\sigma)}{\sigma}\right) \leq \left(\frac{1-|k|}{1+|k|}\right)\frac{1}{z} + \inf_\sigma \left(\frac{g(\sigma)}{\sigma}\right),$$

and

$$\inf_\sigma \left(\frac{g(\sigma)}{\sigma}\right) \geq -\frac{1}{z}\left(\frac{1-|k|}{1+|k|}\right),$$

then the equilibrium solution $v_1 = 0$ of Equation (3.8) is globally asymptotically uniformly stable.

The proof of this theorem is straightforward, although the detailed computations are involved. In essence, the Liapunov functional $V(\varphi) = \frac{1}{2}[D\varphi]^2 + \alpha \int_{-r}^{0} \varphi^2(\theta)d\theta$ is used and conditions for the existence of a nonnegative α such that $\dot{V}(t,\varphi) \leq -\beta[D\varphi]^2$, $\beta > 0$, are determined. These conditions yield the sector criterion quoted in the theorem.

From what has been said above, these sector criteria naturally also imply the nonexistence of oscillations in the original problem. It is of interest to note that these criteria are sharp in the following sense. If the problem is linear, that is, $g(\sigma) = -\gamma\sigma$, then it is a simple exercise to determine that the condition $-\gamma \geq -\frac{1}{z}\left(\frac{1-|k|}{1+|k|}\right)$ is a necessary and sufficient condition for the non-existence of oscillations. But in the linear case, this is precisely the condition given by Theorem 3.3, which implies that a type of Aizerman conjecture is valid for this problem.

4. A Bifurcation Problem

A number of applications, especially those arising from chemical reactor stability problems [1] give rise to a problem of the following nature. Consider the partial differential equation

$$u_t = u_{xx} + \lambda f(u), \quad \lambda \geq 0, \quad 0 \leq x \leq \pi, \quad t > 0 \tag{4.1}$$

which satisfies the boundary and initial conditions

$$\begin{aligned} u(0,t) &= u(\pi,t) = 0, && t \geq 0, \\ u(x,0) &= \Phi(x), && 0 \leq x \leq \pi \end{aligned} \tag{4.2}$$

where f is a given function defined on the real line, $f(0) = 0$, $uf(u) > 0$ for

$u \neq 0$ and $f(u)u^{-1} \to 0$ as $|u| \to \infty$. Assume for simplicity that f is C^2 smooth, odd and $\operatorname{sgn} f''(u) = -\operatorname{sgn} u$. With the given hypotheses $u \equiv 0$ is an equilibrium solution of this problem. For $\lambda = 0$ it is well known that this solution of the heat equation is stable in any usual meaning of the word, and the qualitative behavior of the solutions of (4.1), (4.2) is clear. What is of interest here is to determine how this picture changes as λ is allowed to increase from zero value; if the equilibrium solution $u \equiv 0$ loses its property of stability, do there appear any new equilibrium solutions which inherit this property? This problem has been investigated by Matkowsky [15] using formal asymptotic methods under hypothesis differing somewhat from these given here. The viewpoint here is to interpret (4.1), (4.2) as a dynamical system in an appropriate Banach space and to apply Liapunov methods of the type developed in [9, 11, 14]. Again, the details are omitted for the sake of the brevity of exposition. This specific application is more fully described in [5].

The first task here is to show that (4.1) - (4.2) defines a dynamical system. As a first step in this direction, consider the Banach space X of functions $\phi: [0,\pi] \to R$ continuously differentiable on $[0,\pi]$ with $\phi(0) = \phi(\pi) = 0$ and with norm $\|\phi\|_1 = \sup\{|\phi'(x)|: 0 \leq x \leq \pi\}$. Define also the norms $\|\phi\|_0 = \sup\{|\phi(x)|: 0 \leq x \leq \pi\}$ and $\|\phi\|_{W_2^1} = (\int_0^\pi \phi'(x)^2 dx)^{1/2}$, and note that $\|\phi\|_0 \leq \sqrt{\pi}\,\|\phi\|_{W_2^1} \leq \pi\|\phi\|_1$. Let $B_0(r)$ be open balls centered at zero with radius r in the $\|\ \|_0$ norm. Then it is possible to prove [5]:

<u>Theorem 4.1.</u> For any $\phi \in X$ and $\lambda \in [0,\infty)$, Equations (4.1), (4.2) have unique solutions $u(x,t;\phi,\lambda)$ denoted by $u(\phi,\lambda)(t) \in X$ defined for $0 < t < s(\phi,\lambda) \leq \infty$. Furthermore, if $u(\phi,\lambda)(t) \in B_0(r)$ for some r then $s(\phi,\lambda) = \infty$, the map $(t,\phi) \to u(\phi,\lambda)(t)$ defined for all $\phi \in X$ is a dynamical system in X with $\|\ \|_1$ and furthermore, the positive orbit $O^+(\phi,\lambda)$ of $u(\phi,\lambda)(t)$ is relatively compact in this space.

Note that, except for the hypothesis that the orbits are bounded in the

$\| \, \|_0$ norm, the theorem states that we are dealing with a dynamical system; furthermore, that the dynamical system is self-compactifying. This last property is precisely the expected result, given the smoothing properties of the heat equation which, this theorem states, are not affected by the nonlinearity.

Let us now define for every $\lambda \in [0,\infty)$ the Liapunov functional $V_\lambda(\phi) = \int_0^\pi \{\frac{1}{2} \phi'(x)^2 - \lambda \int_0^{\phi(x)} f(\xi)d\xi\}dx$ for $\phi \in X$. Note that V_λ is continuous on X relative to $\| \, \|_1$ and $\| \, \|_{W_2^1}$ and that, given the assumptions it is not too difficult to see that $V_\lambda(\phi) \to \infty$ as $\|\phi\|_0 \to \infty$. Furthermore, it is of interest to see that $\frac{d}{dt} V_\lambda(u(\phi,\lambda)(t)) = -\int_0^\pi u_t^2(x,t;\ \phi,\lambda)^2 dx \leq 0$, for $0 < t < s(\phi,\lambda)$. These observations lead to

<u>Theorem 4.2</u>. For any $\phi \in X$ and $\lambda \in [0,\infty)$ the map $t,\phi \to u(\phi,\lambda)(t)$ is a dynamical system in X normed by $\| \, \|_1$. Furthermore, the positive orbit $0^+(\phi,\lambda)$ is relatively compact in this space.

Note that the use of the Liapunov functional was essential in proving global existence. But now, since the Liapunov function has already been constructed it is possible to conclude much more.

Indeed, all of the conditions for the entire Theorem 2.1 are satisfied. Note that the largest invariant set M within our context is the set of equilibrium solutions. Hence

<u>Theorem 4.3</u>. Every solution of (4.1) - (4.2) approaches an equilibrium solution in the norm $\| \, \|_1$.

Actually, much more can be said about the qualitative picture by analyzing the equilibrium solutions, which are the solutions of the two point boundary value problem

$$u''(x) + \lambda f(u(x)) = 0, \quad u(0) = u(\pi) = 0, \quad 0 \leq \lambda < \infty. \tag{4.3}$$

Using methods inspired by the work of Urabe [20] it is possible to prove

Theorem 4.4. Let $\lambda_n = \frac{n^2}{f'(0)}$, $n = 1,2,\ldots$. Then, for any $\lambda \in [\lambda_n,\infty)$ Equation (4.3) has two solutions $u_n^{\pm}(\lambda) \in B_0(r_0)$ with the properties that

(i) $u_n^{\pm}(\lambda_n) = 0$

(ii) $u_n^{\pm}(\lambda)$ have exactly $n + 1$ zeros in $[0,\pi]$

(iii) $u_n^{\pm}(\lambda)$ varies continuously in λ relative to the norm $\| \|_1$ with $\|u_n^{\pm}(\lambda)\|_1 \to \infty$ as $\lambda \to \infty$.

Furthermore, for any $\lambda \in [0,\infty)$, (4.1) have no equilibrium points in X other than the origin $u_0 \equiv 0$ and those elements $u_n^{\pm}(\lambda)$, $n \geq 1$, for which $\lambda_n < \lambda$.

It is then quite clear that if $\lambda \leq \lambda_1$ then for every $\Phi \in X$ the corresponding solution $u(\Phi,\lambda)(t) \to 0$ as $t \to \infty$, the convergence naturally being in the norm $\| \|_1$. The question arises, given $\Phi \in X$ and $\lambda \in (\lambda_1,\infty)$ to which equilibrium point $u(\Phi,\lambda)(t)$ will converge. Again, it is possible to answer, at least partially, this query by an appropriate analysis of the Liapunov functional. Indeed, we have

Theorem 4.5. For each integer $n \geq 1$, let $u_n^{\pm}(\lambda)$, $\lambda_n \leq \lambda < \infty$ be as in Theorem 4.4. Then for any $\lambda \in (\lambda_1,\infty)$ the origin $u_0 = 0$ is unstable. For any $\lambda \in [\lambda_1,\infty)$, $u_1^{\pm}(\lambda)$ is asymptotically stable and for $\lambda \in [\lambda_n,\infty)$, $n \geq 2$, $u_n^{\pm}(\lambda)$ is unstable. (All these assertions are valid in X normed by $\| \|_1$).

These five theorems give a rather clear picture of the qualitative behavior of the solutions. All solutions will, in general, approach either u_0 or $u_1^{\pm}(\lambda)$

5. The General Problem of Thermoelasticity

In the previous problem it was possible to find a Banach space in which the dynamical system was self-compactifying. It was this property that was heavily exploited and which is essential in the application of invariance principles. It is to be suspected that such self-compactifying properties can be expected in dynamical systems which arise from functional differential equations of the retarded type and partial differential equations of parabolic nature. For hyperbolic partial differential equations clearly this property would be very surprising. The example presented now is of hyperbolic nature, yet it is possible, through a little more work, to still apply the principle.

Elastic stability is usually discussed from strictly mechanical considerations; here the concern is with thermodynamic properties of elastic materials. More specifically, one may ask what effects the second law of thermodynamics has on the asymptotic stability of equilibrium of otherwise non-dissipative materials [7].

A material point is identified by $x = (x_1, x_2, x_3)$ in its state of equilibrium (no stresses, constant temperature $= \gamma_0$). The displacement field at some time t following an initial disturbance at time $t = 0$ is given by $u(x,t)$ and the temperature deviation by $T(x,t)$; $\rho(x)$ denotes the density at x in the equilibrium state. Let Ω be an open, bounded, connected set in E^3 which is properly regular [8]; let $\partial\Omega$ denote the boundary of Ω. The constitutive equations of thermoelasticity are taken then in the form

$$\rho\ddot{u}_i = (C_{ijk\ell})_{,j} - (m_{ij}T)_{,j}, \tag{5.1}$$

$$\rho C_0\dot{T} + m_{ij}\gamma_0\dot{u}_{i,j} = (K_{ij}T_{,j})_{,i}; \tag{5.2}$$

where body forces and heat sources have been excluded. In these equations $C_{ijk\ell} = C_{jik\ell} = C_{k\ell ij}$, $m_{ij} = m_{ji}$, $K_{ij} = K_{ji}$ and C_D, ρ, $C_{ijk\ell}$, m_{ij} and K_{ij} are assumed to be smooth functions of x.

Let now $t_0 > 0$. By a classical solution of the mixed initial-boundary value problem in $\Omega \times (0,t_0)$ we mean a pair (u,T) satisfying equation (5.1) and (5.2) together with the boundary conditions

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0,t_0) \quad \text{(clamped boundary)}, \tag{5.3}$$

$$T = 0 \quad \text{on} \quad \partial\Omega \times (0,t_0) \quad \text{(constant temperature)}; \tag{5.4}$$

and with initial conditions

$$(u(x,0),\dot{u}(x,0),T(x,0)) = (u_0(x),\dot{u}_0(x),T_0(x)), \tag{5.5}$$

where $u_0(x)$, $\dot{u}_0(x)$ and $T_0(x)$ are given functions on Ω.

The generalized solutions of the mixed initial boundary value problem described above can be viewed on an appropriate Banach space as a dynamical system. Once this is done, the application of Theorem 2.1 permits us to draw immediate conclusions on the asymptotic behavior of the solutions of our problem.

Consider the Sobolev spaces $W_2^{(k)}(\Omega)$ and $W_{20}^{(k)}(\Omega)$, $k = 1,2,\ldots$. Assume that

$$\text{ess inf } \rho(x) > 0, \text{ ess inf } C_D(x) > 0, \tag{5.6}$$

$$K_{ij}\xi_i\xi_j \geq C_1\xi_i\xi_i, \; C_1 > 0 \quad \text{constant}, \tag{5.7}$$

(the second law of thermodynamics requires K_{ij} positive semidefinite at $x \in \Omega$; we make the stronger assumption of positive definiteness). Also for all $v_i \in W_{20}^{(1)}(\Omega)$

$$\int_\Omega C_{ijk\ell}v_{i,j}v_{k,\ell}dx \geq C_2 \int_\Omega v_{i,j}v_{i,j}dx, \quad C_2 > 0 \quad \text{constant} \tag{5.8}$$

Define now the spaces $H_0(\Omega) \approx W_{20}^{(1)}(\Omega) \times L_2(\Omega) \times L_2(\Omega)$ with norm $|(v_i,w_i,R)|_0^2 = \int_\Omega [\rho w_i w_i + C_{ijk\ell}v_{i,j}v_{k,\ell} + \frac{\rho C_D}{\gamma_0} R^2]dx$ and $H(\Omega) = W_{20}^{(1)}(\Omega) \times W_{20}^{(1)}(\Omega) \times W_{20}^{(1)}(\Omega)$. Define the map $P: H_0(\Omega) \overset{\text{onto}}{\to} H_1(\Omega)$ sending $(v_i, w_i, R) \in H_0(\Omega)$ onto $(u_i, v_i, T) \in H(\Omega) \subset H(\Omega)$ where $(u_i,T) \in W_{20}^{(1)}(\Omega) \times W_{20}^{(1)}(\Omega)$ is defined by the solution of the system

$$\int_\Omega C_{ijk\ell}u_{k,\ell}\theta_{i,j}dx = -\int_\Omega [\rho w_i\theta_i - m_{ij}T\theta_{i,j}]dx$$

$$\int_\Omega K_{ij}T_{,j}D_{,i}dx = -\int_\Omega [\rho C_D R + m_{ij}\gamma_0 v_{i,j}]D\ dx$$

for every $D_{,}\theta_i \in W_{20}^{(1)}(\Omega)$. The mapping P_m is linear, well-defined on $H_0(\Omega)$ and one to one. Hence, defining $P_m = P^\circ P^\circ \ldots {}^\circ P$ let $H_m(\Omega)$ denote the range of the map P_m. It is clear that P_m^{-1} exists and maps $H_m(\Omega)$ onto $H_0(\Omega)$. Let $\psi \in H_m(\Omega)$ and define $|\psi|_m = |P_m^{-1}\psi|_0$. Then [6],

Lemma 5.1. H_m is a Banach space with norm $|\cdot|_m$. $H_0(\Omega) \supset H(\Omega) \supset \ldots \supset H_m(\Omega)$ algebraically and topologically. Furthermore, $H_m(\Omega)$ is dense in $H_\ell(\Omega)$ for $m > \ell$ and the imbedding $I: H_m(\Omega) \to H_\ell(\Omega)$ is compact.

Let us now define appropriately a generalized solution of our problem:

Definition 5.1. $(u_{i,} \dot{u}_{i,} T)$ will be called a generalized solution of (5.1) - (5.5) on $\Omega \times (0,t_0)$ if for all smooth test functions $(v_{i,} R)$ with compact support on Ω and v_i vanishing on $\Omega \times 0$

$$\begin{aligned}
&\int_0^{t_0}\int_\Omega \{(t-t_0)[\rho\dot{u}_i\ddot{v}_i - C_{ijk\ell}u_{k,\ell}\dot{v}_{i,j} + m_{ij}T\dot{v}_{i,j} + \\
&+ \frac{\rho C_D}{\gamma_0}T\dot{R} + m_{ij}u_{i,j}\dot{R}] + \rho\dot{u}_i\dot{v}_i + \rho\frac{C_D}{\gamma_0}TR + \\
&+ m_{ij}u_{i,j}R - \frac{1}{\gamma_0}\int_0^t (K_{ij}R_{,i})_{,j}T\ dt\}dxdt \\
&= -t_0\int_\Omega [\rho\dot{u}_{0_i}v_i|_{t=0} + \frac{\rho C_D}{\gamma_0}T_0R|_{t=0} + m_{ij}u_{0_{i,j}}R|_{t=0}]dx.
\end{aligned} \tag{5.9}$$

With this definition it follows that [6]:

Theorem 5.1. Under assumptions (5.1) - (5.3) the triple $(u_{i,} \dot{u}_{i,} T)$ describes a dynamical system on $H_m(\Omega)$, $m = 0,1,2,\ldots$, where $(u_{i,} \dot{u}_{i,} T)$ is the generalized solution to the equations of linear thermoelasticity satisfying equation (5.9). Furthermore, for t in $(0,t_0)$

$$|(u_i,\dot{u}_i,T)(t)|_m^2 + \frac{1}{\gamma_0}\int_0^t\int_\Omega K_{ij}T^{(m)}_{,i}T^{(m)}_{,j}\,dx\,d\tau = |(u_{i_0},\dot{u}_{i_0},\dot{T}_0)|_m^2 \qquad (5.10)$$

where $T^{(m)}(x,t)$ denotes the generalized m^{th} derivative in time of $T(x,t)$.

The problem of termoelastic stability has now been put in a setting appropriate for the application of Theorem 2.1 which allows us to obtain stability results in a simple and direct manner.

For the trajectory $(u_i,\dot{u}_i,T)$ in $H_m(\Omega)$ define $P^\circ(u_i,\dot{u}_i,T) \equiv (\overline{u}_i,\dot{\overline{u}}_i,\overline{T})$. It follows from the definition of the map P that $(\overline{u}_i,\dot{\overline{u}}_i,\overline{T})$ is a dynamical system on $H_{m+1}(\Omega)$ with initial data $P^\circ(u_{0_i},\dot{u}_{0_i},T_0)$ in $H_{m+1}(\Omega)$ satisfying (5.9) and Theorem 5.1. Therefore, Theorem 2.1 and (5.10) imply that for any initial data $(u_{0_i},\dot{u}_{0_i},T_0)$ in $H_m(\Omega)$ the trajectory $(\overline{u}_i,\dot{\overline{u}}_i,\overline{T})(t)$ will lie in a bounded set of $H_m(\Omega)$ for all $t \geq 0$. Hence by Lemma 5.1 the trajectory $(\overline{u}_i,\dot{\overline{u}}_i,\overline{T})$ will lie in a compact set G of $H_\ell(\Omega)$, $\ell \leq m$. But then all the hypotheses of Theorem 2.1 are met with $\mathcal{B} = H_\ell(\Omega)$. For simplicity let $\ell = 1$ and $V = |(\overline{u}_i,\dot{\overline{u}}_i,\overline{T})|_1^2$. From (5.7) and (5.10) it immediately follows that $\dot{V} = \frac{-1}{\gamma_0}\int_\Omega K_{ij}\,\overset{(1)}{T}_{,i}\,\overset{(1)}{T}_{,j}\,dx \leq -c_3|(0,0,\overline{T})|_1^2$. The set S is then $S = \{(\overline{u}_i,\dot{\overline{u}}_i,\overline{T}) \in H_1(\Omega)\,|\,\overline{T} = 0\}$. The determination of M, the largest invariant set in S, which is not trivial, then leads to [18]:

<u>Theorem 5.2</u>. For any initial data $(u_{0_i},\dot{u}_{0_i},T_0)$ in $H_m(\Omega)$, $m \geq 1$, and under assumptions (5.6) - (5.7), $(u_i,\dot{u}_i,T)(t)$ approaches the set $M = \{(w_i,\dot{w}_i,Y)$ in $H_0(\Omega)\,|\,m_{ij}w_{i,j} = 0,\ Y = 0,\ \int_0^{t_0}\int_\Omega \{(t-t_0)[\rho\dot{w}_i\ddot{v}_i - C_{ijk\ell}w_{k,\ell}\dot{v}_{i,j}] + \rho\dot{w}_i\dot{v}_i\}dx\,dt = -t_0\int_\Omega \rho\dot{w}_{0_i}\dot{v}_i|_{t=0}dx$ for all v_i test functions with compact support on Ω and vanishing on $\Omega \times 0$ in the norm of the space $H_0(\Omega)$ as $t \to \infty$.

It is of interest to note that in this case there is an infinity of solutions in the set M and that the use of the Liapunov functional allows a very nice characterization of them; they are the isothermal oscillations of the body, representing pure shear stresses. It should be noted that to obtain the needed compactification it is necessary for the problem to represent a dynamical system in two Banach spaces, here, for example, H_1 and H_0 with the imbedding of H_1 into H_0 completely continuous. The boundedness of the trajectories in H_1 then imply that the trajectory is in a compact set in H_0 and allows the application of the theorem. In this problem, which is linear, the generation of the H_n spaces is quite natural, they are velocity spaces. For nonlinear problems, unfortunately, this is far from easy.

6. Summary

In this brief lecture an attempt has been made to indicate the power and difficulties of application of Liapunov stability theory, with emphasis on the invariance principle. Looking back over the three examples, it is quite clear that the construction of the Liapunov functional is, in general, necessary to obtain the boundedness results required by a dynamical system. Once this functional is known, then if its derivative is negative definite in an appropriate domain, then only one equilibrium point will be stable. If the derivative is negative semidefinite, but the trajectory lies in a compact orbit, then the invariant subset of the set $\dot{V} = 0$ will be the set approached by the solutions. In the second example, the equations of motion were self-compactifying - in the last one they were not and one had to give initial conditions in a subspace which had the property that bounded set in it are compact in the larger space.

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