STABILITY OF DISSIPATIVE SYSTEMS WITH APPLICATIONS TO FLUIDS AND MAGNETOFLUIDS

E.M. Barston

Courant Institute of Mathematical Sciences
New York University, New York, New York

Abstract

An energy principle is presented which gives necessary and sufficient conditions for exponential stability for a useful class of continuous linear dissipative systems. The maximal growth rate Ω of an unstable system is shown to be the least upper bound of a certain functional, providing a variational expression for Ω. Applications to the problems of the stability of a stratified viscous incompressible fluid in a gravitational field and the resistive, viscous, incompressible magnetohydrodynamic sheet pinch are dicussed.

I. Introduction

In attempting to determine the stability characteristics of a given (usually nonlinear) physical system, one is often led to consider the stability of a derived (approximate) linear system. Perhaps it is known that the stability or instability of the original problem can in fact be inferred from the results obtained for the linearized problem; even if this information is not available, the lack of a general systematic method for the construction of Lyapunov functions often leaves one with no alternative, and so one proceeds with a study of the stability of the linear system, at least as a preliminary step in the solution of the problem.

Unfortunately, the solution of the derived linear problem itself is often formidable, even for autonomous systems, when the dimension is sufficiently large. This is particularly true for continuous systems where the linearized equations contain partial differential operators with spatially varying coefficients. Perhaps the best one can hope for in such cases is the existence of an energy principle which gives necessary and sufficient conditions for (exponential) stability. The existence of such an energy principle for determining the linear stability of the equilibrium states of a conservative dynamical system is well-known, and has been the cornerstone of almost every investigation of the stability of non-trivial equilibria in perfectly conducting, invicid, magneto-hydrodynamics [5],[6]. In 1903, Kelvin and Tate[8] proposed an extension of the energy principle to a class of real, finite-dimensional, dissipative linear systems (Kelvin and Tate did not prove their assertion; a proof using Lyapunov methods can be found in Ref. [7]). In recent years, the energy principle has been extended to a general class of continuous linear dissipative systems, and in the process, a maximum principle for the maximal growth rate of an unstable system has been obtained [1],[3]. We shall briefly discuss these developments and some applications in this paper. For a more complete discussion and further applications references [1]-[4] should be consulted.

We shall begin with a discussion of the problem of the gravitational stability of a stratified viscous incompressible fluid, which will serve to motivate as well as illustrate the theory. After developing the energy and maximum principles, we briefly dicuss the application of these results to the problem of the stability of the resistive, viscous, incompressible magneto-hydrodynamic sheet pinch.

II. Equations for a Viscous Incompressible Fluid in a Gravitational Field

Perhaps the most familiar example of a continuous dissipative system of the type we shall analyze is the problem of the gravitational stability of a stratified, viscous, incompressible fluid. Let us then consider such a fluid occupying a bounded region U (a simply connected open set) with surface ∂U, satisfying the following set of equations in U:

$$\nabla \cdot \vec{v} = 0 \tag{2.1}$$

$$\frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla} \rho = 0 \tag{2.2}$$

$$\rho\left\{\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right\} = -\vec{\nabla}p - \rho g\,\vec{e}_z + \nu\nabla^2\vec{v} \tag{2.3}$$

The quantity $\rho(\vec{x},t)$ denotes the mass density, $\vec{v}(\vec{x},t)$ the fluid velocity, $p(\vec{x},t)$ the scalar pressure, ν the viscosity (a positive constant), g the gravitational acceleration, and $\vec{e}_z$ the unit vector in the z-direction (assumed vertical). The equilibrium values of the fluid variables, denoted by the subscript o, are given as follows: $\vec{v}_o \equiv 0$; $\rho_o = \rho_o(z) > 0$ on $[z_1,z_2]$, $\rho_o \in C^1[z_1,z_2]$, where $z_1 \equiv \inf_{\vec{x}\in U} z$, $z_2 = \sup_{\vec{x}\in U} z$; and $p_o(z)$ is given by $p_o(z) = -g\int_{z_1}^{z} \rho_o(u)\,du + \text{const.}$ We linearize Eqs.(2.1)-(2.3) about the equilibrium state (in the sequel, the variables $\vec{v}$, p, and ρ without the subscript o will denote <u>linearized</u> quantities) and obtain, after introducing the (linear) displacement vector $\vec{\xi}(\vec{x},t) \equiv \int_0^t \vec{v}(\vec{x},\tau)\,d\tau + \vec{\xi}(\vec{x},0)$ where $\nabla\cdot\vec{\xi}(\vec{x},0) = 0$ and $\rho(\vec{x},0) = -\vec{\nabla}\rho_o\cdot\vec{\xi}(\vec{x},0)$,

$$\nabla\cdot\vec{\xi} = 0 \tag{2.4}$$

$$\rho_o\ddot{\vec{\xi}} - \nu\nabla^2\dot{\vec{\xi}} - g\frac{d\rho_o}{dz}\,\xi_z\,\vec{e}_z + \vec{\nabla}p = 0\ . \tag{2.5}$$

We take ∂U to be a rigid surface, so that the appropriate boundary condition is that $\vec{\xi}$ vanish on ∂U. We assume, of course, that all quantities are sufficiently smooth so that the indicated operations are well-defined; in particular, we consider the class of solutions $\vec{\xi}(\vec{x},t)$ of Eq. (2.5) such that $\vec{\xi}$ and $\dot{\vec{\xi}}$ are both in the class D and $\ddot{\vec{\xi}} \in C(\bar{\Omega})$ for each $t \geq 0$, where D is defined as the set of all functions $\vec{f}(\vec{x})$ with the properties that $\nabla\cdot\vec{f} = 0$ in U, $\vec{f} = 0$ on ∂U, and $\vec{f}$ is twice continuously differentiable on U. It is easy to see that the operators P,K, and H defined by $P\vec{\xi} \equiv \rho_o\vec{\xi}$, $K\vec{\xi} \equiv -\nu\nabla^2\vec{\xi}$, and $H\vec{\xi} \equiv -g\frac{d\rho_o}{dz}\xi_z\,\vec{e}_z$ are formally self adjoint on D with respect to the inner product $(\vec{f},\vec{g}) = \int_U \vec{f}^*\cdot\vec{g}\,d^3x$ ($\vec{f}^*$ denotes the complex conjugate of $\vec{f}$) and that P and K are positive. We note that $(\vec{\nabla}p,\vec{\xi}) = (\vec{\nabla}p,\dot{\vec{\xi}}) = 0$ for our solutions $\vec{\xi}$ of Eq. (2.5). This follows from the divergence theorem, since $\nabla\cdot\vec{\xi} = 0$ and $\vec{\xi}$ vanishes on ∂U.

<u>III. The Energy and Maximum Principles</u>

The preceeding problem is a special case of the more general system

$$P\ddot{\xi} + K\dot{\xi} + H\xi(t) + F_\xi = 0 \ , \qquad t \geq 0 \qquad (3.1)$$

where $\xi, \dot{\xi}, \ddot{\xi}$ and F_ξ are elements of an inner product space E for each fixed $t \geq 0$; P,K, and H are time-independent linear formally self-adjoint operators from E into E with domains of definition D_P, D_K, and D_H, respectively; $P \geq 0$ on D_P and $K \geq 0$ on D_K; and F_ξ, defined for each solution $\xi(t)$ of Eq. (3.1), has the property that $(F_\xi, \dot{\xi}) = (F_\xi, \xi) = 0$, $t \geq 0$. In the sequel, we restrict our attention to the class S of solutions $\xi(t)$ of Eq. (3.1) satisfying the following ten conditions:

$$\xi(t) \in D \equiv D_P \cap D_K \cap D_H \ , \qquad t \geq 0 \qquad (3.2)$$

$$\dot{\xi}(t) \in D_P \cap D_K \ , \qquad t \geq 0 \qquad (3.3)$$

$$\ddot{\xi}(t) \in D_P \ , \qquad t \geq 0 \qquad (3.4)$$

$$P\ddot{\xi} + K\dot{\xi} + H\xi + F_\xi = 0 \ , \qquad t \geq 0 \qquad (3.1)$$

$$\frac{d}{dt}(\dot{\xi}, P\dot{\xi}) = (\ddot{\xi}, P\dot{\xi}) + (\dot{\xi}, P\ddot{\xi}) \ , \qquad t \geq 0 \qquad (3.5)$$

$$\frac{d}{dt}(\dot{\xi}, P\xi) = (\ddot{\xi}, P\xi) + (\dot{\xi}, P\dot{\xi}) \ , \qquad t \geq 0 \qquad (3.6)$$

$$\frac{d}{dt}(\xi, P\xi) = (\dot{\xi}, P\xi) + (\xi, P\dot{\xi}) \ , \qquad t \geq 0 \qquad (3.7)$$

$$\frac{d}{dt}(\xi, K\xi) = (\dot{\xi}, K\xi) + (\xi, K\dot{\xi}) \ , \qquad t \geq 0 \qquad (3.8)$$

$$\frac{d}{dt}(\xi, H\xi) = (\dot{\xi}, H\xi) + (H\xi, \dot{\xi}) \ , \qquad t \geq 0 \qquad (3.9)$$

$$(F_\xi, \xi) = (F_\xi, \dot{\xi}) = 0 \ , \qquad t \geq 0 \qquad (3.10)$$

The class S may be thought of as the class of suitably "smooth" solutions of Eq. (3.1). Equations (3.5)-(3.9) are merely the usual rules for differentiating inner products; Eqs. (3.2)-(3.4) offer no restrictions on the solutions of Eq. (3.1) provided $D_P \supset D_K \supset D_H$, but become additional "smoothness" requirements should the above relation not hold.

The precise definition of the t-derivative $\dot{\xi}$ is not important in the sequel, provided that the usual rules for differentiating sums and products (of vectors and scalars) are valid. Thus one can think of $\dot{\xi}$ as being defined in the norm-topology of E, or if E is an n-fold Cartesian product of L_2-spaces (as is usually the case in applications), $\dot{\xi}$ can be taken to be the n-vector obtained by computing the partial derivative with respect to t of each of the n components of $\xi(t)$.

In addition to restricting the analysis to solutions $\xi(t) \in S$, we assume that H is bounded below on D and that $\inf_D \frac{(\eta,[\omega P+K]\eta)}{(\eta,\eta)} > 0$ for all $\omega > 0$. In the circumstance that $\inf_D \frac{(\eta,H\eta)}{(\eta,\eta)} < 0$, we define $\tilde{D} \equiv \{\eta | \eta \in D, (\eta,H\eta) < 0\}$, require $P > 0$ on $\tilde{D}$, set

$$Q_\eta \equiv \frac{1}{2}\left\{\left[\frac{(\eta,K\eta)^2}{(\eta,P\eta)^2} - 4\,\frac{(\eta,H\eta)}{(\eta,P\eta)}\right]^{1/2} - \frac{(\eta,K\eta)}{(\eta,P\eta)}\right\}$$

$$\text{for } \eta \in \tilde{D},\quad \Omega \equiv \sup_{\tilde{D}} Q_\eta,$$

$Y \equiv \{\phi |$ for each $\omega \in (0,\Omega)$, there exists $\xi(t) \in S$ and $\psi \in D_P \cap D_K$ such that $P\psi = 0$, $\xi(0) = \phi$, $\dot{\xi}(0) = \omega\phi+\psi\}$, and assume that $\sup_{Y\cap\tilde{D}} Q_\eta = \sup_{\tilde{D}} Q_\eta = \Omega$.

The stability of the solutions $\xi(t)$ of Eq. (3.1) will be discussed in terms of $\|\xi\| = (\xi,\xi)^{1/2}$. The function $\xi(t)$, defined for $t \geq 0$, is said to be exponentially stable if for every $\varepsilon > 0$, there exists a constant M_ε such that $\|\xi(t)\| \leq M_\varepsilon e^{\varepsilon t}$ for $t \geq 0$. If $\xi(t)$ is not exponentially stable, we say it is exponentially unstable. If every solution $\xi(t) \in S$ is exponentially stable, the system (3.1) is called exponentially stable.

With the preceeding definitions and hypothesis, we have the following theorem:

<u>Theorem 1:</u>

(A) Let $\inf_D \frac{(\eta,H\eta)}{(\eta,\eta)} > 0$. Then for each $\xi(t) \in S$, there exists a constant B such that $\|\xi(t)\| \leq B$ for all $t \geq 0$.

(B) Let $\inf_D \frac{(\eta,H\eta)}{(\eta,\eta)} = 0$. Then system (3.1) is exponentially stable.

(C) Let $\inf_D \frac{(\eta,H\eta)}{(\eta,\eta)} < 0$. Then the system is exponentially unstable with maximal growth rate Ω, i.e., given any $\omega \in (0,\Omega)$, there exists $\xi(t) \in S$ and a positive constant M such that $\|\xi(t)\| \geq M e^{\omega t}$ for all

$t \geq 0$, and given any $\xi(t) \in S$ and $\varepsilon > 0$, there exists a constant M_ε such that $\|\xi(t)\| \leq M_\varepsilon \, e^{(\Omega+\varepsilon)t}$, $t \geq 0$.

<u>Proof</u>: Let $\xi(t) \in S$. Then

$$\begin{aligned}\frac{d}{dt}\{(\dot\xi,P\dot\xi) + (\xi,H\xi)\} &= (P\ddot\xi+H\xi,\dot\xi) + (\dot\xi,P\ddot\xi+H\xi)\\ &= -2(\dot\xi,K\dot\xi) - (F_\xi,\dot\xi) - (\dot\xi,F_\xi)\\ &= -2(\dot\xi,K\dot\xi) \leq 0\ , \qquad t \geq 0,\end{aligned}$$

so that

$$(\dot\xi,P\dot\xi) + (\xi,H\xi) \leq (\dot\xi_o,P\dot\xi_o) + (\xi_o,H\xi_o)\ , \quad t \geq 0 \quad (3.11)$$

where $\xi_o \equiv \xi(0)$, $\dot\xi_o \equiv \dot\xi(0)$. Let $\Delta \equiv \inf\limits_D \frac{(\eta,H\eta)}{(\eta,\eta)}$. If $\Delta > 0$, Eq. (3.11) gives

$$\Delta\|\xi\|^2 \leq (\xi,H\xi) \leq (\dot\xi_o,P\dot\xi_o) + (\xi_o,H\xi_o), \qquad t \geq 0 \quad (3.12)$$

which proves (A). Let $\omega > 0$, $\xi(t) \in S$, and set $\zeta(t) \equiv \xi(t)e^{-\omega t}$, $t \geq 0$. Then $\xi(t) = \zeta(t)e^{\omega t}$, and a straightforward calculation yields

$$P\ddot\zeta + K_\omega\dot\zeta + H_\omega\zeta + f_\zeta = 0\ , \qquad t \geq 0 \quad (3.13)$$

where $K_\omega \equiv 2\omega P+K$, $H_\omega \equiv \omega^2 P+ \omega K+H$, and $f_\zeta \equiv F_\xi \, e^{-\omega t}$, so that $(f_\zeta,\zeta) =$ $= (f_\zeta,\dot\zeta) = 0$ for $t \geq 0$. By analogy with the derivation of Eq. (3.11) we have

$$(\dot\zeta,P\dot\zeta) + (\zeta,H_\omega\zeta) \leq (\dot\zeta_o,P\dot\zeta_o) + (\zeta_o,H_\omega\zeta_o)\ , \quad t \geq 0 \quad (3.14)$$

Let $\Delta = \inf\limits_D \frac{(\eta,H\eta)}{(\eta,\eta)} = 0$. Then since $\inf\limits_D \frac{(\eta,[\omega P+K]\eta)}{(\eta,\eta)} > 0$, we conclude that $\Delta_\omega \equiv \inf\limits_D \frac{(\eta,H_\omega\eta)}{(\eta,\eta)} > 0$, so that Eq. (3.14) implies

$$\|\xi(t)\| = \|\zeta(t)\|e^{\omega t} \leq \left[\frac{(\dot\zeta_o,P\dot\zeta_o)+(\zeta_o,H\zeta_o)}{\Delta_\omega}\right]^{1/2} e^{\omega t},\ t \geq 0$$

which holds for any $\omega > 0$. Thus statement (B) is verified. Now suppose that $\Delta < 0$. Then $\tilde D$ is nonempty, and for each $\eta \in \tilde D$, $Q_\eta > 0$, so that $\Omega > 0$. Let $\omega \in (0,\Omega)$. Since $\sup\limits_{Y\cap\tilde D} Q_\eta = \Omega$, there exists $\phi \in Y$ such that

$$\omega < Q_\phi \leq \Omega, \text{ and a } \xi(t) \in S \text{ such that } \xi_o = \phi\ ,$$

$$\dot\xi_o = \omega\phi+\psi\ , \text{ where } P\psi = 0\ . \text{ Set } \zeta(t) \equiv \xi(t)e^{-\omega t}\ .$$

Then $\zeta_o = \phi$, $\dot\zeta_o = \dot\xi_o-\omega\xi_o = \psi$, and Eq. (3.14) yields

$$\Delta\|\zeta(t)\|^2 \le (\phi, H_\omega \phi) \ , \qquad t \ge 0 \quad (3.15)$$

The quadratic function $g(\alpha) \equiv (\phi, H_\alpha \phi)$ is a strictly increasing function of α for $0 < \alpha < \infty$ and vanishes for $\alpha = Q_\phi$; thus $(\phi, H_\omega \phi) < 0$. We therefore conclude from Eq. (3.15) that

$$\|\xi\| = \|\zeta\| e^{\omega t} \ge \left[\frac{(\phi, H_\omega \phi)}{\Delta}\right]^{1/2} e^{\omega t} \ , \qquad t \ge 0 \ .$$

Thus the growth rate Ω can be approached arbitraily closely for some $\xi(t) \in S$. Finally, suppose that Ω is finite and let $\varepsilon > 0$. Since $\inf_D \frac{(\eta, [\omega P + K]\eta)}{(\eta,\eta)} > 0$ for $\omega > 0$, it follows that $\Delta_{\Omega+\varepsilon} = \inf_D \frac{(\eta, H_{\Omega+\varepsilon}\eta)}{(\eta,\eta)} > 0$. Let $\xi(t) \in S$ and set $\zeta(t) = \xi(t) e^{-(\Omega+\varepsilon)t}$. Then Eq. (3.14) gives

$$\|\xi\| = \|\zeta\| e^{(\Omega+\varepsilon)t} \le \left[\frac{(\dot{\zeta}_o, P\dot{\zeta}_o) + (\zeta_o, H_{\Omega+\varepsilon}\zeta_o)}{\Delta_{\Omega+\varepsilon}}\right]^{1/2} e^{(\Omega+\varepsilon)t}, \ t \ge 0 \ ,$$

which completes the proof.

The derivation of the energy principle given herein has the advantage of being free from any assumptions of completeness imposed on the eigenfunctions of the linear system; in fact, the results are valid for systems with no proper eigenfunctions. This is important in applications to systems with a continuous spectrum. We have basically made the much weaker assumption that the system (3.1) admits smooth solutions for smooth initial data, and do not require the existence of any solutions of the form $\xi(t) = \eta e^{\omega t}$, where η is independent of t. It should be clear that in general, Ω will not lie in the discrete spectrum, i.e., the theorem only guarantees that the growth rate Ω can be approached arbitrarily closely, but does not imply that it can actually be achieved.

IV. Applications

The energy and maximum principles of Theorem 1 are applicable to any system satisfying an equation of the form (3.1) and the associated hypothesis imposed in Sec. III. (It should be observed from the proof of Theorem 1 that relatively little of that hypothesis is required to prove exponential stability once $H \geq 0$ on D_H is known; the entire hypothesis was used, however, in the proof of the instability results and the maximum principle). There are two approaches to the rigorous application of the energy and maximum principles to a given problem. The first, and usually most difficult, requires an existence theorem guaranteeing the existence of the required smooth solutions for smooth initial data. The second approach, applicable to unstable systems where the maximal growth rate Ω lies in the discrete spectrum, is to demonstrate the existence of an eigenvector η (independent of t) such that $\xi(t) = \eta e^{\Omega t}$ is a solution of (3.1). This approach is valid for the resistive sheet pinch [2]. It is to be expected, however, that in most applications the investigator will simply assume that his system is well-behaved and that the energy and maximum principles apply. If the system is based on sound physical principles and the equilibrium data are sufficiently smooth, one would generally expect smooth solutions for smooth initial data. Then the only problem remaining is the choice of the domain $D = D_P \cap D_K \cap D_H$. A guiding principle here is to take D to be the "maximal" linear manifold satisfying the conditions that P,K, and H are all well-defined and formally self-adjoint on D (P is formally self-adjoint on D if and only if $(\eta, P\zeta) = (P\eta, \zeta)$ for all $\eta, \zeta \in D$) and that $P\eta$, $K\eta$, and $H\eta$ are reasonably smooth for all $\eta \in D$. Of course we require $P \geq 0$ and $K \geq 0$ on D. Returning to the problem discussed in Sec. II, we identify P with ρ_o, K with $-\nu\nabla^2$, and H with $-g \frac{d\rho_o}{dz} (\vec{e}_z \cdot)\vec{e}_z$. Due to the side condition (2.4) and the boundary condition $\vec{\xi} \equiv 0$ on ∂U, we take D to be the linear manifold of all vector functions $\vec{f}(\vec{x})$ such that $\nabla \cdot \vec{f} \equiv 0$ in U, $\vec{f} \equiv 0$ on ∂U, $\vec{f}$ is twice continuously differentiable in U, and the functions defined by the first and second partials of $\vec{f}$ can be extended to ∂U so that they are continuous on the closure of U. For $\eta = \vec{f} \in D$, we have $(\eta, H\eta) = -g \int_U \frac{d\rho_o}{dz} |f_z|^2 d^3x$; thus if $\frac{d\rho_o}{dz} \leq 0$ on U, $H \geq 0$ on D and we have exponential stability. If, on the other hand, $\frac{d\rho_o}{dz} > 0$ on some open sphere in U, then we can choose an $\eta \in D$ such that $(\eta, H\eta) < 0$, and we then "conclude" that the system is exponentially unstable with the maximal growth rate $\Omega = \sup_{D} Q_\eta$. The maximal growth rate Ω will of course depend on the viscosity ν, the mass density ρ_o, and the domain U.

The remainder of this section will be devoted to a brief discussion of the application of the energy principle to the resistive viscous, incompressible, magnetohydrodynamic sheet pinch. A detailed discussion can be found in [2]. (For an application to the electrohydrodynamic Rayleigh-Taylor bulk instability [9], see [4]).

We consider an infinite horizontal layer of an incompressible, viscous fluid satisfying the usual incompressible magnetohydrodynamic equations with a viscosity term added to the equation of motion, except for a simple "Ohm's Law" of the form $\vec{E} + \vec{v} \times \vec{B} = \eta\vec{J}$ and the addition of a conservation equation $\frac{\partial \eta}{\partial t} + \nabla\cdot(\eta\vec{v}) = 0$ for the resistivity η. The equilibrium quantities are assumed to be functions of the vertical coordinate z only, with the equilibrium fluid velocity identically zero, and the equilibrium magnetic field $\vec{B}_o(z)$ horizontal. The boundaries of the fluid (located at z = 0 and z = a) are assumed to be rigid, perfect insulators. The system equations require that the equilibrium electric field $\vec{E}_o$ be constant and horizontal, while $\vec{B}_o(z)$ and $\eta_o(z)$ are related by

$$\vec{B}_o(z) = \vec{B}_o(0) + \eta_o\vec{E}_o \times \vec{e}_z \int_0^z \eta_o^{-1}(u)\,du ,$$

where $\vec{B}_o(0)$ is a constant horizontal magnetic field and μ_o is the permeability of free space (mks units). The system equations are linearized about the equilibrium and the linearized variables are Fourier analyzed in the horizontal plane. After a great leal of algebra, the following 2×2 matrix equation is obtained, which determines the stability of the system:

$$P\ddot{\xi} + K\dot{\xi} + H\xi = 0 , \qquad (4.1)$$

where $\xi = \begin{pmatrix} \xi_1(z,t) \\ \xi_2(z,t) \end{pmatrix}$ with ξ_1 the Fourier coefficient of the z component of the perturbed displacement vector and ξ_2 the Fourier coefficient of the z component of the perturbed magnetic field; the 2×2 matrix operators P,K, and H have the form

$$P = \begin{pmatrix} L_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} L_2 & 0 \\ 0 & 0 \end{pmatrix} + B_1, \quad H = \begin{pmatrix} 0 & 0 \\ 0 & L_3 \end{pmatrix} + B_2$$

where L_1 and L_3 are second-order linear differential operators in z, L_2 is a fourth-order differential operator, and B_1 and B_2 are 2 × 2 Hermitian matrix operators whose elements are continuous functions of z on [0,a] (we assume that all equilibrium quantities are twice continuously differentiable functions of z on [0,a]). Consideration of the boundary conditions and smoothness requirements leads us to require that for each $t \geq 0$, $\xi_1(z,t) \in D_1 \equiv \{f(z) \mid f \in C^4[0,a],\ f(0) = f'(0) = f(a) = f'(a) = 0\}$ and $\xi_2(z,t) \in D_2 \equiv \{f(z) \mid f \in C^2[0,a],\ f'(a)+kf(a) = 0 = f'(0)-kf(0)\}$, where k denotes the magnitude of the horizontal wave number vector. Thus we take $D = D_1 \times D_2$, and find that P,K, and H are all formally self-adjoint on D with $P \geq 0$ and $K > 0$. The energy principle is applicable (here $F_\xi \equiv 0$), and the result is that unless η_o is a constant, the pinch ($\vec{E}_o \neq 0$) is always exponentially unstable (for sufficiently small k).

The theory of Sec. III leads us to expect that if the sheet pinch is unstable at the wave number k, then the maximal growth rate of perturbations with this wave number will be given by $\Omega(k) = \sup_{\tilde{D}} Q_\eta$. (The maximal growth rate for arbitrary disturbances, i.e., disturbances of arbitrary wave number, would then be given by sup $\Omega(k)$, where the supremum is over all k for which $\Omega(k) > 0$.) We now show that the maximal growth rate $\Omega(k)$ is actually achieved for the unstable (at wave number k) sheet pinch, i.e., we demonstrate the existence of a nonzero eigenvector $\psi \in D$ such that $\psi e^{\Omega t}$ satisfies Eq. (4.1). Let $k > 0$, and suppose that $\inf_D \frac{(\xi,H\xi)}{(\xi,\xi)} < 0$ (i.e., the system is unstable for wave number k). The operators L_1 and L_2 are strictly positive on D_1, and $B_1 \geq 0$ on $E \equiv \mathcal{L}_2[0,a] \times \mathcal{L}_2[0,a]$, so that $F(\omega) \equiv \inf_D \frac{(\xi,H_\omega\xi)}{(\xi,\xi)}$, $0 \leq \omega < \infty$, is strictly increasing on $[0,\infty)$. Now $\Omega = \sup_{\tilde{D}} Q_\eta > 0$, and we have $F(\omega) < 0$ on $[0,\Omega)$, $F(\Omega) \geq 0$. The operator L_3 has a positive compact Hermitian inverse K defined on $\mathcal{L}_2[0,a]$ such that

$K(\mathcal{L}_2[0,a]) \subset C[0,a]$, $L_3K = I$ on $C[0,a]$, and $KL_3 = I$ on D_2. For each $\omega > 0$, the operator $\omega^2 L_1 + \omega L_2$ has a positive compact Hermitian inverse K_ω defined on $\mathcal{L}_2[0,a]$ such that $K_\omega(\mathcal{L}_2[0,a]) \subset C[0,a]$, $(\omega^2 L_1 + \omega L_2)K_\omega = I$ on $C[0,a]$, $K_\omega(\omega^2 L_1 + \omega L_2) = I$ on D_1, and K_ω is continuous in ω on $(0,\infty)$. Thus for $\omega > 0$, $S_\omega \equiv \begin{pmatrix} \omega^2 L_1 + \omega L_2 & 0 \\ 0 & L_3 \end{pmatrix}$ with domain D admits the positive compact Hermitian inverse $T_\omega \equiv \begin{pmatrix} K_\omega & 0 \\ 0 & K \end{pmatrix}$ from E into E such that T_ω is continuous in ω, $T_\omega S_\omega = I$ on D, $S_\omega T_\omega = I$ on $C[0,a] \times C[0,a]$, and $T_\omega(E) \subset C[0,a] \times C[0,a]$. For $\omega > 0$, let $r_\omega \equiv T_\omega^{1/2}$, $B_\omega \equiv -\omega B_1 - B_2$, and $G(\omega) \equiv \inf_E \frac{(\phi,[I-r_\omega B_\omega r_\omega]\phi)}{(\phi,\phi)}$. Note that for $\zeta \in D$, $(\phi,[I-r_\omega B_\omega r_\omega]\phi) = (\zeta, H_\omega \zeta)$, where $\phi \equiv r_\omega S_\omega \zeta$. Therefore $F(\omega) < 0$ on $[0,\Omega)$ implies $G(\omega) < 0$ on $[0,\Omega)$, and since $\overline{r_\Omega S_\Omega(D)} = E$, $F(\Omega) \geq 0$ implies $G(\Omega) \geq 0$. It therefore follows from the continuity of $G(\omega)$ on $(0,\infty)$ that $G(\Omega) = 0$, i.e., $\sup_E \frac{(\phi, r_\Omega B_\Omega r_\Omega \phi)}{(\phi,\phi)} = 1$. The operator $r_\Omega B_\Omega r_\Omega$ is compact and Hermitian, so that there exists $\zeta \in E$, $\|\zeta\| = 1$, such that $\zeta = r_\Omega B_\Omega r_\Omega \zeta$. Hence $\psi \equiv r_\Omega \zeta = T_\Omega B_\Omega \psi \in C[0,a] \times C[0,a]$, so that $B_\Omega \psi \in C[0,a] \times C[0,a]$, which implies that $\psi = T_\Omega B_\Omega \psi \in D$. Therefore $S_\Omega \psi = S_\Omega T_\Omega B_\Omega \psi = B_\Omega \psi$, i.e., $H_\Omega \psi = 0$, and the proof is complete.

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Bibliography

1. Barston, E.M., Comm. Pure and Appl. Math. 22, 627 (1969).
2. Barston, E.M., Phys. Fluids 12, 2162 (1969).
3. Barston, E.M., J. Fluid Mech. 42, 97 (1970).
4. Barston, E.M., Phys. Fluids 13, 2876 (1970).
5. Bernstein, I.B., Frieman, E.A., Kruskal, M.D., and Kulsrud, R.M., Proc. Roy. Soc. A 244, 17 (1958).
6. Chandrasekhar, S., Hydrodynamic and Hydromagnetic Stability, Chap. 14 (Oxford University Press, 1961).

7. Chetaev, N.G., The Stability of Motion, Chap. 5 (Pergamon Press, London 1961).

8. Thompson, W., (Lord Kelvin), and Tait, P.G., Treatise on Natural Philosophy, Part I, Secs. 339-345 (Cambridge University Press, London, 1903).

9. Turnbull, R.J., and Melcher, J.R., Phys. Fluids 12, 1160 (1969).