A GUIDE TO LIE SUPERALGEBRAS*

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Abstract

We give an elementary presentation of the Lie superalgebras, their classification and some properties of their representations. A sketch of the classical Lie supergroup is also given.

Introduction

Lie superalgebras are also known in the literature as graded Lie algebras and pseudo Lie algebras. Before this colloquium Kac has convinced me to subscribe to the first name since already Kaplansky [1] did so, I gave in and I hope that other people will do the same.

Two other reviews of the subject are available [2,3]. The first one, by Corwin, Ne'eman and Sternberg, presents the situation in the field in the fall of 1974. The second one, by Kac is an excellent survey presented in a language which is not always accessible to physicists. Here we try to give a self contained catalogue of the main properties of superalgebras (for physical applications see Refs. [4,5]). I am aware that an enumeration of results (a detailed presentation could cover a book) makes the text hard to read and much of the beauty of the subject gets lost. It is very much like staying in Tübingen and reading a Paris Michelin guide without seeing the city.

Consider M generators Q_n $(n=1,2,\ldots M)$ and N generators R_α $(\alpha=1,\ldots N)$ that we can think of as matrices which satisfy the following commutation and anticommutation relations

$$[Q_m,Q_n] = f^p_{mn}Q_p \tag{1a}$$

$$[Q_m,R_\alpha] = F^\beta_{m\alpha}R_\beta \tag{1b}$$

$$\{R_\alpha,R_\beta\} = A^m_{\alpha\beta}Q_m \tag{1c}$$

where

$$[A,B] = AB-BA,\ \{A,B\} = AB+BA$$

The structure constants satisfy generalized Jacobi identities

$$f^p_{nr}f^q_{mp}+f^p_{rm}f^q_{np}+f^p_{mn}f^q_{rp}=0 \tag{2a}$$

$$F^\gamma_{n\alpha}F^\delta_{m\gamma}-F^\gamma_{m\alpha}F^\delta_{n\gamma} = f^p_{mn}F^\delta_{p\alpha} \tag{2b}$$

$$F^\delta_{m\gamma}A^n_{\beta\delta}+F^\delta_{m\beta}A^n_{\gamma\delta} = f^n_{mp}A^p_{\beta\gamma} \tag{2c}$$

$$A^p_{\beta\gamma}F^\delta_{p\alpha}+A^p_{\gamma\alpha}F^\delta_{p\beta}+A^p_{\alpha\beta}F^\delta_{p\gamma}=0 \tag{2d}$$

Equations (1) and (2) define a Lie superalgebra S. From Eqs. (1a) and (2a) we see that the Q_m generators define a Lie algebra S_0. We denote by S_1 the set of generators

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R_α; obviously

$$S = S_0 + S_1 \tag{3}$$

From Eqs. (1b) and (2) follows that R_α are tensor operators corresponding to a certain N-dimensional representation (in general reducible) of the Lie algebra S_0.

At a first glance the construction of a classification theory of superalgebras looks very difficult. One may start with a given Lie algebra (f^p_{mn} in Eq. (1a)), choose a representation in general reducible ($F^\beta_{m\alpha}$ in Eq. (1b)) and end up with an infinite number of possibilities for the coefficients $A^m_{\alpha\beta}$ in Eq. (1c). It took some time to understand that fortunately the structure of superalgebras is very similar to that of Lie algebras and in many respects all what we have to do is to generalize the concepts used for Lie algebras. When this job is done, we realize that Lie algebras can be looked upon as a special case of superalgebras.

Although superalgebras were known in mathematics for twenty years and rediscovered in physics in 1966 [20], most of the progress was made starting in 1974 when it was understood that superalgebras have major applications in physics. The content of the next sections will show that the problem of superalgebras is now essentially solved.

In Sec. 2 we define the Killing form metric, the Casimir operators and give a generalization of the Schur Lemma.

Simple superalgebras are defined in the same way as simple Lie algebras. In Sec. 3 we present all simple superalgebras (this corresponds to the Cartan classification of simple Lie algebras). It turns out that simple superalgebras belong to two classes. If the representation S_1 is completely reducible one gets the classical superalgebras; those are defined considering the algebra of matrices (19). There are essentially four series of classical superalgebras; ($sp\ell(m,n)$, $osp(m,n)$, $P(m)$ and $Q(m)$) and three exceptional ones ($F(4)$, $G(3)$ and $osp(4,2;\alpha)$). If the representation S_1 is not completely reducible one gets the Cartan type superalgebras ($W(n)$, $S(n)$, $\tilde{S}(n)$ and $H(n)$) which are defined using Fermi-Dirac creation and annihilation operators (see Eq. (30)).

Semisimple superalgebras (defined as S/I where S is a superalgebra and I the maximum solvable ideal (see Sec. 4)) can be expressed in terms of simple superalgebras. As opposed to the Lie algebra case, solvable superalgebras may have irreducible representation which are not one-dimensional.

The representation theory of simple superalgebras is given in Sec. 5. From all simple superalgebras only the representations of $osp(1,n)$ are completely reducible (the $osp(1,2)$ example is presented in detail). The irreducible representations can be labelled by the highest weight (like for simple Lie algebras) but the Casimir operators do not specify anymore the presentation (there are different irreducible representations corresponding to the same eigenvalues of the Casimir operators; this point is illustrated in the example of $spl(2,1)$). Finally the hermitian representations have their equivalent in the star and superstar representations.

Supergroups of linear transformations are defined in Sec. 6. The parameters of supergroups are elements of Grassman algebras. For matrices with both commuting and anticommuting elements we show how the trace, determinant, transpose and adjoint operations can be generalized. In this way we define the classical supergroups.

2. Some Properties of Superalgebras

It is convenient to introduce a compact notation for Eqs. (1) and (2). Let X_μ comprise the sets of generators Q_m (m=1,...M) and R_α (α=1,...N)

$$X_\mu = Q_m, R_\alpha \quad (\mu=1,\ldots, M+N)$$

Further, define the degree $g(X_\mu)$ of a generator by

$$g(Q_m) = 0, \; g(R_\alpha) = 1$$

If g = 0(1) the corresponding generator will be called <u>even</u> (<u>odd</u>). We define $\langle X_\mu, X_\nu \rangle$ by

$$\langle X_\mu, X_\nu \rangle = X_\mu X_\nu - (-1)^{g(X_\mu)g(X_\nu)} X_\nu X_\mu \tag{4}$$

With this notation the Eqs. (1) and (2) can be written as:

$$\langle X_\mu, X_\nu \rangle = c^\omega_{\mu\nu} X_\omega \qquad (c^\omega_{\mu\nu} = -(-1)^{g(X_\mu)g(X_\nu)} c^\omega_{\nu\mu}) \tag{5}$$

$$\langle X_\mu, \langle X_\nu, X_\omega \rangle\rangle (-1)^{g(X_\mu)g(X_\omega)} + \text{cyclical permutations} = 0 \tag{6}$$

A superalgebra S is <u>Z-graded</u> if it is decomposed into a direct sum of m+n+1 subspaces G_i (i=-n, -n+1, ... 0,1, ... m) such that

$$S = \bigoplus_i G_i, \quad \langle G_i, G_j \rangle \subset G_{i+j} \tag{7}$$

Obviously any superalgebra is <u>Z_2 graded</u> because of Eqs. (1) and (3) we have

$$\langle S_i, S_j \rangle \subset S_{i+j} \qquad (i,j = 0,1)$$

where $S_0 = \bigoplus_i G_{2i}$, $S_1 = \bigoplus_i G_{2i+1}$.

<u>Example 1.</u> Consider the Lie superalgebra

$$[Q, R_{\pm 1}] = 0; \quad \{R_{\pm 1}, R_{\pm 1}\} = 0 \qquad \{R_1, R_{-1}\} = Q \tag{8}$$

The generators have the following 2-dimensional irreducible representation:

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad R_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad R_{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{9}$$

we also have

$$S_0 = Q; \quad S_1 = R_1 \oplus R_{-1}; \quad G_{-1} = R_{-1}, \; G_0 = Q, \; G_1 = R_1$$

<u>Example 2.</u> Consider the Lie superalgebra:

$$[Q, R_{\pm 1}] = 0; \quad \{R_{\pm 1}, R_{\pm 1}\} = 0; \quad \{R_1, R_{-1}\} = Q \tag{10}$$

$$[Q, R_0] = 0, \quad \{R_0, R_{\pm 1}\} = 0, \quad \{R_0, R_0\} = Q$$

The generators have the following 4-dimensional irreducible representation

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad R_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \quad R_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad R_0 = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & -1 & 0 \\ 0 & -1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{11}$$

$$S_0 = Q; \quad S_1 = R_{-1} \oplus R_0 \oplus R_1$$

This example is interesting because although the representation is irreducible, there exists a matrix K

$$K = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix} \qquad (12)$$

which commutes with all the generators (11) although it is not a multiple of the unit matrix; thus for superalgebras the Schur lemma is not always valid.

The generators Q_m and R_α act in a Z_2-graded vector space $V = V_0 \oplus V_1$ having the general form

$$Q_m = \begin{pmatrix} A_m & 0 \\ 0 & D_m \end{pmatrix}; \quad R_\alpha = \begin{pmatrix} 0 & B_\alpha \\ C_\alpha & 0 \end{pmatrix} \quad ; \quad X_\mu = \begin{pmatrix} A_\mu & B_\mu \\ C_\mu & D_\mu \end{pmatrix} \qquad (13)$$

where A_m and D_m are square matrices, B_α and C_α are rectangular matrices. Q_m transforms an even (odd) vector into an even (odd) vector, R_α transforms an even (odd) vector into an odd (even) vector.

We define the supertrace of the generator X_μ:

$$\text{str}\, X_\mu = \text{tr}\, A_\mu - \text{tr}\, D_\mu \qquad (14)$$

The following properties can easily be shown:

$$\text{str}\,(<X_\mu, X_\nu>) = 0; \quad \text{str}(<X_\mu, X_\nu> X_\omega) = \text{str}(X_\mu <X_\nu, X_\omega>)$$

Let us consider the superalgebra (5) and let X_μ^{Ad} represent the matrices corresponding to the adjoint representation and X_μ^R the matrices corresponding to a certain representation R. We define the Killing form metric [6]

$$g_{\mu\nu} = \text{str}(X_\mu^{Ad}\, X_\nu^{Ad}) = c^\sigma_{\omega\mu}(-1)^{g(X_\omega)} c^\omega_{\sigma\nu} \qquad (15)$$

and the supertrace form metric corresponding to a certain representation R[7,8]:

$$g^R_{\mu\nu} = \text{str}(X^R_\mu X^R_\nu); \quad g^R_{\mu\nu} = (-1)^{g(X_\mu)g(X_\nu)} g^R_{\nu\mu} \qquad (16)$$

If $\det g^R_{\mu\nu} \neq 0$ we can construct Casimir operators [6,9]

$$K_n = \text{str}(X^R_{\sigma_1} \ldots X^R_{\sigma_n}) X^{\sigma_n} \ldots X^{\sigma_1} \qquad (X^\sigma = g^{\sigma\nu} X_\nu) \qquad (17)$$

$$[K_n, X_\mu] = 0$$

Before proceeding further let us observe that for Lie superalgebras the Schur Lemma generalizes as follows [3,10].

Let R be an irreducible representation of the Lie superalgebra S acting in a vector space $V = V_0 \oplus V_1$ and K a matrix which commutes with all the generators of S then either K is a multiple of the unit matrix or if $\dim V_0 = \dim V_1$, K can be a non-singular matrix which permutes V_0 and V_1 (see example 2).

3. Classification Of Simple Lie Superalgebras

A superalgebra S contains an ideal $I \subset S$ if

$$\langle X,Y\rangle \subset I \qquad (X \subset I,\ Y \subset S) \tag{18}$$

A superalgebra S is called simple if it contains no ideals. As opposed to simple Lie algebras for which the Killing form metric is nonsingular, for simple Lie superalgebras we encounter three possibilities: a) the Killing form metric is nonsingular, b) the Killing form metric is identically zero but the supertrace metric form is nonsingular, c) the supertrace metric form vanishes (one cannot find a representation R for which $g^R_{\mu\nu}$ defined by Eq. (16) does not vanish identically). We have thus to specify for each simple superalgebra which possibility applies.

Simple Lie superalgebras fall into two classes. The first one corresponds to the case when the odd generators R_α (see Eq. (1b)) correspond to a completely reducible representation of the Lie algebra S_0. In this case one can show that S_0 is reductive (semisimple plus abelian Lie algebras). The superalgebras belonging to this class are called classical superalgebras. For the superalgebras belonging to the second class and which are called Cartan type superalgebras the odd generators belong to a representation which is not completely reducible.

We now list the simple Lie superalgebras.

Classical superalgebras

Consider matrices of the form (13):

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{19}$$

where A is an m x m matrix and D an n x n matrix.

spl(m,n) $(m\neq n,\ m> n\geq 1)$ superalgebras are defined by:

$$\mathrm{tr}A = \mathrm{tr}D \tag{20}$$

There are $(m+n)^2-1$ generators, the Lie algebra S_0 is $sl(m)\oplus sl(n)\oplus gl(1)$ and $\det g_{\mu\nu}\neq 0$.

The spl(m,m) superalgebras are not simple but $spl(m,m)/z_m$ are. The center is

$$z_m = \lambda \begin{pmatrix} I_m & 0 \\ 0 & I_m \end{pmatrix}$$

where I_m is the m x m unit matrix.

There are $4m^2-2$ generators, $S_0=sl(m)\oplus sl(m)$, $g_{\mu\nu}\equiv 0$, $\det g^R_{\mu\nu}\neq 0$.

osp(m,n) $(m\geq 1,\ n=2p)$ superalgebras are defined by,

$$D^TG + GD = 0;\quad A^T = -A;\quad B = C^TG \tag{21}$$

where A^T is the transpose of A and

$$G = \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix} \tag{22}$$

There are $\frac{1}{2}[(m+n)^2+n-m]$ generators, $S_0 = o(m)\oplus sp(n)$, $\det g_{\mu\nu}\neq 0$ (except osp(n+2,n) when $g_{\mu\nu} \equiv 0$, det $g^R_{\mu\nu} \neq 0$); osp(2,2) is isomorphic to spl(2,1).

P(m) (m ≥ 3) superalgebras. Take m = n in Eq. (19) and

$$A^T + D = 0; \quad B = B^T; \quad C = -C^T; \quad \mathrm{tr}\, A = 0 \tag{23}$$

There are $2m^2-1$ generators, S_0 = sl(m), $g^R_{\mu\nu} \equiv 0$.

Q(m) (m ≥ 3) superalgebras. Take m = n in Eq. (19) and

$$A = D, \; B = C; \quad \mathrm{tr}\, B = 0 \tag{24}$$

These superalgebras ($\tilde{Q}(m)$) are not simple but they become simple if we divide them by z_m($Q(m) = \tilde{Q}(m)/z_m$ are simple). There are $2m^2-2$ generators, S_0 = sl(m) and $g^R_{\mu\nu} \equiv 0$. These Q(m) superalgebras are well known to physicists; they are the (f,d) algebras of Gell-Mann, Michel and Radicati [11] the commutation relations are:

$$\begin{aligned} [Q_\alpha, Q_\beta] &= f_{\gamma\alpha\beta} Q_\gamma \\ [Q_\alpha, R_\beta] &= f_{\gamma\alpha\beta} R_\gamma \\ \{R_\alpha, R_\beta\} &= d_{\gamma\alpha\beta} Q_\gamma \end{aligned} \tag{25}$$

where $\alpha, \beta, \gamma = 1,\ldots,m^2-1$, $f_{\alpha\beta\gamma}$ ($d_{\alpha\beta\gamma}$) are the usual totally skew-symmetric (symmetric) symbols.

The Exceptional Classical Superalgebras

F(4) superalgebra. Has forty generators. S_0=sl(2) + o(7), det $g_{\mu\nu} \neq 0$. We give the commutation relations: The even generators are $Q_i (1 \leq i \leq 3)$ and $\tilde{Q}_{pq} (1 \leq p,q \leq 7$; $\tilde{Q}_{pq} = -\tilde{Q}_{qp}$). The odd generators are $R_{\alpha\mu}$ ($\alpha=\pm 1$, $1 \leq \mu \leq 8$)

$$\begin{aligned} &[Q_j, Q_K] = i\varepsilon_{jK\ell} Q_\ell, \quad [Q_i, \tilde{Q}_{pq}] = 0 \\ &[\tilde{Q}_{pq}, \tilde{Q}_{rs}] = -\delta_{pr}\tilde{Q}_{qs} + \delta_{qr}\tilde{Q}_{ps} - \delta_{qs}\tilde{Q}_{pr} + \delta_{ps}\tilde{Q}_{qr} \\ &[Q_j, R_{\alpha\mu}] = \tfrac{1}{2}\tau^j_{\gamma\alpha} R_{\gamma\mu} \\ &[\tilde{Q}_{pq}, R_{\alpha\mu}] = \tfrac{1}{2} (\Gamma_p \Gamma_q)_{\nu\mu} R_{\alpha\nu} \\ &\{R_{\alpha\mu}, R_{\beta\nu}\} = 2\tilde{C}_{\mu\nu}(C\tau^j)_{\alpha\beta} Q_j + \tfrac{1}{3} C_{\alpha\beta}(\tilde{C}\Gamma_p\Gamma_q)_{\mu\nu}\tilde{Q}_{pq} \end{aligned}$$

where

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{26}$$

The Γ_p, $(1 \leq p \leq 7)$are a family of 8x8 matrices which satisfy $[\Gamma_p, \Gamma_q] = 2\delta_{pq}$ $\tilde{C}$ is the corresponding charge conjugation matrix with

$$\tilde{C}^T = \tilde{C}; \quad \Gamma_p^T \tilde{C} = -\tilde{C}\Gamma_p$$

For a convenient choice of Γ_p and $\tilde{C}$ see [12].

G(3) superalgebra. Has thirty one generators. S_0= sl(2) ⊕ G_2, det $g_{\mu\nu} \neq 0$.

The even generators are $Q_i (1 \leq i \leq 3)$ and $\tilde{Q}_{pq} (1 \leq p, q \leq 7)$, $\tilde{Q}_{pq} = -\tilde{Q}_{qp}$, $\xi_{pqr}\tilde{Q}_{pq} = 0$). The odd generators are $R_{\alpha p}$($\alpha = \pm 1$, $1 \leq p \leq 7$). ξ_{pqr} is a totally skew-symmetric G_2-invariant tensor. If (i,j,k) is one of the triples

$$(1,2,3),\ (1,4,5),\ (1,7,6),\ (2,4,6),\ (2,5,7),\ (3,4,7),\ (3,6,5) \tag{27}$$

then $\xi_{ijk} = 1$. If there is no permutation of (1,...,7) which transforms (i,j,k) into the triples (27) then $\xi_{ijk}=0$.
Commutation relations:

$$
\begin{aligned}
&[Q_j,Q_k]=i\varepsilon_{ijk}Q_k,\ [Q_j,\tilde{Q}_{pq}]=0\\
&[\tilde{Q}_{pq},\tilde{Q}_{rs}]=3\delta_{pr}\tilde{Q}_{qs}-3\delta_{qr}\tilde{Q}_{ps}+3\delta_{qs}\tilde{Q}_{pr}-3\ {}_{ps}\tilde{Q}_{qr}-\xi_{pqu}\xi_{rsv}\tilde{Q}_{uv}\\
&[\tilde{Q}_{pq},R_{\alpha r}]=2\delta_{pr}R_{\alpha q}-2\delta_{qr}R_{\alpha p}-\eta_{pqrs}R_{\alpha s}\\
&[Q_j,R_{\alpha p}]=\frac{1}{2}\tau^j_{\alpha'\alpha}R_{\alpha' p}\\
&\{R_{\alpha p},R_{\beta q}\} = 2\delta_{pq}(C\tau^j)_{\alpha\beta}Q_j-\frac{C_{\alpha\beta}}{2}\tilde{Q}_{pq}
\end{aligned}
\tag{28}
$$

where η_{ijpq} is totally skew-symmetric. $\eta_{ijpq} = 1$ if (i,j,p,q) is one of the quadruples

$$(1,2,4,7),\ (1,2,6,5),\ (1,3,6,4),\ (1,3,7,5),\ (2,3,4,5),\ (2,3,7,6),\ (4,5,7,6) \tag{29}$$

and $\eta_{ijpq}=0$ if there is no permutation of (1,...7) which transforms (i,j,p,q) in one of the quadruples (29).

<u>osp(4,2;α) superalgebras</u>. Have seventeen generators (as osp(4,2)) $S_0=s\ell(2)\oplus s\ell(2)\oplus s\ell(2)$

$g_{\mu\nu}\equiv 0$, det $g^R_{\mu\nu}\neq 0$. The even generators are Q^m_j ($1\leqslant m, j\leqslant 3$) and the odd generators are $R_{\alpha\beta\gamma}$ ($\alpha,\beta,\gamma=\pm 1$).
Commutation relations:

$$
\begin{aligned}
&[Q^m_j,Q^n_k]= i\delta_{mn}\varepsilon_{jk\ell}Q^m_\ell\\
&[Q^1_j,R_{\alpha\beta\gamma}] = \frac{1}{2}\tau^j_{\alpha'\alpha}R_{\alpha'\beta\gamma};\quad [Q^2_j,R_{\alpha\beta\gamma}] = \frac{1}{2}\tau^j_{\beta'\beta}R_{\alpha\beta'\gamma};\quad [Q^3_j,R_{\alpha\beta\gamma}]=\frac{1}{2}\tau^j_{\gamma'\gamma}R_{\alpha\beta\gamma'}\\
&\{R_{\beta\gamma\delta},R_{\alpha'\beta'\delta'}\} = \alpha_1C_{\gamma\gamma'}C_{\delta\delta'}(C\tau^j)_{\beta\beta'}Q^1_j+\alpha_2C_{\beta\beta'}C_{\delta\delta'}(C\tau^j)_{\gamma\gamma'}Q^2_j\\
&\qquad\qquad +\alpha_3C_{\beta\beta'}C_{\gamma\gamma'}(C\tau^j)_{\delta\delta'}Q^3_j
\end{aligned}
$$

where α_1, α_2, α_3 are arbitrary non-zero numbers which satisfy $\alpha_1+\alpha_2+\alpha_3=0$.
<u>Exercise</u>: Find the values of the parameters α_i for wich osp(4,2;α) is isomorphic to osp(4,2).

<u>Cartan type simple superalgebras</u> [3,13]

Consider 2n Fermi-Dirac creation and annihilation operators a_i and a^+_i (i=1,...n).

$$\{a^+_i,a^+_j\} = \{a_i,a_j\} = 0;\quad \{a^+_i,a_j\} = \delta_{ij} \tag{30}$$

Construct 2^n vectors

$$|0\rangle :\ a^+_1|0\rangle,\dots a^+_n|0\rangle;\ a^+_1a^+_2|0\rangle,\dots;\ a^+_1a^+_2\dots a^+_n|0\rangle\ (a_i|0\rangle=0) \tag{31}$$

<u>W(n) superalgebras (n⩾ 3)</u>.

$$W(n) = G_{-1}\oplus G_0\oplus G_1\oplus\dots\oplus G_{n-1}$$

$$\langle G_i,G_j\rangle \subset G_{i+j}$$

where

$$G_{-1} = a_i$$
$$G_0 = a_i^+ a_j$$
$$G_1 = a_i^+ a_j^+ a_k \qquad (i \neq j) \tag{32}$$
$$G_{n-1} = a_i^+ a_j^+ \ldots a_\ell^+ a_m \qquad (i \neq j \neq \ldots \neq \ell)$$

G_0 is isomorphic to $g\ell(n)$, $W(n)$ has $n2^n$ generators. $W(2)$ is isomorphic to $sp\ell(2,1)$.

S(n) superalgebras ($n \geq 3$).

$$S(n) = G_{-1} \oplus G_0 \oplus G_1 \oplus \ldots \oplus G_{n-2}$$

where

$$G_{-1} = a_i$$
$$G_0 = a_1^+ a_1 - a_j^+ a_j \qquad (j \neq 1)$$
$$a_i^+ a_j \qquad (i \neq j)$$
$$G_1 = a_i^+ (a_1^+ a_1 - a_j^+ a_j) \qquad (i \neq j \neq 1)$$
$$a_1^+ (a_2^+ a_2 - a_j^+ a_j) \qquad (j \neq 1,2) \tag{33}$$
$$a_i^+ a_j^+ a_k \qquad (i \neq j \neq k)$$
$$G_2 = a_i^+ a_j^+ (a_1^+ a_1 - a_k^+ a_k) \qquad (i \neq j \neq k \neq 1)$$
$$a_k^+ a_1 (a_2^+ a_2 - a_j^+ a_j) \qquad (k \neq j \neq 1,2)$$
$$a_1^+ a_2^+ (a_3^+ a_3 - a_k^+ a_k) \qquad (k \neq 1,2,3)$$
$$a_i^+ a_j^+ a_k^+ a_\ell \qquad (i \neq j \neq k \neq \ell) \text{ etc. } \ldots$$

G_0 is isomorphic to $s\ell(n)$, $S(n)$ has $(n-1)2^n+1$ generators.

$\tilde{S}(n)$ superalgebras ($n \geq 4$, n even).

These superalgebras are identical to the S(n) except for G_{-1}:

$$G_{-1} = (1 + a_1^+ a_2^+ \ldots a_n^+) a_i$$

G_0 is isomorphic to $s\ell(n)$, $\tilde{S}(n)$ has $(n-1)2^n+1$ generators.

H(n) superalgebras ($n \geq 4$).

$$H(n) = G_{-1} \oplus G_0 \oplus G_1 \oplus \ldots \oplus G_{n-3}$$

where

$$G_{-1} = a_i$$
$$G_0 = a_i^+ a_j - a_j^+ a_i \tag{34}$$
$$G_1 = a_i^+ a_j^+ a_k - a_i^+ a_k^+ a_j - a_k^+ a_i^+ a_j + a_j^+ a_k^+ a_i + a_k^+ a_i^+ a_j - a_k^+ a_j^+ a_i$$

etc. ...

G_0 is isomorphic to $so(n)$, $H(n)$ has 2^n-2 generators.

This completes the list of all simple superalgebras, the real simple superalgebras can be found in [3].

The simple Lie superalgebras with a nonsingular supertrace metric form have been found by Freund and Kaplansky [7], the classical superalgebras have been discussed by Nahm, Rittenberg and Scheunert [14,15] and all simple superalgebras have been found by Kac [16]. Very important contributions have been made by Djoković and Hochschild [17]. Here we give a table of the notations used by different authors.

Present Paper	[3]	[7]	[15]
$sp\ell(m,n)$	$A(m-1,n-1)$	$sp\ell(m,n)$	$sp\ell(m,n)$
$osp(m,n)$	$osp(m,n)$	$osp(n,m)$	$osp(n,m)$
$P(n)$	$P(n-1)$		$b(n)$
$Q(n)$	$Q(n-1)$		$d(n)/Z_n$
$F(4)$	$F(4)$		$s\ell(2) \times o(7)$
$G(3)$	$G(3)$		$s\ell(2) \times G_2$
$osp(4,2;\alpha)$	$D(2,1,\alpha)$		$s\ell(2) \times s\ell(2) \times s\ell(2)$

4. Semisimple Superalgebras

A superalgebra S is called solvable if

$$\langle S,S\rangle = S^{(1)};\ \langle S^{(1)},S^{(1)}\rangle = S^{(2)};\ldots \langle S^{(n-1)},S^{(n-1)}\rangle = S^{(n)} = 0 \qquad (35)$$

The superalgebras of examples 1 and 2 (see Eqs. (8) and (12)) are solvable. For solvable Lie algebras the only finite-dimensional irreducible representations are one-dimensional. This is not true anymore for solvable Lie superalgebras, one can show however the following theorems [3]:

a) Let $S=S_0 \oplus S_1$ be a solvable Lie superalgebra. All its irreducible representations are one-dimensional if and only if $\langle S_1,S_1\rangle \subset \langle S_0,S_0\rangle$.

b) Let $V=V_0\oplus V_1$ be the space of irreducible finite-dimensional representations of a solvable Lie algebra. Then either dim V_0=dim V_1 and dim $V=2^s$, $0<s\leq \dim S_1$ or dim V=1.

Consider now a superalgebra S which can be written as follows

$$\langle S^{(1)},S^{(1)}\rangle = S^{(1)};\ \langle S^{(1)},S^{(2)}\rangle = S^{(1)};\ \langle S^{(2)},S^{(2)}\rangle = S^{(1)} + S^{(2)}$$

then we define the quotient superalgebra $S^{(2)} \equiv S/S^{(1)}$ by dropping $S^{(1)}$ in the last commutation relation.

(if $\langle S^{(2)}, S^{(2)}\rangle = S^{(2)}$, $S = S^{(1)} \oplus_s S^{(2)}$, S is the semi-direct sum of $S^{(1)}$ and $S^{(2)}$; if $\langle S^{(2)},S^{(2)}\rangle = S^{(2)}$ and $\langle S^{(1)},S^{(2)}\rangle = 0$, $S=S^{(1)}\oplus S^{(2)}$, S is the direct sum of the Lie superalgebras $S^{(1)}$ and $S^{(2)}$).

For Lie algebras there are three equivalent definitions of semisimplicity:

a) If S_0 is a Lie algebra and I_0 the maximal solvable ideal then $\bar{S}_0=S_0/I_0$ is semisimple and $S_0=I_0 \oplus_s \bar{S}_0$.

b) If $\bar{S}_0$ is a Lie algebra whose metric tensor g_{mn} is nonsingular, then $\bar{S}_0$ is semisimple.

c) If $\bar{S}_0$ is a Lie algebra whose all finite-dimensional representations are completely-reducible, then $\bar{S}_0$ is semisimple.

For a semisimple Lie algebra $\bar{S}_0$ we have

$$\bar{S}_0 = \bigoplus_i S_0^{(i)}$$

where $S_0^{(i)}$ are simple.

For Lie superalgebras the three definitions stop being equivalent

a) If S is a Lie superalgebra and I the maximal solvable ideal than $\bar{S}=S/I$ is semi=simple [3]. The relations $S= I \oplus_s \bar{S}$ and $\bar{S}= \bigoplus_i S^{(i)}$ where $S^{(i)}$ are simple do not hold. A semisimple Lie superalgebra can be expressed in terms of simple superalgebras but the algorithm is more complicated [3,10].

b) If S is a Lie superalgebra and the killing form metric (15) is nonsingular, then $S=\bigoplus_i S^{(i)}$ where $S^{(i)}$ are simple superalgebras with det $g_{\mu\nu}\neq 0$ (simple Lie algebras and spl(m,n) ($m\neq n$), osp(m,n) ($m\neq n+2$), F(4), G(3)) [3,14].

c) If S is a Lie superalgebra and if all its finite-dimensional representations are completely reducible, then $S=\bigoplus_i S^{(i)}$ where $S^{(i)}$ are simple Lie algebras and osp (1,n) simple superalgebras [17]. (From all the simple Lie superalgebras only the osp(1,n) have the property that all the finite-dimensional representations are fully reducible.)

Let us now consider the superalgebra $S=S_0\oplus S_1$ and assume that its Lie algebra S_0 is semisimple, what can be said about the superalgebra S?

If $S=S_0\oplus S_1$ is a Lie superalgebra, with S_0 semisimple, then S is an elementary extension of a direct sum of Lie algebras or one of the Lie superalgebras $spl(m,m)/\mathbb{Z}_m$, osp(m,n) ($m\neq 2$), osp(4,2;α), F(4), G(3), Q(n),derQ(n) or $G(S_1,..S_r;L)$ [3].

(The Lie superalgebras der Q(n) and $G(S_1,..S_r;L)$ are defined in [3]. If $S=S_0\oplus S_1$ is a superalgebra with S_1 completely reducible, $S=S\oplus_s T$ is called its elementary extension if T is an odd commutative ideal and $\langle S_1,T\rangle=0$.)

5. Representations of Simple Lie Superalgebras

In the case of simple Lie algebras the finite-dimensional representations are completely reducible, the irreducible representations are equivalent to hermitian representations and they can be labelled either by the highest weight or by the eigenvalues of the Casimir operators. Which of these properties remain valid for simple Lie superalgebras? (We have seen already that we have complete-reducibility only for the osp(1,n) superalgebras.) We consider two examples and we will mention which properties are of general validity.

osp(1,n) superalgebra [6,18] This superalgebra is defined by the commutation relations:

$$[Q_3,Q_\pm]=\pm Q_\pm,\ [Q_+,\ Q_-]=\ 2Q_3;$$
$$[Q_3,R_\pm]=\ \pm\frac{1}{2}\ R_\pm\ ,\ [Q_\pm,R_\pm]=0,\ [Q_\pm,R_\mp]=\ R_\pm \tag{36}$$
$$\{R_\pm,R_\pm\}=\ \pm\frac{1}{2}\ Q_\pm;\ \{R_+,\ R_-\}=-\ \frac{1}{2}\ Q_3$$

This is a Z graded superalgebra $S=G_{-2}\oplus G_{-1}\oplus G_0\oplus G_1\oplus G_2$ with $G_{\pm 2}=Q_\pm$, $G_{\pm 1}=V_\pm$, $G_0=Q_3$.

We label the state vectors of an irreducible representation by $|\lambda,q,q_3>$

$$\vec{Q}^2|\lambda;q,q_3>=q(q+1)|\lambda;q,q_3>(\vec{Q}^2=Q_1^2+Q_2^2+Q_3^2,\ Q_\pm=Q_1\pm iQ_2)$$
$$Q_3|\lambda;q,q_3>\ =\ q_3|\lambda;q,q_3>\ (q=\lambda,\lambda-1/2;\ \lambda=0,\ 1/2,1,..,\ q_3=-q,..,q)$$

Thus the irreducible representation can be labelled by the highest weight $\underline{\lambda}$. This property is valid for all simple Lie superalgebras: the irreducible representations of simple Lie superalgebras can be labelled by the highest weight [3].

There is only one Casimir operator

$$K_2=\ \vec{Q}^2\ +\ R_+R_--R_-R_+$$

its eigenvalues are

$$K_2|\lambda;q,q_3>=\ \lambda(\lambda+1/2)|\lambda;q,q_3>.$$

The Clebsch-Gordan series reads

$$\lambda\otimes\lambda'\ =\ |\lambda-\lambda'|,\ |\lambda-\lambda'|+\ 1/2,\ \ldots,\ \lambda+\lambda'.$$

The Clebsch-Gordan coefficients are known explicitly [18] and the Wigner-Eckart theorem was proven [19] .

For a certain irreducible representation the generators are matrices which can be written in the block form

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

one block xorresponds to the $|\lambda,\lambda,q_3\rangle$ states, the other to the $|\lambda,\lambda-1/2,q_3\rangle$ states. We define the <u>superadjoint</u> (X^{s+}) of the matrix X:

$$X^{s+} = \begin{pmatrix} A^+ & -C^+ \\ B^+ & D^+ \end{pmatrix} \qquad (37)$$

where A^+ is the usual adjoint of a matrix A. We can then show that the irreducible representations of the osp(1,2) superalgebra are equivalent to <u>superstar</u> representations [19] for which

$$Q_m^+ = Q_m \ , \ V_+^{s+} = -V_- , \ V_-^{s+} = V_+ \qquad (38)$$

In the case of osp(1,2) the superstar representations represent the generalization of hermitian representations of Lie algebras.

<u>spl(2,1) superalgebra</u> [18] ,

$$\begin{aligned} &[Q_m,Q_n]=i\varepsilon_{mnp}Q_p, \ [Q_m,B] = 0 \quad (m=1,2,3) \\ &[Q_m,R_\alpha]= \tfrac{1}{2}\hat{\tau}^m_{\beta\alpha} R_\beta ; \ [B,R_\alpha]= \tfrac{1}{2}\hat{\varepsilon}_{\beta\alpha}R_\beta \quad (\alpha,\beta=1,2,3,4) \qquad (39) \\ &\{R_\alpha, R_\beta\}=(\hat{C}\tau^m)_{\alpha\beta}Q_m-(\hat{C}\hat{\varepsilon})_{\alpha\beta}B \end{aligned}$$

where the 4x4 matrices $\hat{\tau}^m$, $\hat{C}$ and $\hat{\varepsilon}$ are

$$\hat{\tau}^m = \begin{pmatrix} \tau^m & 0 \\ 0 & \tau^m \end{pmatrix} \ , \ \hat{C} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} \ , \ \hat{\varepsilon}= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (40)$$

τ^m and C are given by Eq. (36).

This superalgebra has been discovered by Stavraki in 1966! in a very different context [20].

The eigenstates are $|\lambda,\beta;q,q_3 b\rangle$

$$\begin{aligned} &B|\lambda,\beta;q,q_3,b\rangle= b|\lambda,\beta;q,q_3,b\rangle \\ &Q_3|\lambda,\beta;q,q_3,b\rangle= q_3|\lambda,\beta;q,q_3,b\rangle \qquad (41) \\ &\vec{Q}^2|\lambda,\beta,q,q_3,b\rangle= q(q+1)|\lambda,\beta;q,q_3,b\rangle \end{aligned}$$

(λ=0,1/2,1,..; β any real number, q_3=-q,..,q)

The generators act in a vector space $V=V_0 \oplus V_1$ where if $|\beta| \neq \lambda$; $|\lambda,\beta;\lambda,q_3,\beta>$, $|\lambda,\beta;\lambda-1,q_3,\beta>$ are the even vectors and $|\lambda,\beta;\lambda-1/2,q_3,\beta+1/2>$, $|\lambda,\beta;\lambda-1/2,q_3,\beta-1/2>$ are the odd vectors. We obviously have dim V_0=dimV_1=4 λ. Representations for which dim V_0=dim V_1 are called typical. If $\beta=\pm\lambda$ the even vectors are $|\lambda,\pm\lambda;\lambda,q_3,\pm\lambda>$ and the odd ones $|\lambda,\pm\lambda;\lambda-1/2,q_3, \pm(\lambda+1/2)>$. In this case dim $V_0 \neq$dim V_1 and the representations are called nontypical.

There are two Casimir operators

$$K_2= \vec{Q}^2-B^2+ \frac{1}{2} R\hat{C}R$$

$$K_3=BK_2+ \frac{1}{2} BR\hat{C}R + \frac{1}{6} R\vec{Q}\hat{\varepsilon}\vec{\hat{t}}\hat{C}R + \frac{1}{12} R\hat{\varepsilon}\vec{\hat{t}}\hat{C}R\vec{Q} \tag{42}$$

$$K_2|\lambda,\beta;q,q_3,b> = (\lambda^2-\beta^2)|\lambda,\beta;q,q_3 b>$$

$$K_3|\lambda,\beta;q,q_3,b> = \beta\,(\lambda^2-\beta^2)|\lambda,\beta;q,q_3 b>$$

From equation (42) we see that for typical representations ($\lambda^2 \neq \beta^2$) the eigenvalues of the Casimir operators are nonvanishing. If $\lambda^2=\beta^2$ both Casimir operators have zero eigenvalues. Thus in general the e.v. of the Casimir operators do not define the irreducible representations.

For all simple superalgebra (except osp(1,n)) the following property is valid: if a certain representation ρ can be brought into the block form

$$\begin{pmatrix} \rho_{11} & \rho_{12} \\ 0 & \rho_{22} \end{pmatrix}$$

and the ρ_{22} representation corresponds to a typical irreducible representation then $\rho_{12}=0$ [27].

This property can be checked in our example if we consider the Kronecker product of two (1/2,0) representations (λ=1/2, β=0). We have

$$(1/2,0) \otimes (1/2,0) = (1,0) \oplus \rho$$

ρ is a noncompletely reducible representation; the representation (1,0) is indeed typical.

The question arises if there are classes of irreducible representations for which the Kronecker product of two of them is completely reducible.

One can show that for $\beta \geq \lambda$, for all the representations (λ,β) one can choose a basis such that

$$Q_m^+ = Q_m,\ B^+= B,\ R_\alpha^+ = D_{\beta\alpha}R_\beta \tag{43}$$

where

$$D = \begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}$$

these representations form a class of star representations [19]. Another class of star representations (there are no more than two classes) contains representations for which $-\beta \geqslant \lambda$ and

$$Q_m^+ = Q_m \ , \ B_\alpha^+ = B, \ R_\alpha^+ = -D_{\beta\alpha} R_\beta \qquad (43')$$

The Kronecker product of two irreducible star representations belonging to the same class is completely reducible into irreducible star representations belonging to the same class. The star representations (like the superstar representations) represent a generalization of hermitian representations, for more details see [19].

For completeness we list here the dimension (N) of the representations of minimal dimension for the simple Lie superlagebras [21] .

spl(m,n), osp(m,n) (N=m+n); P(m) (N=2m); Q(m)
($N=2m^2-2$); W(n), S(n), $\tilde{S}$(n) ($N=2^n-1$); H(n) ($N=2^n-2$);
F(4) (N=40); G(3) (N=31); osp(4,2;α) (N=17)

Exercise: Find the values of the parameters α_i(i=1,2,3) for which the representation of minimal dimension of osp(4,2;α) is smaller then seventeen.

6. Supergroups

An extensive discussion of supergroups and supermanifolds was given by Kostant [22]. In the present paper we confine ourselves to a presentation of the classical supergroups [21,23,24,25] in a framework first introduced by Berezin [23].

For a usual Lie group the group elements are described by a set of parameters which are real numbers α_m , a multiplication rule is given

$$\alpha''_m = f_m(\alpha'_n,\alpha_p) \quad (f_m(0,\alpha_p)=\alpha_m)$$

and the Lie algebra describes the Lie group for small values of the parameters α_m. A similar situation occurs for supergroups with the difference that the parameters are elements of a Grassman algebra.

If $\theta_k(k=1,..p)$ are the generators of a Grassman algebra $(\theta_i\theta_k+\theta_k\theta_i= 0)$, a general element of the algebra has the form

$$a + a_k\theta_k+a_{ke}\theta_k\theta_e+ \dots+a_{123\dots p}\theta_1\theta_2\dots\theta_p$$

($a_{i\,j\dots\ell}$ are complex numbers).
If an element contains only even powers of the generators it is called even, (it commutes with the other elements), if an element contains only odd powers of the generators it is called odd (it anticommutes with odd elements).

A Lie superalgebra is obtained from a supergroup the same way a Lie algebra is obtained from a Lie group, the even generators corresponding to even parameters (which are even elements of a Grassmann algebra, not numbers!) and the odd generators to odd parameters (which are odd elements of the Grassman algebra).

We now consider supergroups of linear transformations. Consider matrices of the form

$$\mathcal{M} = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix} \qquad (44)$$

where $\mathfrak{A}$ is an m x m matrix whose elements $\mathfrak{A}_{ij}$ are even elements of a certain Grassman algebra, $\mathfrak{D}$ is an n x n matrix whose elements are even, $\mathfrak{B}$ and $\mathfrak{C}$ have matrix elements which are odd

$$\mathfrak{B}_{ij}\mathfrak{B}_{ke}+\mathfrak{B}_{ke}\mathfrak{B}_{ij}=0 \;;\mathfrak{C}_{ij}\mathfrak{C}_{ke}+\mathfrak{C}_{ke}\mathfrak{C}_{ij}=0$$

$$\mathfrak{B}_{ij}\mathfrak{C}_{ke}+\mathfrak{C}_{ke}\mathfrak{B}_{ij}=0 \;;\mathfrak{A}_{ij}\mathfrak{B}_{ke}-\mathfrak{B}_{ke}\mathfrak{A}_{ij}=0 \text{ etc.. .}$$

The matrices (44) which have an inverse define through the usual matrix multiplication a supergroup called the general linear supergroup GL(m,n).

In order to define subsupergroups of GL(m,n) it is useful to define the equivalent of the transpose and determinant for the matrices (44). We define the transpose ($\mathcal{M}^T$), supertranspose ($\mathcal{M}^{sT}$) and for m=n the P-transpose ($\mathcal{M}^P$) of $\mathcal{M}$:

$$\mathcal{M}^T = \begin{pmatrix} \mathcal{A}^T & \mathcal{C}^T \\ \mathcal{B}^T & \mathcal{D}^T \end{pmatrix} \quad ; \quad \mathcal{M}^{sT} = \begin{pmatrix} \mathcal{A}^T & -\mathcal{C}^T \\ \mathcal{B}^T & \mathcal{D}^T \end{pmatrix} \tag{45}$$

$$\mathcal{M}^P = \begin{pmatrix} \mathcal{D}^T & -\mathcal{B}^T \\ \mathcal{C}^T & \mathcal{A}^T \end{pmatrix} \quad (m=n) \tag{46}$$

notice that

$$(\mathcal{M}\mathcal{N})^T \neq \mathcal{N}^T\mathcal{M}^T; \quad (\mathcal{M}\mathcal{N})^{sT} = \mathcal{N}^{sT}\mathcal{M}^{sT}; \quad (\mathcal{M}\mathcal{N})^P = \mathcal{N}^P\mathcal{M}^P \tag{47}$$

We define the supertrace

$$\mathrm{str}\,\mathcal{M} = \mathrm{tr}\,\mathcal{A} - \mathrm{tr}\,\mathcal{D} \tag{48}$$

If $\mathcal{A}=\mathcal{D}$ and $\mathcal{B}=\mathcal{C}$ ($\mathrm{str}\,\mathcal{M}=0$) we define an ω-supertrace [21]

$$\mathrm{str}_\omega \mathcal{M} = \omega\, \mathrm{tr}\, \mathcal{B} \tag{49}$$

where ω is a fixed anticommuting element.

The superdeterminant is defined as follows: if

$$\mathcal{M}^{-1} = \begin{pmatrix} \mathcal{A}' & \mathcal{B}' \\ \mathcal{C}' & \mathcal{D}' \end{pmatrix}$$

$$\mathrm{sdet}\,\mathcal{M} = \det \mathcal{A} \det \mathcal{D}' = e^{\mathrm{str}\ \ln \mathcal{M}} \tag{50}$$

If $\mathcal{A}=\mathcal{D}$ and $\mathcal{B}=\mathcal{C}$ ($\mathrm{sdet}\,\mathcal{M}=1$) we define an ω-superdeterminant [21] .

$$\mathrm{sdet}_\omega \mathcal{M} = 1 + \frac{\omega}{2}\,\mathrm{tr}\ \ln\left[(\mathcal{A}-\mathcal{B})^{-1}(\mathcal{A}+\mathcal{B})\right] = e^{\mathrm{str}_\omega \ln \mathcal{M}} \tag{51}$$

notice that

$$\mathrm{sdet}\,(\mathcal{M}\mathcal{N}) = \mathrm{sdet}\,\mathcal{M}\,\mathrm{sdet}\,\mathcal{N}\,;\ \mathrm{sdet}_\omega(\mathcal{M}\mathcal{N}) = \mathrm{sdet}_\omega\mathcal{M}\,\mathrm{sdet}_\omega\mathcal{N} \tag{52}$$

We now define the equivalent of the adjoint operation of the matrix (44). In order to do so we have to define the complex conjugation operation for anticommuting objects. There are two ways to do it.

a) the (*) operation is defined:

$$(a\theta) = a^*\theta^*, \quad \theta^{**} = \theta; \quad (\theta_1\theta_2)^* = \theta_2^*\theta_1^* \tag{53}$$

(a is a complex number, θ are anticommuting objects)

b) the (x) operation is defined [24]:

$$(a\theta)^{x}= a^{*}\theta^{x},\ \theta^{xx}=-\theta,\ (\theta_1\theta_2)^{x}=\theta_1^{x}\theta_2^{x} \tag{54}$$

(notice the unusual property $\theta^{xx}=-\theta$)

The adjoint ($^{+}$) and the superadoint ($^{s+}$) of the matrix (44) are

$$\mathcal{M}^{+} = (\mathcal{M}^{T})^{*}\ ;\ \mathcal{M}^{s+}= (\mathcal{M}^{sT})^{x} \tag{55}$$

notice that

$$(\mathcal{M}\mathcal{N})^{+}=\mathcal{N}^{+}\mathcal{M}^{+};\ (\mathcal{M}\mathcal{N})^{s+}= \mathcal{N}^{s+}\mathcal{M}^{s+} \tag{56}$$

We now list the classical supergroups (we do not include the exceptional ones).

$$\underline{SPL(m,n)}:\quad \mathcal{M}\in GL(m,n)\ ;\quad \mathrm{sdet}\,\mathcal{M}=1 \tag{57}$$

$$\underline{OSP(m,n)}:\quad \mathcal{M}\in GL(m,n)\ (n=2p);\ \mathcal{M}^{sT}H\mathcal{M}= H \tag{58}$$

where

$$H =\begin{pmatrix} I_m & 0 \\ 0 & G \end{pmatrix} \tag{59}$$

where G is defined in Eq. (22).

$$\underline{P(n)}:\ \mathcal{M}\in GL(m,m),\mathcal{M}\mathcal{M}^{P}=1,\quad \mathrm{sdet}\mathcal{M}=1 \tag{60}$$

$$\underline{\tilde{Q}(n)}:\ \mathcal{M}\in GL(m,m)\ ;\mathcal{A}=\mathcal{D}\ ;\mathcal{B}=\mathcal{C}\ ;\quad \mathrm{sdet}_{\omega}\mathcal{M}=1 \tag{61}$$

In order to define the "compact" forms of the supergroups notice that there exists two unitary groups:

$$\underline{U(m,n)}\ :\ \mathcal{M}\in GL(m,n)\ ;\mathcal{M}\mathcal{M}^{+}=1 \tag{62}$$

$$\underline{sU(m,n)}:\ \mathcal{M}\in GL(m,n)\ ;\mathcal{M}\mathcal{M}^{s+}=1 \tag{63}$$

The "compact" forms of the supergroups SPL(m,n) in OSP(m,n) are

$$\underline{USPL(m,n)}\ :\mathcal{M}\in GL(m,n),\mathrm{sdet}\mathcal{M}=1,\mathcal{M}\mathcal{M}^{+}=1 \tag{64}$$

$$\underline{sOSP(m,n)}\ :\mathcal{M}\in GL(m,n)\ ;\mathcal{M}^{sT}H\mathcal{M}=H;\mathcal{M}\mathcal{M}^{s+}=1 \tag{65}$$

The theory of characters for supergroups can be found in [27] . The problem of the integral over a supergroup is considered in Refs. [28,29,23, 19] .

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