COMPOSITE PARTICLES
AND SYMPLECTIC (SEMI-) GROUPS

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1. Introduction

Interacting nucleons provide up to now one of the most interesting many-body systems. Stable nuclei exist from very small to very large mass numbers. Experimental nuclear physics has explored a great variety of nuclear excitations and brought about detailed knowledge of nuclear systems over wide regions of mass and energy. The theory of nuclear structure has developed along with the experimental studies from simple phenomenological models like the nuclear shell model to sophisticated many-body theories. For a recent account of this theory we refer to the book of Bohr and Mottelson /1/. The theory of nuclear reactions started from phenomenological models like the R-matrix theory which treats separate nuclear fragments as point particles outside the reaction region while allowing for many-body effects only inside this region. The more recent many-body reaction theories are devised from the introduction of appropriate boundary conditions into the full many-body system. The necessary reduction of the degrees of freedom is introduced through a variational procedure. For a presentation of this point of view we refer to the book of Wildermuth and Tang /2/. The so-called resonating group /2/ and the generator coordinate method /3/ are theories of nuclear composite particle interaction and dynamics. With the new data available from heavy-ion reactions one needs an extension of these theories to higher fragment mass numbers /4/. In what follows we shall present an attempt in this direction based on groups and semigroups and their representations in Hilbert space. It is claimed that this approach makes explicit many features used in these theories, provides powerful computational methods and gives new physical insight into composite particle dynamics.

Nuclear composite particle dynamics is intimately related to the fermion character of nucleons. This property is implemented in section 2 via the permutational structure of nuclear states, leading to the concept of exchange and to the quantum number of the orbital partition. In section 3 we review Weyl operators and representations of linear canonical transformations in Bargmann Hilbert space. For a more general discussion of the geometry of canonical transformations we refer to the contributions of Grossmann /5/ and Sternberg /6/. In section 4 we use canonical transformations to describe the general n-body dynamics. It proves necessary to consider semigroup extensions of linear canonical transformations for this purpose. In section 5 we derive the composite particle dynamics and discuss an algorithm to obtain the interaction of composite particles whose constituents are assumed to be in harmonic oscillator states. As a first example we treat in section 6 composite particles with unexcited internal oscillator states. In section 7 we deal with composite particles of internal oscillator shell configurations. The first steps towards the present approach are analyzed in /7/, a full elaboration will be given in /8/.

2. Permutational structure of nuclear states

Nuclear systems are made of fermions with spin and isospin of value one-half. We shall provide separately the orbital or coordinate states and the spin-isospin states with permutational symmetry. Suppose the orbital state transforms according to the irreducible representation of $S(n)$ characterized by the partition $f = [f_1 .. f_j]$ of n. Then to establish the overall partition $[1 \dots 1]$ of n characterizing fermion states we have to couple the orbital states with spin-isospin states characterized by the partition $\hat{f}$ associate to f. The orbital partition provides an orthogonal classification scheme of nuclear states. From the existence of four single-nucleon spin-isospin states it follows that $f_1 \leq 4$. The spin-isospin states may be handled by standard $SU(4)$ technique. From the major central part of the nuclear interaction it can be shown that the lowest orbital partitions f (with rows of maximal length) dominate.

The simplest orbital n-body states which one might consider are built from products of single-particle states. We shall see that for the present purpose it suffices to handle such states.

Suppose we have j different orbital single-particle states and start from a configuration ψ_w with occupation numbers $w = (w_1 \ldots w_j)$ which we call the weight. Clearly this configuration is stable with respect to elements of the group $S(w) = S(w_1) \times \ldots \times S(w_j)$, and applying coset generators c_i of this subgroup of $S(n)$ generates the representation induced by the identity representation of $S(w)$. The multiplicity in the reduction of this representation into irreducible parts may be labelled by a Gelfand scheme q /9/, that is, by applying in each step one factor of $S(w)$ and keeping all intermediate partitions. Explicitly this reduction reads

$$\psi_w \to c(r\,f\,q)\;\psi_w = \sum_i \left[\frac{f}{n!}\,w!\right]^{\frac{1}{2}} d^f_{rq}(c_i)\;c_i\;\psi_w$$

where d^f is the irreducible representation matrix of $S(n)$, f is the dimension of this representation, $w! = w_1!\,w_2!\,..\,w_j!$ and the sum runs over the cosets of $S(w)$. Using standard rules for the Young operators $c(r\,f\,q)$, the scalar products of two similar orbital states may be evaluated according to

$$\begin{aligned}&(c(r\,f\,q)\;\psi_w\,,\,c(r'f'q')\;\psi_{w'})\\ &= \delta(rr')\;\delta(ff')\left[\frac{n!}{f}\right]^{\frac{1}{2}}(\psi_w,\;c(q\,f\,q')\;\psi_{w'})\end{aligned}$$

where

$$c(q\,f\,q') = \left[\frac{f}{n!}\;\frac{1}{w!w'!}\right]^{\frac{1}{2}}\sum_p d^f_{qq'}(p)\;p$$

Now we introduce double cosets with respect to $S(w)$ and $S(w')$ and decompose any permutation as $p = h\,z_k\,h'$ where h and h' are elements of $S(w)$ and $S(w')$ while z_k generates the double coset. The decomposition is not unique, its multiplicity being given by

$$m(k) = k! = \prod_{i,\ell}(k_{i\ell})!$$

where the integers k_{ij} characterize the double coset. Introducing this decomposition one finally gets

$$\begin{aligned}&(c(r\,f\,q)\;\psi_w\,,\,c(r\,f\,q')\;\psi_{w'})\\ &= \left[w!\;\;w'!\right]^{\frac{1}{2}}\sum_k d^f_{qq'}(z_k)\;m^{-1}(k)\;(\psi_w\,,z_k\;\psi_{w'})\end{aligned}$$

The remaining scalar products in this sum are the basic exchange integrals. For a detailed description of the double coset concept and its correspondence to exchange we refer to /10/. For the special case of states $\psi_w , \psi_{w'}$ built as products of single-particle states these scalar products factorize into products of overlap integrals $\varepsilon_{i\ell}$ between single-particle states

$$(\psi_w , z_k \ \psi_{w'}) = \varepsilon^k = \prod_{i\ell} (\varepsilon_{i\ell})^{k_{i\ell}}$$

Considering the scalar product discussed above as a function D^f of the matrix ε yields

$$D^f_{qq'}(\varepsilon) = [w!w'!]^{\frac{1}{2}} \sum_k d^f_{qq'}(z_k) \ m^{-1}(k) \ \varepsilon^k \quad .$$

and we claim:

Theorem: The matrices D^f form the irreducible representation of the group $GL(j, \mathbb{C})$ for the element ε in the canonical form characterized by Gelfand schemes. This representation is unitary when restricted to the subgroup $U(j) < GL(j, \mathbb{C})$.

A proof of this theorem may be based on the observation that the Young operators, in the form chosen above, yield a unitary reduction of the induced representation /8/. By application of this theorem one gets explicit expressions for the scalar products of orbital states in terms of the representations $D^f(\varepsilon)$. These representations are available from combining two approaches: Louck /11/ has shown that the representations D^f are proportional to double Gelfand polynomials, and these polynomials for highest weights have been obtained by Moshinsky /9/ while the polynomials of other weights may be constructed by application of lowering operators as derived by Moshinsky and Nagel /12/. From /9/ one finds with a change of notation for highest weights $w_{max} = w'_{max}$ and corresponding Gelfand labels $q_{max} = q'_{max}$ that

$$D^f_{q_{max} q_{max}}(\varepsilon) = (\Delta_{1,1})^{f_1 - f_2} (\Delta_{12,12})^{f_2 - f_3} (\Delta_{1..j-1,1..j-1})^{f_{j-1} - f_j} (\Delta_{1..j,1..j})^{f_j}$$

where the symbol $\Delta_{1..i,1..i}$ denotes the determinant of the $i \times i$ submatrix of ε containing the first i rows and columns. For later use, we particularize this expression to the cases

$$D^{[4^j]}_{q_{max} q_{max}}(\varepsilon) = (\Delta_{1..j,1..j})^4$$

and

$$D^{[4^{j-1}1]}_{q_{max} q_{max}}(\varepsilon) = (\Delta_{1..j-1,1..j-1})^3 (\Delta_{1..j,1..j})$$

The relations described so far suffice to determine scalar products between configurations of non-orthogonal single-particle states. If we wish to determine the matrix elements of two-body interactions, the decomposition of the n-body states into products of states of n-2 and 2 particles respectively allows one to reduce the n-body matrix elements to two-body matrix elements. This decomposition requires the use of Wigner coefficients of the unitary group in the Gelfand scheme. For the use of this concept in composite particle theory we refer to /8/.

3. Weyl operators and linear canonical transformations in Bargmann Hilbert space

In the Schrödinger representation of the canonical commutation relations for m degrees of freedom one uses a Hilbert space of square integrable functions $\varphi(\xi)$ of m real variables $\xi = (\xi_1 \xi_2 .. \xi_m)$. Following the notation of Bargmann /13/, translations α' of position and α'' of momentum in phase space are unitarily and irreducibly represented by

$$\alpha = (\alpha' \alpha'') \to T_\alpha \quad , \qquad T_\alpha \text{ being the Weyl operator}$$

$$T_\alpha : \varphi \to T_\alpha \varphi \ , \quad (T_\alpha \varphi)(\xi) = \exp\left[-i\, \alpha'' \cdot (\xi - \tfrac{1}{2}\alpha')\right] \varphi(\xi - \alpha')$$

The Weyl operators multiply according to the Weyl relations

$$T_\alpha \circ T_\beta = \exp \tfrac{1}{2} i \{\alpha, \beta\} \, T_{\alpha+\beta}$$

where $\{\alpha,\beta\}$ is the real symplectic form

$$\{\alpha, \beta\} = \alpha \cdot K \beta \ , \qquad K = \begin{bmatrix} 0 & 1_m \\ -1_m & 0 \end{bmatrix}$$

which characterizes the geometry of classical phase space. The real symplectic form $\{\ ,\ \}$ is preserved by real symplectic transformations $g \in Sp(2m, \mathbb{R})$. It follows that the operators $T_{g\alpha}$, $T_{g\beta}$ provide a pair obeying the Weyl relations with the same factor as T_α, T_β. Now a theorem of von Neumann states that two irreducible unitary sets of operators obeying the Weyl relations are equivalent and related by a unitary operator S determined up to a phase factor. In the present case $S = S_g$ must then obey

$$S_g^{-1} \circ T_{g\alpha} \circ S_g = T_\alpha$$

and the operators S_g form a ray representation of $Sp(2m, \mathbb{R})$.

The explicit form of the operatos S_g has been derived from the canonical commutation relations by Moshinsky and Quesne /14/. Bargmann /13/ constructed a Hilbert space F_m of entire functions $f(z)$ of m complex variables $z = (z_1 z_2 .. z_m)$ which yields the Fock representation of the canonical commutation relation obeyed by oscillator annihilation and creation operators. The scalar product in F_m is defined as

$$(f,g) = \int \overline{f(z)}\; g(z)\; d\mu(z)$$

$$d\mu(z) = \Pi^{-m} \exp(-z\cdot\bar{z}) \prod_{i=1}^{m} d\,\mathrm{Re}(z_i)\, d\,\mathrm{Im}(z_i).$$

Moreover he constructed a unitary mapping between H_m and F_m by the integral transform

$$f(z) = \int A(z,\xi)\; \varphi(\xi)\; d\,\xi$$

$$\varphi(\xi) = \int \overline{A(z,\xi)}\; f(z)\; d\mu(z)$$

where

$$A(z,\xi) = \Pi^{-\frac{1}{4}m} \exp\left[-\frac{1}{2}\,\xi\cdot\xi - \frac{1}{2}\,z\cdot z + \sqrt{2}\;\xi\cdot z\right]$$

The Weyl operators in Bargmann space F_m become operators W_{α_c} with the property

$$W_{\alpha_c} : f \to W_{\alpha_c} f\,, \quad (W_{\alpha_c} f)(z) = \exp(\bar{a}\cdot(z - \frac{1}{2}a))\; f(z-a)$$

where $\alpha_c = \binom{a}{\bar{a}}$ is related to $\alpha = \binom{\alpha'}{\alpha''}$ by

$$\alpha_c = R\,\alpha\,, \quad R = \sqrt{\tfrac{1}{2}} \begin{bmatrix} 1_m & i\,1_m \\ 1_m & -i\,1_m \end{bmatrix}$$

The Weyl relations become

$$W_{\alpha_c} \circ W_{\beta_c} = \exp(-\frac{1}{2}\{\alpha_c, \beta_c\})\; W_{\alpha_c+\beta_c}$$

One derives from the properties of R that

$$\{\alpha,\beta\} = i\{\alpha_c,\beta_c\}$$

$$\{\bar{\alpha},\beta\} = i(\alpha_c/\beta_c)$$

where

$$(\alpha_c/\beta_c) = \bar{\alpha}_c \cdot M\beta_c \quad , \qquad M = \begin{bmatrix} 1_m & 0 \\ 0 & -1_m \end{bmatrix}$$

Conversely, if g_c preserves both { , } and (/), it belongs to the group $Sp(2m, \mathbb{C}) \cap U(m,m)$ and can be shown to be equivalent to an element of $Sp(2m, \mathbb{R})$. Again by von Neumann's theorem, W_{α_c} and $W_{g_c\alpha_c}$ are related by

$$S_{g_c}^{-1} \circ W_{g_c\alpha_c} \circ S_{g_c} = W_{\alpha_c}$$

In Bargmann space, operators may be written as integral operators with kernels $K(z,z')$. In particular, the identity operator is

$$S_e(z,z') = \exp(z \cdot \bar{z}')$$

Of particular interest are the coherent states

$$e_a(z) = \exp z \cdot \bar{a}$$

which upon transforming back to H_m become Gaussian states whose average position and momentum are determined by the real and imaginary part of a. Noting the relation of the coherent states to the identity operator one easily finds that the kernel of any operator K may be obtained as a scalar product

$$K(z,z') = (e_z, K e_{z'})$$

The kernels of W_{α_c} and S_{g_c} are derived by Bargmann and read

$$W_{\alpha_c}(z,z') = \exp\left[z\cdot\bar{a} + z\cdot\bar{z}' - a\cdot\bar{z}' - \frac{1}{2} a\,\bar{a}\right]$$

Finally the operators S_{g_c} have the kernel

$$S_{g_c}(z,z') = \left[\det C^{-1}\right]^{\frac{1}{2}} \exp\left[\frac{1}{2}z \cdot Az + z \cdot C\bar{z}' + \frac{1}{2}\bar{z}' \cdot B\,\bar{z}'\right]$$

where

$$g_c = \begin{bmatrix} \lambda & \mu \\ \nu & \rho \end{bmatrix} = \begin{bmatrix} 1_m & 0 \\ A & 1_m \end{bmatrix} \begin{bmatrix} {}^tC^{-1} & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} 1_m & -B \\ 0 & 1_m \end{bmatrix}$$

and ${}^tA = A$, ${}^tB = B$ guarantees that all factors are complex symplectic (but in general do not belong to $U(m,m)$). An equivalent result was derived in /7/ by passing from H_m to F_m .

We shall require a factorization of Weyl operators acting on finite dimensional linear subspaces L^N of F_m . We denote by L^N the polynomial states of degree up to N or in other words, all linear combinations of oscillator states up to excitation N . In addition we introduce the linear subspaces $W_{\alpha_c} L^N$, $W_{\beta_c} L^N$ as the images of L^N under W_{α_c} or W_{β_c} .

Then we define /8/ :

$$V_a : f \to V_a f \ , \ (V_a f)(z) = f(z-a)$$

$$\Lambda_a : f \to \Lambda_a f \ , \ (\Lambda_a f)(z) = (W_{\alpha_c} \circ V_{-a} f)(z)$$
$$= \exp(z \cdot \bar{a}) \ f(z)$$

and claim the following properties for V_a , Λ_a :

V_a is a bounded operator on the domain $W_{\beta_c} L^N$ with range $W_{\beta_c} L^N$.

Two operators V_a , V_b multiply as $V_a \circ V_b = V_{a+b}$.

The inverse of V_a is $V_a^{-1} = V_{-a}$.

Λ_a is a bounded operator on the domain $W_{\beta_c} L^N$ with range $W_{\alpha_c+\beta_c} L^N$.

Two operators Λ_a , Λ_b multiply as $\Lambda_a \circ \Lambda_b = \Lambda_{a+b}$.

The inverse of Λ_a is $\Lambda_a^{-1} = \Lambda_{-a}$.

The next properties apply to combinations of Λ and V :

$$V_a \circ \Lambda_b = \Lambda_b \circ V_a \quad \exp(-a \cdot \bar{b})$$

$$W_{\alpha_c} = \Lambda_a \circ V_a \quad \exp(-\tfrac{1}{2} a \cdot \bar{a})$$
$$= V_a \circ \Lambda_a \quad \exp(\tfrac{1}{2} a \, \bar{a})$$

$$\Lambda_a^{+} = V_{-a} \ , \quad V_a^{+} = \Lambda_{-a}$$

$$\Lambda_a \circ V_b \circ \Lambda_{a'} \circ V_{b'} = \Lambda_{a+a'} \circ V_{b+b'} \ \exp(-b \cdot \bar{a}')$$

All these relations are to be understood with appropriate arrangements of range and domain. The proof /8/ is essentially based on the fact that V_a applied to a polynomial cannot raise the degree.

Since in later applications we shall use Weyl operators up to a factor, we introduce them by the definition

$$\hat{W}_{\alpha_c} = W_{\alpha_c} \exp \frac{1}{2} a \cdot \bar{a} = \Lambda_a \circ V_a$$

4. Canonical transformations for interacting n-body systems

As a first step towards a theory of composite particles we shall introduce appropriate coordinates which will always be taken by linear orthogonal transformations from the single-particle coordinates ξ . The same orthogonal transformation applies to the momenta and the corresponding $2n \times 2n$ matrix in phase space is

$$g(B) = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \qquad {}^{t}B\,B = 1_n$$

For complex coordinates in Bargmann space this matrix becomes

$$g_c(B) = R\,g(B)\,R^{-1} = g(B) = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$$

and hence (for orthogonal B!) the complex coordinates transform as the real ones. As an example we shall consider a splitting of the n particles into two sets of n_1 and n_2 particles. If we introduce standard orthogonal Jacobi coordinates for both sets, we may permute them by a matrix Q to get the two center coordinates

$$z_1 = [n_1]^{-\frac{1}{2}} \sum_{1}^{n_1} x_i \quad , \qquad z_2 = [n_2]^{-\frac{1}{2}} \sum_{n_1+1}^{n} x_i$$

plus $n-2$ additional relative coordinates y . We write the orthogonal transformation as

$$x = J\,Q \binom{z}{y} \quad .$$

From these new coordinates we may pass to relative coordinates s between the centers and maintain the coordinates y ,

$$\binom{z}{y} = W \binom{s}{y} .$$

For two centers the relevant part of W is described by

$$s_1 = \left[\frac{n_2}{n}\right]^{\frac{1}{2}} z_1 - \left[\frac{n_1}{n}\right]^{\frac{1}{2}} z_2$$

$$s_2 = \left[\frac{n_1}{n}\right]^{\frac{1}{2}} z_1 + \left[\frac{n_2}{n}\right]^{\frac{1}{2}} z_2$$

and s_2 is the overall c.m. vector. To transform states from single-particle to center or cluster plus internal coordinates, we employ the operator S_{g_c} to get

$$\ell(x) = (S_{g_c^{-1}(JQ)}\, \ell)(zy) = (S_{g_c^{-1}(JQW)}\, \ell)(sy)$$

where for orthogonal B the kernel of $S_{g_c(B)}$ is found to be

$$S_{g_c(B)}(z,z') = \exp \sum_{i,j}^{m} z_i B_{ij} \bar{z}'_j$$

To project states of permutational symmetry we shall employ Young operators $c(r\,f\,q)$. Since in single-particle coordinates a permutation is represented by an orthogonal permutation matrix P , we get the Young operators in the form

$$c(r\,f\,q) = \left[\frac{f}{n!}\,\frac{1}{w!}\right]^{\frac{1}{2}} \sum_{p\in S(n)} d^{f}_{rq}(p)\, S_{g_c(P)}$$

Let us analyze in one dimension the complex symplectic 2 x 2 matrix

$$g = \begin{bmatrix} 1 & 0 \\ i\,q & 1 \end{bmatrix}$$

By constructing formally the corresponding operator in H_1 we obtain the bounded Gaussian operator

$$S_g(\xi,\xi') = \exp(-\tfrac{1}{2} q\, \xi^2)\ \delta(\xi-\xi')$$

This allows us to formally describe a Gaussian interaction between particles $n-1$ and n in terms of the 2 x 2 matrix

$$J'' = \sqrt{\tfrac{1}{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

and the n x n matrices

$$F = \begin{bmatrix} 1_{n-2} & 0 \\ 0 & J'' \end{bmatrix}, \qquad C = \begin{bmatrix} 0_{n-1} & 0 \\ 0 & q \end{bmatrix}$$

by a symplectic matrix

$$\begin{bmatrix} {}^tF & 0 \\ 0 & {}^tF \end{bmatrix} \begin{bmatrix} 1_n & 0 \\ i\,C & 1_n \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix} = \begin{bmatrix} 1_n & 0 \\ i\,{}^tFCF & 1_m \end{bmatrix} = h$$

The matrix h may be transformed by the matrix R into a matrix h_c which is symplectic but does not belong to $U(n,n)$. Decomposing h_c in a similar fashion as g_c allows one to define three blocks A , B , C and to formally write down a kernel in Bargmann space. The question how these manipulations may be justified was studied in /15/16/ and we briefly mention the main results. On a $2n$ dimensional linear complex space equipped with the scalar product (/) mentioned in section 3, one may introduce the length increasing semigroups $U^{>}(n,n)$ and $U^{\geq}(n,n)$ respectively by the condition that the elements j and j' of these semigroups obey respectively the inequalities

$$(j\,\xi/j\,\xi) - (\xi/\xi) > 0$$
$$(j'\xi/j'\xi) - (\xi/\xi) \geq 0$$

for any element $\xi \neq 0$ of the linear space. The geometry and factorization of such semigroups is developed in /15/.

If now one defines the semigroups

$$U^{>}(n,n) \cap Sp(2n, \mathbb{C}) \quad \text{and} \quad U^{\geq}(n,n) \cap Sp(2n, \mathbb{C})$$

it is shown in /16/ that the kernels corresponding to the first semigroup form a representation by Hilbert-Schmidt operators while the kernels corresponding to the second semigroup form a representation by bounded operators.

These results have an immediate application to the many-body system since they allow the interpretation of Gaussian interactions as representations of the second semigroup. Combining all the steps we may write the operator of a Gaussian two-body interaction between projected n-body states depending on center coordinates in the form

$$S_{g_c^{-1}(J\,Q)} \circ c(q\,f\,r) \circ S_k \circ c(r'f'q') \circ S_{g_c(J'Q')}$$

This expression may be expanded into a sum of terms each of which is the representation of a semigroup element.

5. Dynamics of interacting composite systems

The methods developed in the last section allow us to express general two-body interactions in an n-body system in terms of operators acting on functions of the coordinates (z,y) or, with simple modifications, of (s,y). The variational derivation of the dynamics of composite particles is based on the assumption that a reasonable but fixed choice of the state dependence on the internal coordinates y be made. A possible choice of these internal states would be eigenstates of the corresponding internal Hamiltonian. We shall work with the much simpler assumption that the internal states are described by harmonic oscillator configurations. This assumption may be justified by the observation that these configurations describe indeed a reasonable fraction of a nuclear state when effective nucleon-nucleon forces are employed /17/. It offers the great advantage that one obtains closed expressions for the composite particle interactions as we shall see in this and the following sections.

The dynamics of j composite particles will be written in a Hilbert space F_j. The projection from F_n to F_j will be obtained by a variational principle from variational states which in the $z\,y$ coordinates are assumed in the form

$$(S_{g_c^{-1}(JQ)}\, \ell)(z\,y) = u(z)\, v(y)$$

subject to the restricted variation

$$(\delta S_{g_c^{-1}(JQ)}\, \ell)(z\,y) = (\delta u)(z)\, v(g) .$$

For simplicity we consider the case $j = 2$ of two composite particles. The variational principle for stationary states

$$(\delta\; c(r\,f\,q)\, \ell \;,\; (H - E) \circ c(r\,f\,q')\, \ell\;) = 0$$

leads to an integral equation for $u(z)$ of the form

$$\int \left[\mathcal{H}(z,z') - E\, \mathcal{N}(z,z') \right] u(z')\, d\mu(z') = 0$$

The kernels $\mathcal{H}(z,z')$ and $\mathcal{N}(z,z')$ act in F_2 but may be obtained, comparing the remarks of section 3, as scalar products in F_n upon replacing

$u(z')\, v(y)$ by $e_{z'}(z)\, v(y)$ in the form

$$\mathcal{H}(z,z') = (\ c(rfq)\circ S_{g_c(JQ)}\ (e_z v), H \circ c(rfq')\circ S_{g_c(JQ)}\ (e_{z'} v))$$

$$\mathcal{N}(z,z') = (\ c(rfq)\circ S_{g_c(JQ)}\ (e_z v),\ c(rfq')\circ S_{g_c(JQ)}\ (e_{z'} v))$$

Note that these operators incorporate both interactions and orbital projections. We call $\mathcal{H}$ the interaction and $\mathcal{N}$ the normalization operator.

We shall now propose a method which allows the evaluation of the kernels $\mathcal{H}\ (z,z')$ and $\mathcal{N}(z,z')$ by integration over n single-particle coordinates. Assume that an n-particle state is built from the occupation of single-particle states for the first and second set of particles respectively. To this state we apply a Weyl operator W_{τ_c} where

$$\tau'_c = \binom{t'}{\bar{t}'}$$

and

$$t' = \begin{bmatrix} t'_1 \\ \vdots \\ t'_1 \\ t'_2 \\ \vdots \\ t'_2 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} t \\ t \\ t \end{matrix}}\right\} n_1 \\ \left.\vphantom{\begin{matrix} t \\ t \\ t \end{matrix}}\right\} n_2 \end{matrix}$$

The translation τ_c may be passed to the coordinates (z,y) by computing

$$g_c^{-1}\ \tau'_c = \begin{bmatrix} (JQ)^{-1}\ t' \\ (JQ)^{-1}\ \bar{t}' \end{bmatrix}$$

$$(JQ)^{-1}\ t' = \begin{bmatrix} \sqrt{n_1}\quad t'_1 \\ \sqrt{n_2}\quad t'_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} t \\ t \end{matrix}}\right\} 2 \\ \left.\vphantom{\begin{matrix} t \\ t \\ t \end{matrix}}\right\} n-2 \end{matrix}$$

From the operator relations given in section 3 we have

$$S_{g_c^{-1}(JQ)} \circ \hat{W}_{\tau'_c} = \hat{W}_{g_c^{-1}\tau_c} \circ S_{g_c^{-1}(JQ)}$$

Therefore the states in (zy) coordinates become

$$(\hat{W}_{\tau'_c}\ \ell)(x) = (S_{g_c^{-1}(JQ)} \circ \hat{W}_{\tau'_c}\ \ell)(z\ y)$$

$$= (\hat{W}_{g_c^{-1}\tau'_c} \circ S_{g_c^{-1}(JQ)}\ \ell)(z\ y)$$

$$= (S_{g_c^{-1}(JQ)}\ \ell)(z_1 - \sqrt{n_1}t'_1,\ z_2 - \sqrt{n_2}t'_2,\ y)\ \times$$

$$\times\ \exp(z_1\ \sqrt{n_1}\ \bar{t}'_1 + z_2\ \sqrt{n_2}\ \bar{t}'_2)$$

Now we shall make the assumption that the original choice of $\ell(x)$ in single-particle coordinates is such that, after projection of orbital symmetry, no excitation of the center-of-mass vectors z_1 z_2 occurs. This implies that the states in the last equation depend on $z_1\, z_2\, t'_1\, t'_2$ only through the exponentials. Then these states are precisely of the form encountered in the expressions for the operators $\mathcal{H}$ and $\mathcal{N}$. If we put

$$\sqrt{n'_1}\ t'_1 \to z'_1 \qquad \sqrt{n'_2}\ t'_2 \to z'_2$$

$$\sqrt{n_1}\ t_1 \to z_1 \qquad \sqrt{n_2}\ t_2 \to z_2$$

we find

$$\mathcal{H}(z,z') = (c(r\,f\,q) \circ \hat{W}_{\tau_c}\ \ell\ ,\ H \circ c(r\,f\,q') \circ \hat{W}_{\tau'_c}\ \ell)$$

$$\mathcal{N}(z,z') = (c(r\,f\,q) \circ \hat{W}_{\tau_c}\ \ell\ ,\ c(r\,f\,q') \circ \hat{W}_{\tau'_c}\ \ell)$$

In this form the n-dimensional matrix elements appear as a problem of n-body states built from non-orthogonal single particle states, but these matrix elements carry, through the vectors appearing in τ_c and τ'_c, the dynamics of composite particles. The single-particle aspects allow the full application of concepts like fractional parentage, double cosets, etc., and we shall make use of these concepts in what follows. The expressions have an obvious generalization to more than two composite particles.

As a final comment we remark that the passage to the relative coordinates s_i may be done as the very last step since the internal coordinates y are not affected. In the next sections we give explicit results for the operators $\mathcal{H}$ and $\mathcal{N}$.

6. Configurations of simple composite particles

In Bargmann space the single-particle harmonic oscillator states take the form

$$\ell_N(x) = [N!]^{-\frac{1}{2}} z^N$$

or, for particles in three dimensions,

$$\ell_{NLM}(x) = P^N_{LM}(x) = A_{NL}(x \cdot x)^{\frac{1}{2}(N-L)} \mathcal{Y}_{LM}(x)$$

where the polynomials P^N_{LM} are given in /18/.

We shall now discuss a configuration where at center i there are w_i nucleons in the unexcited level $N = 0$,

$$\ell_{000}(x) = 1$$

Application of the Weyl operator to this state yields

$$(\hat{W}_{\tau_{ci}} \ell_{000})(x) = \exp(x \cdot \overline{t}_i)$$

and the overlaps become

$$\varepsilon_{ij} = (\hat{W}_{\tau_{ci}} f_{000} , \hat{W}_{\tau'_{cj}} f_{000}) = \exp(t_i \cdot \overline{t}'_j)$$
$$= \exp([n_i \cdot n'_j]^{-\frac{1}{2}} z_i \cdot \overline{z}'_j)$$

The technique of $GL(j, \mathbb{C})$ developed in section 2 gives for the n-particle state

$$\mathcal{N}(z,z') = D^f_{qq'}(\varepsilon) = [w!\, w'!]^{\frac{1}{2}} \sum_k d^f_{qq'}(s_k)\, m^{-1}(k)\, \varepsilon^k$$
$$= [w!\, w'!]^{\frac{1}{2}} \sum_{k_{ij}} d^f_{qq'}(s_k) [\Pi\, k_{ij}!]^{-1} \exp(k_{ij} [n_i n'_j]^{-\frac{1}{2}} z_i \cdot \overline{z}'_j)$$

This is an operator acting in a Bargmann space F_j , and again it may be interpreted as a linear combination of operators which describe complex extensions of linear canonical transformations. Moreover, $\mathcal{N}(z,z')$ commutes with the oscillator hamiltonian of j particles and its eigenstates must belong to fixed total excitations. With Gaussian interactions one still gets representations of linear canonical transformations. States of up to three such composite particles have been applied to the structure of light nuclei up to $n = 10$ /17,8/. One obtains very good agreement with experimentally known data and predictions of new resonances in these nuclei.

7. Complex two-center configurations

The computations for complex clusters are greatly simplified if the Weyl operators are modified. With respect to oscillator states we found the factorization of Weyl operators (see section 3):

$$\hat{W}_{\tau'_c i} = \Lambda_{t'_i} \circ V_{t'_i}$$

If for the scalar products ε_{ij} we modify this operator according to

$$\Lambda_{t'_i} \circ V_{t'_i} \rightarrow \Lambda_{t'_i} \circ V_{t_i}$$

for the ket states and according to

$$\Lambda_{t_i} \circ V_{t_i} \rightarrow \Lambda_{t_i} \circ V_{t'_i}$$

for bra states, then this does not affect the n-body matrix element after orbital projection. But since

$$[\Lambda_{t_i} \circ V_{t'_i}]^+ \circ \Lambda_{t'_i} \circ V_{t_i}$$

$$= \Lambda_{-t'_i} \circ V_{-t_i} \circ \Lambda_{t'_i} \circ V_{t_i} = 1 \exp t_i \cdot \bar{t}'_i$$

this modification yields a biorthogonal basis at each center i /8/.

A second useful transformation is obtained if the projection operator P_1 is introduced which projects on all states occupied at the center 1 (undisplaced). Then the states at center two may be modified according to

$$(\Lambda_{t'_2} \circ V_{t_2} f)(x)$$

$$\rightarrow ([1 - \Lambda_{t'_1} \circ V_{t_1} \circ P_1 \circ \Lambda_{-t'_1} \circ V_{-t_1} \quad \Lambda_{t'_2} \circ V_{t_2} f)(x) \exp(-t_1 \cdot \bar{t}'_1)$$

$$= (\Lambda_{t'_1} \circ V_{t_1} \circ [1 - P_1] \circ \Lambda_{t'_2 - t'_1} \circ V_{t_2 - t_1} f) \exp(-t_1 \cdot \bar{t}'_1 + t_1 \cdot \bar{t}'_2)$$

These states are biorthogonal to all states at center one while their scalar product with similar bra states at center two is determined by the matrix element of the operator

$$\Lambda_{t'} \circ V_t \circ [1 - P_1] \circ \Lambda_{-t'} \circ V_{-t} \exp(t_2 \cdot \bar{t}'_2 - t \cdot \bar{t}')$$

where $t = t_1 - t_2$, $t' = t'_1 - t'_2$. Finally we note the following properties of the transformation to relative coordinates s_1, s_2 :

$$n_1\ t_1\cdot\bar{t}'_1 + n_2\ t_2\cdot\bar{t}'_2 = s_1\cdot\bar{s}'_1 + s_2\cdot\bar{s}'_2$$

$$s_1 = \left[\frac{n_1 n_2}{n}\right]^{\frac{1}{2}}\ t$$

The second transformation amounts to a transformation of the overlap matrix ε by triangular matrices such that the determinant is unchanged.

We examine first a single nucleon interacting with a closed oscillator shell of N_1 quanta of excitation at center 1 corresponding to ^4He, ^{16}O or ^{40}Ca . The number of single-particle states at center 1 we call $j-1$. With the modified states, the matrix ε is now diagonal with entries

$$\varepsilon_{i\ell} = \delta_{i\ell}\ \exp(t_i\cdot\bar{t}'_\ell) \qquad i,\ell = 1\ ..\ j-1$$

while for $i = \ell = j$ we get

$$\varepsilon_{jj} = (f_{000},\ \Lambda_{t'}\circ V_t\circ[1-P_1]\circ\Lambda_{-t'}\circ V_{-t}\ f_{000})\exp(t_2\cdot\bar{t}'_2 - t\cdot\bar{t}')$$

$$= \exp(t_2\cdot\bar{t}'_2 - t\cdot\bar{t}')\ q_{N_1+1}\ (t\cdot\bar{t}')$$

where we introduced the function

$$q_m(z) = \sum_{\ell=m}^{\infty} (\ell!)^{-1}\ z^\ell$$

The only permutational symmetry of physical interest is $f = [4^{j-1}\ 1]$ with the representation

$$D^{[4^{j-1}\ 1]}_{q_{max},q_{max}}(\varepsilon) = (\Delta_{1..j-1,1..j-1})^3\ (\Delta_{1..j,1..j})$$

and where the choice $q = q_{max}$ means that we have a single nucleon interacting with a closed oscillator shell. Now

$$\Delta_{1..j-1,1..j-1} = \exp[\ (j-1)t_1\cdot\bar{t}'_1\ ]$$

and

$$\Delta_{1..j,1..j} = \exp[4(j-1)t_1\cdot\bar{t}'_1 + t_2\cdot\bar{t}'_2 - t\cdot\bar{t}'\ ]\ q_{N_1+1}\ (t\cdot\bar{t}')$$

which gives

$$\mathcal{N}(t_1 t_2, t'_1 t'_2) = \exp[\,4(j-1)\ t_1\cdot\bar{t}'_1 + t_2\cdot\bar{t}'_2 - t\cdot\bar{t}'\ ]\ q_{N_1+1}\ (t\cdot\bar{t}')$$

or, after transforming to cluster coordinates and noting that

$n_1 = 4(j-1)$,

$$\mathcal{N}(s_1 s_2, s'_1 s'_2) = \exp\left[s_2\cdot\bar{s}'_2 + s_1\cdot\bar{s}'_1 - \frac{4(j-1)+1}{4(j-1)} s_1\cdot\bar{s}'_1\right] q_{N_1+1}\left(\frac{4(j-1)+1}{4(j-1)} s_1\cdot\bar{s}'_1\right)$$

This expression yields the normalization kernel in explicit form. The dependence on the overall c.m. vector s_2 has the form of a reproducing kernel describing the unit operator. The dependence on s_1, s'_1 is only through $s_1\cdot\bar{s}'_1$ which means that the eigenstates of $\mathcal{N}$ are the oscillator states with respect to the vector s_1 . The eigenvalues may be explicitly obtained upon expanding $\mathcal{N}$ in the form

$$\mathcal{N}(s_1 s_2 s'_1 s'_2) = \exp\left[s_2\cdot\bar{s}'_2\right] \sum_{N=N_1+1}^{\infty} \lambda_N \frac{(s_1\cdot\bar{s}'_1)^N}{N!}$$

since clearly λ_N is the eigenvalue for an oscillator state of excitation N . The values N below N_1+1 are forbidden by the Pauli principle. The eigenvalue for N_1+1 is clearly

$$\lambda_{N_1+1} = \left[\frac{4(j-1)+1}{4(j-1)}\right]^{N_1+1}$$

which for not too small values of j is close to unity. For very high excitations the kernel $\mathcal{N}$ with respect to s_1 behaves like a unit operator. An estimate of the lowest possible value N_{min} is obtained as follows: If all nucleons are put into a single center, one obtains a minimum total excitation N_{12} . If we subtract from this number the excitations N_1 and N_2 at the separated centers, we get the estimate

$$N_{min} = N_{12} - N_1 - N_2$$

The next example which we shall analyze is the same type of configuration at center one but a ^{4}He configuration at center two. The single particle states are the same ones as before but now we have $f = [4^j]$ and

$$D^f_{qq}(\varepsilon) = (\Delta_{1..j,1..j})^4 = \mathcal{N}(z,z')$$

Rewriting $\Delta_{1..j,1..j}$ in terms of s_1, s_2 yields

$$\Delta_{1..j,1..j} = \exp\left(\frac{1}{4} s_1\cdot\bar{s}'_1 + \frac{1}{4} s_2\cdot\bar{s}'_2 - \frac{j}{4(j-1)} s_1\cdot\bar{s}'_1\right) q_{N_1+1}\left(\frac{j}{4(j-1)} s_1\cdot\bar{s}'_1\right)$$

The general properties of $\mathcal{N}(s_1 s_2, s'_1 s'_2)$ are similar to the ones of the previous system. The lowest non-vanishing eigenvalue is clearly

$$\lambda_{4N_1+4} = \left[\frac{j}{4(j-1)} \right]^{4N_1+4}$$

which may be a small number. Obviously the effect of the Pauli principle extends to higher excitations as can be seen from a graphical representation of λ_N .

The last example to be considered is the system $^{16}O + ^{16}O$. The number of single-particle states at each center is $\nu = 4$ corresponding to the configuration $s\,p^3$. The overlap matrix is an 8 x 8 matrix and the orbital symmetry is $f = [4^{2\nu}]$. We have

$$D^{f}_{qq}(\varepsilon) = (\Delta_{1..2\nu,1..2\nu})^4$$

and the computation yields

$$\Delta_{1..j,1..j} = \exp(4\, t_1 \cdot \bar{t}'_1 + 4\, t_2 \cdot \bar{t}'_2 - 2\, t \cdot \bar{t}')$$

$$\cdot \left[q_6(2\,t\cdot\bar{t}') + q_6(-2\,t\cdot\bar{t}') - q_6(t\cdot\bar{t}') - q_6(-t\cdot\bar{t}') - (t\cdot\bar{t}')^2 (q_4(t\cdot\bar{t}') + q_4(-t\cdot\bar{t}')) \right]$$

Without taking the fourth power, the determinant would describe the composite system $sp^3 + sp^3$. The combination of terms in Δ means that these composite particles behave like bosons under exchange. As a consequence $\det \varepsilon$ starts with $N = 6$, not with $N_{min} = N_{12} - N_1 - N_2 = 11 - 3 - 3 = 5$. If we take the fourth power to describe $^{16}O + ^{16}O$, $(\Delta_{1..2\nu,1..2\nu})^4$ starts with $N = 6 \cdot 4 = 24$ instead of $N_{min} = 4 \cdot 5 = 20$. This means that by a collision of $^{16}O + ^{16}O$ nuclei one cannot reach the lower levels of the combined system and gets a rather selective excitation. Similar viewpoints were developed from a completely different angle by Harvey /19/.

8. Conclusion

The interplay between nuclear composite particle dynamics and linear canonical transformations may be summarized as follows: Semi-group extensions of these transformations have bounded operator representations in n-body Hilbert space. The n-body dynamics, including transformations to appropriate coordinates and application of permutations, is expressible in terms of linear combinations of these operators with coefficients from representations of the symmetric group. By employing Weyl translation operators one arrives at explicit expressions for the composite particle interaction in the corresponding reduced Hilbert space. The groups and semigroups are not symmetries of this interaction but could be called dynamical semigroups.

Since the present approach provides explicit expressions for the composite particle dynamics, one is in a position to analyze the structure of these interactions. The fermion nature of the underlying n-body system is represented by the normalization operator. This operator prevents the composite particles from moving into forbidden states, modifies the interaction for the lowest allowed oscillator states and exhibits the boson (or fermion) nature of identical composite particles. More information is accessible from the interaction operators which were not analyzed in this contribution. Keeping in mind the close relationship between the nuclear collective model and the symplectic group as developed by D. J. Rowe /20/, one would like to link composite particle dynamics to this model. It is believed that there is a large range of new applications of the present approach to nuclear models and to nuclear reactions.

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