PHASE TRANSITIONS, APPROACH TO EQUILIBRIUM, AND STRUCTURAL STABILITY*

by Gérard G. Emch

Departments of Mathematics and Physics

The University of Rochester, NY 14627, USA

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I. INTRODUCTION

In Equilibrium Statistical Mechanics, the occurence of phase transitions is most often accompanied by the phenomenon of spontaneous symmetry breaking: the invariance group of the pure thermodynamical phases is only a subgroup of the group of symmetries under which the Hamiltonian is invariant [1].
In Non-equilibrium Statistical Mechanics, the reversibility of the microscopic hamiltonian time-evolution is apparently broken by a general macroscopic tendency to approach some equilibrium states; whereas the microscopic evolution is described by a group (conservative laws of mechanics), the macroscopic evolution can at best be described by a semi-group (dissipative transport equations) [2].
When Equilibrium Statistical Mechanics allows the coexistence of several thermodynamical phases, the problem moreover arises to describe the respective attraction domains of each of these phases.
Our aim here is to present a model for which this problem can be solved. This being a general (i.e. hopefully pedagogical) lecture, the choice of the model should be guided by the following three considerations:

(a) the model should be "simple" enough to be mathematically exactly solvable;

(b) the model should be "sophisticated" enough to exhibit the physical phenomena one is interested in;

(c) the model should be general enough to escape the most obvious criticisms that it be an isolated mathematical accident; more precisely, the model should be "stable" against structural perturbations, so that it can qualify as a representative of a wider class of physical, if not exactly solvable, models.

II. DESCRIPTION OF THE WEISS-ISING MODEL

The quantum Weiss-Ising model consists of a denumerable collection $\{\sigma_n | n \in \mathbb{Z}\}$ of spin-half particles, each pinned to a fixed position in space, say on the vertices of a one-dimensional lattice $\mathbb{Z}$. As usual in quantum mechanics, the observables relative to a single particle are in one-to-one correspondance with the self-adjoint two-by-two matrices. Upon taking their complex linear combinations, we obtain the 4-dimensional complex vector space $\mathfrak{M}(2,\mathbb{C})$ spanned by

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We denote by A^* the hermitian adjoint of the matrix A, by $\| A \|$ its norm, and notice that $\| A^*A \| = \| A \|^2$: $\mathfrak{M}(2,\mathbb{C})$ is a C*-algebra. Let now $\mathfrak{F} = \{\Lambda \subset \mathbb{Z} | \text{card}(\Lambda) < \infty\}$. To every $\Lambda \in \mathfrak{F}$ we associate the C*-algebra of observables related to this finite subset of $\mathbb{Z}$:

$\mathfrak{A}(\Lambda) = \otimes_{n \in \Lambda} \mathfrak{A}_n$ with $\mathfrak{A}_n \simeq \mathfrak{M}(2,\mathbb{C})$, so that:

$\mathfrak{A}(\Lambda) \simeq \mathfrak{M}(2^{N(\Lambda)},\mathbb{C})$ with $N(\Lambda) = \mathrm{card}(\Lambda)$.

A state ϕ of this finite collection Λ of spin-half particles is a positive linear functional $\phi : A \in \mathfrak{A}(\Lambda) \mapsto < \phi; A > \in \mathbb{C}$ with $< \sigma; I> = 1$. We denote by $\mathfrak{S}(\Lambda)$ the convex set of all states on $\mathfrak{A}(\Lambda)$. To every $\phi \in \mathfrak{S}(\Lambda)$ corresponds uniquely a positive matrix $\rho \in \mathfrak{M}(2^{N(\Lambda)},\mathbb{C})$, such that $<\phi;A> = \mathrm{Tr}\ \rho\ A$ for all $A \in \mathfrak{A}(\Lambda)$, and $\mathrm{Tr}\ \rho = 1$.
Let now $\alpha_\Lambda : t \in \mathbb{R} \mapsto \mathrm{Aut}(\mathfrak{A}(\Lambda))$ be a continuous one-parameter group of automorphisms of $\mathfrak{A}(\Lambda)$, to be interpreted as giving the evolution for the region Λ when isolated from the rest of the system. To every such $\alpha_\Lambda(\mathbb{R})$, one can associate a self-adjoint element $H(\Lambda) \in \mathfrak{A}(\Lambda)$, called the Hamiltonian, such that:

$$\alpha_\Lambda(t)[A] = e^{-i\ H(\Lambda)t}\ A\ e^{i\ H(\Lambda)t} \quad \text{for all } A \in \mathfrak{A}(\Lambda).$$

It is a remarkable (and easily verifyable) fact that the canonical equilibrium state

$$\phi_\Lambda \quad \text{with} \quad \rho_\Lambda = e^{-\beta\ H(\Lambda)} \ / \ \mathrm{Tr}\ e^{-\beta\ H(\Lambda)}$$

is uniquely determined by the following condition [3].
<u>KMS-Condition</u>: For every $A,B \in \mathfrak{A}(\Lambda)$ there exists a function $F_{AB}(z)$ analytic in, and continuous on the boundaries of, the strip $\{0 < \mathrm{Im}(z) < \beta\}$ such that:

$$F_{AB}(t) = \langle\phi_\Lambda;\ A\ \alpha_\Lambda(t)[B]\rangle$$

$$F_{AB}(t + i\beta) = \langle\phi_\Lambda;\ \alpha_\Lambda(t)[B]\ A\rangle$$

In physical terms, the definition of the model is thus completed when we specify the form of $H(\Lambda)$ for every $\Lambda \in \mathfrak{F}$. We take:

$$H(\Lambda) = -B \sum_{n \in \Lambda} \sigma_n^z - \frac{1}{2}\frac{J}{N(\Lambda)} \sum_{n,m \in \Lambda} \sigma_n^z \otimes \sigma_m^z$$

III. PHASE TRANSITIONS

If we take the KMS condition as the definition of an equilibrium state, as we should on thermodynamical grounds [4], then every isolated, finite region Λ of the model is unable to exhibit any phase transition. Indeed, since there exists for any $\Lambda \in \mathfrak{F}$ exactly one state ϕ_Λ satisfying the KMS condition, coexistence of several thermodynamical phases is obviously ruled out. This physical disease has been recognized long ago [5], although on different premises. The commonly accepted cure is to take the so-called thermodynamical limit, in which Λ is let grow to become infinite. In the approach followed here, this rises the question of giving a meaning to:

$$\alpha(t) = \lim_{\Lambda \to \mathbb{Z}} \alpha_\Lambda(t) \quad ,$$

so that one can characterize the extremal KMS states for this $\alpha(\mathbb{R})$. It is at this point that our earlier remark, to the effect that $\mathfrak{A}(\Lambda)$ are C*-algebras, is of help. Indeed the first step is to construct the observables of the infinite system as well-defined mathematical objects. To this end, we further remark that $\{\mathfrak{A}(\Lambda) | \Lambda \in \mathfrak{F}\}$ satisfies the following two basic properties:

(i) $\Lambda_1 \subseteq \Lambda_2 \Rightarrow \exists$ isometric embedding $i_{21} : \mathfrak{A}(\Lambda_1) \to \mathfrak{A}(\Lambda_2)$ such that $i_{21}(I_1) = I_2$;

(ii) $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_3 \Rightarrow i_{31} = i_{32} \circ i_{21}$

This allows [6] to prove that there exists a C*-algebra $\mathfrak{A}$ and a collection $\{i_\Lambda : \mathfrak{A}(\Lambda) \to \mathfrak{A} | \Lambda \in \mathfrak{F}\}$ of isometric embeddings such that

$$\mathfrak{A} = \overline{\cup_{\Lambda \in \mathfrak{F}} i_\Lambda(\mathfrak{A}(\Lambda))}^n \quad \text{and} \quad i_\Lambda(I_\Lambda) = I.$$

$\mathfrak{A}$ is the algebra of "quasi-local observables" on the infinite system. The second step is to remark that, whereas $\mathfrak{A}$ is an abstract C*-algebra, to every state ϕ on $\mathfrak{A}$ one can uniquely associate [7] a representation π_ϕ of $\mathfrak{A}$ as a concrete C*-algebra of bounded operators acting on a Hilbert space $\mathfrak{H}_\phi$ which admits a vector Φ such that:

(i) $\overline{\pi_\phi(\mathfrak{A})\Phi} = \mathfrak{H}_\phi$

(ii) $\langle\phi;A\rangle = (\pi_\phi(A)\Phi,\Phi)$ for all $A \in \mathfrak{A}$.

This freedom in choosing a representation of $\mathfrak{A}$ adapted to the situation (i.e. state) at hand allows to solve the above problem. Namely, with:

$$M = \text{w-op} \lim_{\Lambda \to \mathbb{Z}} \frac{1}{N(\Lambda)} \sum_{n \in \Lambda} \sigma_n^z ,$$

the pure thermodynamical phases (i.e. extremal KMS states) are characterized [8] by the solutions of:

$$\langle M \rangle_{\beta,B} = \tanh \beta (B + J \langle M \rangle_{\beta,B}) .$$

This is the well-known [9] "self-consistency" equation in which $J \langle M \rangle_{\beta,B}$ is interpreted as the "mean-free field" acting on the spin σ_n, due to the collective action of all other spins. As is also well-known [10], this model illustrates the occurence of a phase transition accompanied by a spontaneous symmetry breaking. Indeed, for all $\Lambda \in \mathfrak{F}$, the Hamiltonians $H(\Lambda)$ are all invariant under the internal symmetry group generated by the flip-flop operation:

$$\tau : \sigma_n^z \mapsto - \sigma_n^z$$

$$\tau : \sigma_n^{x,y} \mapsto \sigma_n^{x,y}$$

carried out simultaneously on all spins σ_n. If the pure thermodynamical phases were to inherit this symmetry, we would have:

$$\langle M \rangle_{\beta,B=0} = 0.$$

This however is only the case at high temperature. Indeed, at low temperature, i.e. as soon as $\beta J > 1$, two additional solutions do appear, as illustrated in Fig. 1.

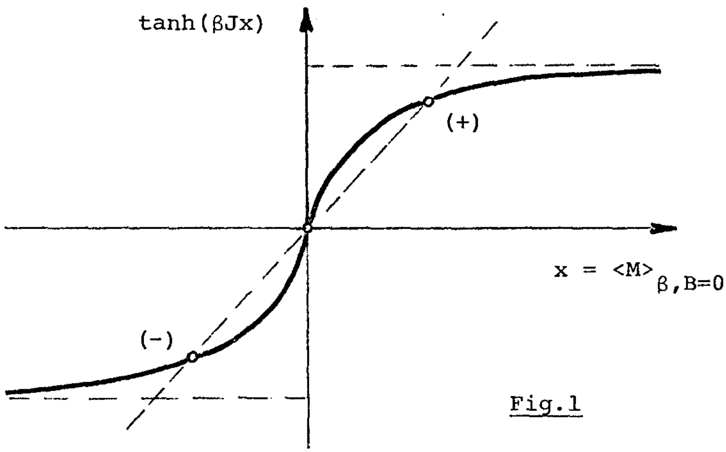

Fig.1

In this model the breaking of symmetry can be understood, and computed, as a structural instability, as is clear from Fig. 2, where we draw the graph of a (piecewise analytic) isotherm for $\beta J > 1$.

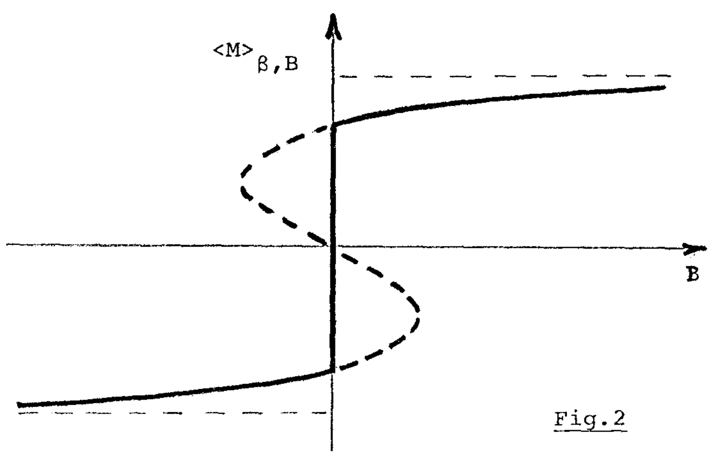

Fig.2

The flip-flop symmetry is evidently restored if one considers the state:

$$< \ldots > = \lim_{\Lambda \to \mathbb{Z}} \phi_\Lambda = \tfrac{1}{2} (<\ldots>_+ + <\ldots>_-) .$$

This state is a mixture of the two pure thermodynamical phases $<\ldots>_+$ and $<\ldots>_-$; its free energy is lower than that of the extremal KMS state $<\ldots>_o$ corresponding to the solution $<M>_o = 0$ of the self-consistency equation. In that sense $<\ldots>_\pm$ are stable equilibrium states, whereas $<\ldots>_o$ is unstable [11]. A deeper meaning for this statement will be given in the next sections.

IV. APPROACH TO EQUILIBRIUM

The aim of this section is to propose a dynamical meaning for the assertion that the pure thermodynamical phases $<\ldots>_\pm$ (resp. $<\ldots>_o$) are <u>stable</u> (resp. <u>unstable</u>) equilibrium states of the quantum Weiss-Ising model. This is achieved by considering the Weiss-Ising ferromagnet as an open system (W) interacting with a thermal bath (R). The latter is devised in such a manner as to provide [12] an hamiltonian justification for the Glauber stochastic dynamics [13] governing the evolution of the observables of interest.

We again set up the model starting with finite regions $\Lambda \in \mathfrak{F}$. We thus have to specify the terms entering in the total Hamiltonian for every finite region Λ:

$$H_{\Lambda,\lambda} = H_W(\Lambda) + H_R(\Lambda) + \lambda H_I(\Lambda).$$

$H_W(\Lambda)$ is the Weiss-Ising Hamiltonian $H(\Lambda)$ of section II. $H_R(\Lambda)$ is the Hamiltonian governing the evolution of the thermal bath, and $H_I(\Lambda)$ is an interaction between the Weiss-Ising system and the bath, which we control through a coupling constant λ.

The thermal bath is constructed as follows. To every $n \in \Lambda$ we associate a quasi-free Fermi system Σ_n in thermal equilibrium. Each Σ_n is taken to be infinite, in the sense that its test function space $\mathfrak{I}_n$ is infinite dimensional; for sake of definiteness we take $\mathfrak{I}_n = \mathcal{L}^2(\mathbb{R}^3)$. To say that Σ_n is quasi-free is to say that the time-dependence of the field operators is given by:

$$\psi(f) \mapsto \psi_t(f) = \psi(u_t f)$$

where $\{u_t = \exp(-iht) \mid t \in \mathbb{R}\}$ is a continuous, one-parameter group of unitary operators acting on $\mathfrak{I}_n$; again for sake of definiteness we take for the one-particle Hamiltonian: $h = p^2/2m$. To say that Σ_n is in thermal equilibrium is to say that it is in a state ϕ_n which satisfies the KMS condition with respect to this evolution. One checks easily that this condition determines ϕ_n uniquely, and that ϕ_n is characterized by its two-point functions:

$$<\phi_n;\ \psi(f)\ \psi(g)> \text{ with } f,g \text{ running over } \mathfrak{J}_n.$$

The function

$$C(t) = <\phi_n;\ \psi(e^{-iht}\ f)\ \psi(f)>$$

will play a central role in the sequel. We note that its Fourier transform satisfies the conditions:

(i) $\tilde{C}(x) \geq 0$

(ii) $\tilde{C}(-x) = \exp(\beta x)\ \tilde{C}(x)$.

We finally assume that the individual thermal baths Σ_n do not interact directly with one another, so that the initial state of the thermal bath $\{\Sigma_n | n \in \Lambda\}$ is:

$$\phi_R(\Lambda) = \otimes_{n \in \Lambda} \phi_n.$$

We finally specify the interaction between the Weiss-Ising system and the thermal bath to be:

$$H_I(\Lambda) = \sum_{n \in \Lambda} \sigma_n^x \otimes \psi_n(f)$$

where f is a fixed element in $\mathfrak{J}_n$.

We thus have a situation where each spin interacts only with its own "private bath" and with the other spins, whereas the individual thermal baths are otherwise isolated from one another.

The dynamics of the total system (Weiss-Ising + Bath) is thus completely specified for every $\Lambda \in \mathfrak{F}$. In particular we could in principle compute the time-dependent expectation value $< M >_{\Lambda,\lambda}(t)$ of the magnetization

$$M = \frac{1}{N(\Lambda)} \sum_{n \in \Lambda} \sigma_n^z$$

when the interaction is switched on at $t = 0$, i.e. when the initial state $\phi_o(\Lambda)$ of the total system is such that the Weiss-Ising system and the thermal bath are uncorrelated at $t = 0$, which is to say that

$$\phi_o(\Lambda) = \phi_W(\Lambda) \otimes \phi_R(\Lambda), \text{ with } \phi_W(\Lambda) \text{ arbitrary on } \mathfrak{U}(\Lambda).$$

We however want to focus our attention on the explicit equation of motion for the quantity:

$$m(\tau) = \lim_{\Lambda \to \mathbb{Z}} \lim_{\substack{t \to \infty,\ \lambda \to 0 \\ \lambda^2 t = \tau}} \langle M \rangle_{\Lambda,\lambda}(t) .$$

The first limit is the so-called van Hove limit [14]. Its purpose is to extract the long-time cumulative effect of the interaction in the weak-coupling limit. In this model its existence is guaranteed by the fact that $h = p^2/2m$ implies $C(t) = 0(t^{-3/2})$ as $t \to \infty$ (spreading of the wave packet). Note that:

$$|C(t)| \leq a \,/\, (1 + |t|^{1 + \varepsilon}) \text{ with } \varepsilon > 0$$

would have sufficed, thus allowing for more general one-particle Hamiltonians h. The proof moreover holds for more general Hamiltonians $H_W(\Lambda)$ than the Weiss-Ising Hamiltoniar and actually insures the existence, in the Schrödinger picture, of the reduced evolution for the spin system. In particular, consider the commutative C*-algebra $\mathfrak{A}^z(\Lambda)$ generated by $\{\sigma_n^z \mid n \in \Lambda\}$; a state ϕ_W on the spin algebra $\mathfrak{A}(\Lambda)$ is then said to be classical if the corresponding density matrix ρ belongs to $\mathfrak{A}^z(\Lambda)$. Such a state can thus be regarded as a probability distribution μ on the configuration space:

$$\Omega = \{\omega : \Lambda \mapsto \{+1,-1\}\}.$$

One can show [12] that if ϕ_W is classical, then so is the state $\phi_W(\tau)$ defined by:

$$<\phi_W(\tau) ; A> =$$

$$\lim_{\substack{t \to \infty, \lambda \to 0 \\ \lambda^2 t = \tau}} <\phi_W \otimes \phi_R ; e^{-iH_{\Lambda,\lambda}t} A e^{iH_{\Lambda,\lambda}t}> \quad \forall A \in \mathfrak{A}(\Lambda) ,$$

and that the reduced evolution $\mu(0) \to \mu(\tau)$ can then be written as a Markov process on Ω.

The second limit appearing in the definition of $m(\tau)$ is the usual thermodynamical limit. Its existence can be proven for a natural [12] collection of "macroscopic" initial states ϕ_W, and one finds the following equation of motion for $m(\tau)$:

$$\dot{m} = 2\,\gamma(B + J\,m)\,[\,\tanh \beta\,(B + J\,m) - m]$$

$$\text{with}\quad \gamma(x) = \tilde{C}(2x) + \tilde{C}(-2x).$$

Upon going back to the definition of $\tilde{C}$ as the Fourier transform of the two-point correlation function for the bath, one can see that its behaviour is entirely determined by the function f chosen in the definition of $H_I(\Lambda)$. In particular, we can choose f in such a manner that $\tilde{C}$ is a strictly positive, slowly varying function. In this case the only stationary points of the differential equation for m are the solutions of the equation:

$$m = \tanh \beta \ (B + J\ m)$$

which is exactly the self-consistency equation of section III. As in that section, we discuss only the case $B=0$, the case $B \neq 0$ being obtained as a straightforward generalization. At high temperature, i.e. for $\beta J < 1$, the unique solution of the self-consistency equation is easily seen to be a global attractor for the differential equation governing the evolution of $m(\tau)$. At low temperature, i.e. as soon as $\beta J > 1$, the self-consistency equation has three solutions: one source (m_o) and two attractors $(m_{\pm})$ with domains of attraction as indicated in the flow profile of Fig. 3:

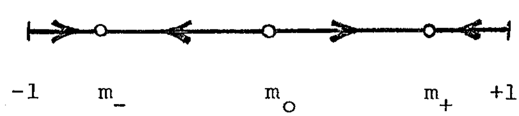

Fig. 3

This gives a dynamical confirmation to the interpretation of $\langle\ldots\rangle_{\pm}$ (resp. $\langle\ldots\rangle_{o}$) as stable (resp. unstable) thermodynamical phases of the Weiss-Ising model [15].

V. STRUCTURAL STABILITY

The purpose of this section is to explore the stability, under structural perturbations, of the model discussed in the preceding section. The open quantum Weiss-Ising model is first generalized as follows [16].

1. Instead of considering only one-body $(-B\sigma_n^z)$ and two-body $(-J\sigma_n^z\sigma_m^z/N)$ interactions, we now allow many-body forces to enter the Weiss-Ising model, the Hamiltonian of which thus becomes (with D a finite positive integer):

$$H_W(\Lambda) = \sum_{d=0}^{D} \frac{a_d}{(d+1)!} H_W^d(\Lambda) \quad \text{with :}$$

$$H_W^d(\Lambda) = \sum_{n_1,\ldots n_{d+1}} J_{n_1,\ldots,n_{d+1}} \sigma_{n_1}^z \otimes \cdots \otimes \sigma_{n_{d+1}}^z$$

$$J_{n_1,\ldots,n_{d+1}} = \begin{cases} -J/N(\Lambda)^d & \text{if } n_j \neq n_k \text{ for } j \neq k \\ 0 & \text{otherwise} \end{cases}$$

a_d and J real coupling constants.

2. We partition Λ into K subsets Λ_k $(k=1,\ldots,K)$;

let $N(\Lambda_k) = \mathrm{card}(\Lambda_k)$ and $\nu_k = N(\Lambda_k) / N(\Lambda)$. The quantities of interest will now be the partial magnetizations

$$M_k(\Lambda) = \frac{1}{N(\Lambda_k)} \sum_{n \in \Lambda_k} \sigma_n^z$$

rather than just the total magnetization

$$M(\Lambda) = \sum_{k=1}^{K} \nu_k \; M_k(\Lambda)$$

of the preceding section.

3. We do not require the temperatures of the individual thermal baths to be uniform, but only that $\beta_n = \beta_k$ for all $n \in \Lambda_k$.

4. Similarly the interaction between the Weiss-Ising system and the thermal bath is taken to be:

$$H_I(\Lambda) = \sum_{n \in \Lambda} \sigma_n^x \; \psi(f_n)$$

with $f_n = f_k$ for all $n \in \Lambda_k$.

This generalized version of the model of section IV is still exactly solvable in the sense that we can again take the van Hove limit and the thermodynamical limit, keeping now ν_k fixed.

One then finds the following system of differential equations for the time-dependent expectation value of the partial magnetizations:

$$\left.\begin{aligned} \dot{m}_k &= 2\,\gamma_k(J\,p(m))\,\left[\tanh(\beta_k\,Jp(m)) - m_k\right] \\ \text{where}\quad m &= \sum_{k=1}^{K} \nu_k\, m_k \\ p(x) &= \sum_{d=0}^{D} a_d\, x^d \\ \gamma_k(x) &= \tilde{C}_k(2x) + \tilde{C}_k(-2x) \end{aligned}\right\} \quad (1)$$

We now approximate the strictly positive, slowly varying functions γ_k by $\gamma_k(x) = c/2$, this approximation being justified a posteriori by the last result of this section. We therefore study the system

$$\dot{m}_k = c\left[\tanh(\beta_k\, Jp(m)) - m_k\right] \qquad (2)$$

from which we get:

$$\dot{m} = c\left[\sum_{k=1}^{K} \nu_k \tanh(\beta_k\, Jp(m)) - m\right] \qquad (3)$$

One verifies easily the following two assertions:

(i) The stationary points $\underline{M} = (M_1,\ldots,M_K)$ of (2) are uniquely determined by the stationary points M of (3), and vice versa;

(ii) M is an attractor (resp. a source, an hyperbolic point, or a non-hyperbolic point) exactly when $\underline{M}$ is an attractor (resp. a saddle point, an hyperbolic point, or a non-hyperbolic point).

Hence the "partial magnetization" description confirms exactly, and actually refines, the "total magnetization" picture.

We say that the K-dimensional flow generated by (2) is hyperbolic if all its stationary points are hyperbolic. Let us assume that this is the case; the typical situation is then illustrated in Fig. 4 for the case $K = 2$, $p(x) = x$ (i.e. the usual Weiss-Ising model with $B = 0$) and $\beta J > 1$ with $\beta = \sum_k \nu_k \beta_k$:

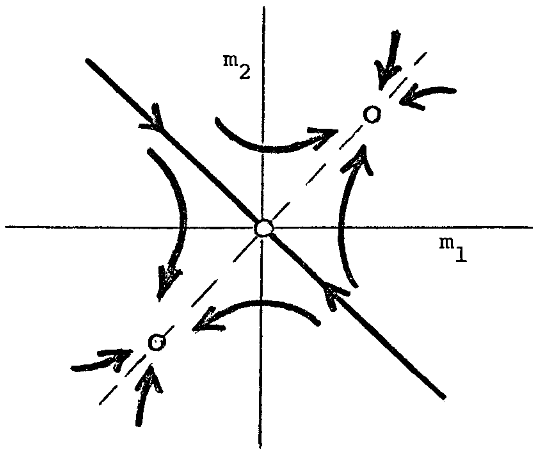

Fig. 4

We now turn to the question of the structural stability of the system of differential equations (2), following [16]. We first notice that the flow defined by (2) is physically constrained in the hypercube $C^K = \left\{-1 \leq m_k \leq +1 \mid k = 1,\ldots,K\right\}$, a compact manifold with boundaries. Since nothing special happens on the boundaries, this flow can be mathematically extended to $\mathbb{R}^K$. The resulting flow has no other stationary point than those already contained in the hypercube C^K. Let then D^K be the hypersphere $\left\{ \sum_k m_k^2 \leq K + 1 \right\}$ and $\mathfrak{M} = \mathbb{R}^K \cup \{\infty\}$ be the one-point compactification of $\mathbb{R}^K$. Let further $g \in C^1(\mathbb{R})$ be such that $g(x) = 1$ for $x^2 \leq K + 1$, and $g(x) = x^{-2}$ for $x^2 > K + 2$. Let now X be the vector field on $\mathfrak{M}$ defined by:

$$X(m) = \begin{cases} g(\|m\|^2)\ f(m) & \text{for } m \in \mathbb{R}^K \\ 0 & \text{for } m = \infty \end{cases}$$

where $f : \mathbb{R}^K \to \mathbb{R}^K$ is the vector field

$$f(m)_k = c \left[\tanh(\beta_k\ Jp(m)) - m_k \right] .$$

X is then a C^1-vector field on $\mathfrak{M}$ (a compact manifold without boundary); X coincides with f on the hypercube C^K of physical interest, whereas the orbit space of X is the same as that of f, with only an irrelevant time-scale change due to g. These remarks then allow to use, for the study of (2), the fact that X generates a Morse-Smale system, which as such is structurally stable. Consequently, we can conclude that for all $K \geq 2$, the flow (2) is structurally stable, provided that it is hyperbolic. This is a "best possible" result. Indeed, the condition under which the dynamical system of interest can be proven to be structurally stable means that we are away from the bifurcation (or "critical") points where one has to deal with the onset of a phase transition in which the number of pure thermodynamical phases changes abruptly under an infinitesimal variation of the control parameters (i.e. the coupling constants and the temperatures of the thermal baths). Hence the approach to equilibrium, even in the presence of several coexisting thermodynamical phases, is described by a flow which is structurally stable, except precisely for those special values of the control parameters where the equilibrium description (see section III) already indicates that a structural instability should develop. This result completes the programme outlined in the Introduction [17].

NOTES AND REFERENCES

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[10] R. Brout, Phase Transitions, Benjamin, New York 1965; also E. Lieb Journ. Math. Phys. 7 (1966) 1016-1024.

[11] As it was not central to the questions discussed in this lecture, we did not want to mention in the main text one nevertheless interesting feature of ref.[8]; this paper indeed illustrates, and gets over the fact that the evolution cannot always be uniquely defined, in the thermodynamical limit, as an automorphism of the C*-algebra $\mathfrak{A}$ of the quasi-local observables; a proper definition often involves, as it does in this model, careful consideration of the von Neumann algebra $\pi_\phi(\mathfrak{A})''$ generated by the representation $\pi_\phi(\mathfrak{A})$ constructed from a properly chosen state ϕ. See also D.W. Robinson, preprint, Bielefeld, 1977. In the present model, this phenomenon is due to the long-range nature of the forces involved, and both its physical background and its analytic expression can be physically understood. In continuous, non-relativistic systems, this sort of difficulty also occur, although for a different reason; this has been already pointed out by D.A. Dubin and G. Sewell, Journ.Math.Phys. 11 (1970) 2990-2998. Models have also been constructed for the singular dynamics of classical, infinite mechanical systems; see for instance O.E. Lanford, Lecture Notes, Battelle Rencontres in Mathematics and Physics. Still another kind of con-

tinuous, classical and quantum, mean-free field models have recently been constructed by G. Battle, Ph.D. dissertation, Duke University, 1977; there again the time-evolution has to be defined with much care when the thermodynamical limit is carried out.

[12] Ph.A. Martin, Journ. Stat. Phys. 16 (1977) 149-168.

[13] R.J. Glauber, Journ.Math.Phys. 4 (1963) 294-307.

[14] L. van Hove, Physica 21 (1955) 517-540; 23 (1957) 441-480. See E.B. Davies, Quantum Theory of Open Systems, Academic Press, 1976.

[15] Amongst the other things also proven in [12], it should be worth pointing out here again that the approach to equilibrium being asymptotically exponential, the relaxation time so obtained diverges as $|T - T_c|^{-1}$, thus giving rise to the expected "critical slowing down" characteristic of the bifurcation point $\beta_c J = 1$; moreover, the analysis is there completed by a discussion of finite volume effects and fluctuations.

[16] M. Giuffre, Ph.D. dissertation, University of Rochester, 1977.

[17] The model can [16] further be generalized by letting the lattice constant tend to zero, thus providing a continuous model for which the system of ordinary differential equations governing $\{m_k(\tau) | k=1,...,K\}$

now becomes a partial-differentio integral equation for the "coarse-grained" observable $m(x,\tau)$. Not surprisingly, but still gratifyingly enough, the latter exhibits the same kind of approach to equilibrium in the presence of phase transitions as that shown by the discrete models described in this lecture.