ACTION OF CONTROL SEMI GROUPS ON MANIFOLD AND APPLICATION TO REALIZATION THEORY

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We report on recent progress (⩾1970) in non linear system theory. The main results are exposed here (parts of th. II. 3. 2, th. II. 4. 3, th. II. 4. 5, parts of II. 5. 1, . . .) but many other people have contributed to this theory unless they are not cited ; this paper is neither a survey nor an historical paper. We just tried to explain what a non linear system governed by ordinary differential equation is and how it can be considered as the differentiable action of certain semi-groups on manifold. To help the reader non expert in the classical linear theory to appreciate the new results we recall what are main classical results.

The paper is divided in three parts. In the first one we give a brief sketch of linear theory, in the second we expose the main results of recent non linear theory and show how they may be applied in a third one.

I. CLASSICAL LINEAR SYSTEM THEORY.

We refer interested people to the books [14], [17], [29] .

I. 1. System behaviour.

We define what a linear system is and then describe how it works. The main concepts introduced are those of "Controllability", "Observability", "realization", and "Canonical forms".

I.1.1. DEFINITION : A "linear system" is a triple Σ of matrices (A, B, C) where A is an $n \times n$ matrix, B an $n \times p$ matrix and C a $q \times n$ one. The field considered here is the field of real numbers. We usually write this triple in the following way :

$$(1) \qquad \Sigma = \begin{cases} \frac{dx}{dt} = Ax + Bu \quad ; \quad x \in \mathbb{R}^n \quad ; \quad u \in \mathbb{R}^p \\ y = Cx \qquad\qquad y \in \mathbb{R}^q \; . \end{cases}$$

The input set is $\mathbb{R}^p$ and by definition an "input" is a mapping from some interval $[0, T]$ into $\mathbb{R}^p$, denoted by $t \to \mathcal{U}(t)$. To the input one associate the "response" which by definition is the unique solution of the differential equation :

$$(2) \qquad \begin{aligned} \frac{dx}{dt} &= Ax + B\mathcal{U}(t) \\ x(o) &= x_o \end{aligned}$$

and is denoted by : $t \to x(t, x_o, \mathcal{U})$. The "reachable set" from x_o is the set of all points of $\mathbb{R}^n$:

$$\mathcal{R}(x_o) = \{x(T, x_o, \mathcal{U}) \; ; \; u \in \mathfrak{U}\}$$

where $\mathcal{U}$ ranges over the set $\mathfrak{U}$ of all possibles inputs. To every input one can associate the mapping $t \to Cx(t, 0, \mathcal{U})$, this mapping is denoted by :

$$y(t, \mathcal{U}) = Cx(t, 0, \mathcal{U}) \, ,$$

and is called the "output" mapping. The correspondance between $\mathcal{U}$ and the output is the "input output mapping" denoted here by F_Σ . The current point in $\mathbb{R}^n$ is called the "state", $\mathbb{R}^n$ is thus the "state space".

Remarks.

1) One may notice that in this definition and more generally in all the paper, sometimes we are not rigourous from the mathematical point of view ; for instance it should be reasonable to precise that input functions $t \to \mathcal{U}(t)$ have to be at least measurable if one wants to integrate differential

equation (2). Everybody knows that. When the technical assumptions of theorems are just the expected ones we will just say "under technical assumptions" and will be more specific when the assumptions are not expected or rather restrictive.

2) In the definition of the "output" the initial state is zero, which may seems unnatural but, due to the linear structure, this is not restrictive.

I. 1. 2. DEFINITION : The system Σ is "completely controllable" if for every x_o in $\mathbb{R}^n$ (but it turns out that it is enough to check just a the origin) the reachable set $\mathcal{R}(x_o)$ is the whole state space $\mathbb{R}^n$.

I. 1. 3. THEOREM : The system Σ is completely controllable if and only if the matrix whose columns are those of B and of AB , and of $A^2B, \dots, A^{n-1}B$ is of maximum rank n . We write this :

$$\text{rank}\,(B,\ AB,\ A^2B, \dots,\ A^{n-1}B) = n \ .$$

Proof : It is an easy proof based on the knowledge of an explicit formula for the solutions of (2).

I. 1. 4. DEFINITION : The system Σ is "observable" if for every to states x_o and x_1 , for every input $\mathcal{U}$ one has the equality :

$$C\,x(t, x_o, \mathcal{U}) = C\,x(t, x_1, \mathcal{U}),\quad 0 \leq t \leq T ,$$

then the states x_o and x_1 are equals. It is easy to see that this property is equivalent to :

$$\{ C\,x(t, x_o, 0) = 0 \quad 0 \leq t \leq T \} \Rightarrow x_o = 0 \ .$$

I. 1. 5. THEOREM : The system Σ is observable if and only if :

$$\text{rank}\,({}^tC,\ {}^tA\,{}^tC,\ {}^tA^2\,{}^tC, \dots,\ {}^tA^{n-1}\,{}^tC) = n \ .$$

Proof : See proof of Th. I. 3.

Comments :

1) Complete controllability is a property of the pair (A, B) and observability a property of the pair (A, C). One can show that in a suitable system of coordinates every system (A, B) can be decomposed in two subsystems-completely controllable and completely uncontrollable.

2) The above concepts and results are generalised to a large extend to time varying systems, infinite dimensional linear systems and to some extend to systems governed by linear partial differential equations.

I. 2. Realization theory.

Given an input output mapping F , is there a system Σ whose associated input output mapping F_Σ is equal to F . Is it possible to choose Σ in a minimal way. The answer is yes ; let us be more specific.

I. 2. 1. DEFINITION : A non anticipative input output mapping F is "realizable" if there exist a system Σ such that :

$$F = F_\Sigma .$$

Such a system is called a "realization" ; a realization is "minimal" if the dimension of its state space has the smallest possible value. Notice that any realizable input output mapping has always a minimal realization.

To know whether an input output mapping is realizable is a question a little bit to long to be examined here. Let us just say that trivially the input output mapping has to be linear. One classical way to define such a mapping is to look in the space of Laplace ($\mathcal{L}$ (.)) transforms of the inputs and outputs and define a correspondance of the following type :

$$(z \to \mathcal{L}(\mathcal{U})(z)) \longrightarrow (z \to T(z). \mathcal{L}(\mathcal{U})(z))$$

where $\mathcal{L}$ stands for Laplace transform of and T(z) is an $q \times p$ matrice depending on z . It turns out that F is realizable when the entries of T(z) are rational function of z .

I. 2. 2. <u>THEOREM</u> : <u>Let</u> F <u>be a realizable input output mapping</u>. <u>A realization</u> Σ <u>is minimal if and only if it is both completely controllable and observable. Two minimal realizations are isomorphic through linear changes of coordinates</u>.

<u>Proof</u> : The proof is not too difficult. First, one has to just keep on the controllable part of the system (see I. 1. 5 comment (1)) and then to identify indistinguishable points.

There exist some algorithms to compute effectively minimal realizations. See [14], [17], [29].

I. 3. Classification.

So far we are interested by the input output behaviour of the system but not by its internal structure we can introduce an equivalence relation on systems ; namely :

I. 3. 1. <u>DEFINITION</u> : Two systems $\Sigma = (A, B, C)$ and $\Sigma' = (A', B', C')$ are equivalent if there exist a non singular $n \times n$ matrix P such that :

$$A' = PAP^{-1}$$
$$B' = PB$$
$$C' = CP^{-1}$$

which says that the state x of Σ is related to the state x' of Σ' by the linear change of coordinates defined by P.

One problem of interest is to find for each equivalence classes some representant with "nice form". This is the so called problem of canonical forms. For instance one has :

I. 3. 2. <u>PROPOSITION</u> : The completely controllable system :

$$\frac{dx}{dt} = Ax + Bu \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}$$
$$y = x$$

is equivalent to a system of the form :

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \\ \cdots \\ \frac{dx_n}{dt} \end{pmatrix} = \begin{pmatrix} 0, 0, \ldots, a_1 \\ 1, 0, \ldots, a_2 \\ 0, 1, 0, \ldots, 0, a_3 \\ \cdots\cdots\cdots \\ 0, 0, \ldots, 0, 1, a_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} U$$

$$y = x \; .$$

<u>Proof</u> : White A and B in the basis : B ; AB ; ... ; A^{n-1} B .

Canonical forms for general systems are more difficult to find. The first where given by BRUNOVSKY [3] and since 1960 KALMAN devoted much time to this problem. Recent results of KALMAN-HAZEVINKEL [15] (see this proceedings paper by HAZEVINKEL), A. M. PERDON [23] (may be other people) give an answer in the following way : to the pair (A, B) one associate the matrix :

$$R(A, B) = (B, \; AB, \ldots, \; A^n B)$$

in the space of $n \times [(n+1)]p$ matrices. The input output equivalence relation :

$$(A, B) \simeq (A', \; B')$$

turns out to be :

$$R(A, B) \simeq R(A', B') \Leftrightarrow R(A, B) = P \, R(A, B)$$

which is known has the "geometric equivalence" of $n \times [(n+1)]p$ matrices and then allows us to interpret systems as points of the Grassmann variety. Then classicals tools and results of algebraic geometry applies and give satisfactory answers to natural system questions. For instance no "continuous" (i. e. depending continuously on coefficients of the matrices) can exist in in general. See [15] for other results.

I. 4. Identification.

The previous theory is useful in applications only if one is able to find out from the experiments some resonable approximation of the actual input output mapping. This has to do with (linear) stochastic differential equations for two reasons. The first is due to the unavoidable presence of noises on the actual process and measurements, the second, purely technical, due to the fact that the input output mapping is caracterized by the "correlation" between input and output when one takes a white noise as input. We shall not report on these aspects by lake of place and knowledge unless it is a very popular technique. See [5].

II. NON LINEAR SYSTEM THEORY.

II. 1. What is a natural generalization of a non linear system ?

In this paragraph we intend to explain what are the natural considerations which lead us to define a non linear input output system as :

"A manifold M plus a collection of vector fields on M plus a mapping φ from M into some euclidien space $\mathbb{R}^q$ ",

which may seems rather abstract object for applied purposes of system theory.

Our first claim is that the state space must be a differential manifold for various reasons. Many natural systems have manifold has natural state space, for instance in classical mechanics ; but even an engineer who wants to stabilize a satellite attitude has to deal with the tangent bundle to SO(3), that is the phase space for hortogonal matrices. An other reason, perhaps more definitive is given by the system :

$$\frac{dx}{dt} = 1 \quad x \in \mathbb{R}$$

$$y = \sin x .$$

In this system if two initial states x_o and x_1 differs by 2π then the correspondings outputs $\sin(x_o + t) = \sin(x_o + 2\pi + t) = \sin(x_1 + t)$ are identic. If one wants to make this system observable the state space has to be $\mathbb{R}(\text{mod } 2\pi)$, thus the circle S^1.

The second claim is that at least at the first general level the set of inputs values U has to be just a set,finite or infinite, but no special structure has to be assumed on U . For instance, an electrical network controlled by two swiches is a system controlled by an input with four values, with n swiches by an input with $2n$ values ; in biology or chemistry one deals with experiments in presence (or in absence) of some specific product etc ...

The dynamic now can be represented by a non linear differential equation of the following type :

$$\frac{dx}{dt} = f(x,u) \qquad x \in M \ , \ u \in U$$

where for each u the mapping $x \to f(x,u)$ has to be understood as a vector field on M (i. e. to every x in M one associate a vector in the tangent space TM_x at point x). Thus our dynamic is given by a collection $\{x \to f(x,u) \ ; \ u \in U\}$ of vector fields on M . At this stage it seems better to introduce standard notations from differential geometry and to denote by some capital letter X, Y, Z ... vector fields on M . So we do.

II. 1. 1. <u>Notations</u> : We denote by M any (reasonably smooth) Hausdorf manifold by X a (smooth !) vector field on M . Given an initial condition x_o we denote by $X_t(x_o)$ the value at time t (if it exists) of the unique solution of the differential equation :

$$\frac{dx}{dt} = X(x)$$

$$x(o) = x_o \ .$$

II. 1. 2. <u>DEFINITION</u> : A smooth vector X is "complete" if the mapping :

$$(x,t) \to X_t(x)$$

is defined everywhere. On compact manifold every vector field is complete.

II. 1. 3. Comment : In all the paper we shall assume, for convenience, that all the vector fields are complete, this is by no mean necessary except in realization theory (see II. 4 below).

The observation function φ takes its values in some eucliden space $\mathbb{R}^q$. Actually the theory works as well if the values of φ are in some manifold N without any other difficulties.

It is clear now that a "control dynamical system is". Take a piecewise constant input $\mathcal{U}(t)$; $0 \leqslant t \leqslant T$ with values in U ; for every value of the input integrate the corresponding vector field and observe ; this define an input output relation.

I. 1. 4. Comment : One may ask what happend when instead of a set of inputs values without any structure one has, for instance, a subset of $\mathbb{R}^p$. Then it make sense to speak of continuous piecewise continuous, analytic ... inputs. Are the results modified. The answer is no. The main results and difficulties are in the case of piecewise constant inputs. The swich to other kind of inputs is just matter of technique.

II. 1. 5. Comment : We shall report on the theory by the example of a particular case, namely the case where the set of inputs has just two values and the output takesvalues in $\mathbb{R}$ instead of $\mathbb{R}^q$. This gives us some simplification in the notations and we loose nothing essential, but we emphasize that all the theory works with any finite or infinite set of vector fields.

In the study of functions of one or more variables the derivatives or the taylor expansion play a central role. In the case of a control dynamical system the Lie algebra generated by the family of vector fields plays a central role. Because of its importance in the sequel we recall that the Lie brackett of two vector fields X and Y is the vector field defined (locally in a coordinate chart) by :

$$[X, Y]_i(x) = \sum_j \frac{\partial X_i}{\partial x_j}(x)\ Y_j(x) - \sum_j \frac{\partial Y_i}{\partial x_j}(x)\ X_j(x) .$$

The Lie algebra generated by a family $\mathcal{D}$ of vector fields is the smallest submodulus (over smooth functions) of the modulus of vector fields on M which is closed under brackett operations.

II. 2. Définition of a non linear control system as (semi) groups action on manifolds :

We assume (see comment II. 1. 5) that the input set is a set with two elements $U = \{1, 2\}$.

II. 2. 1. DEFINITION : A manifold M and a family $\mathcal{D} = \{X^1, X^2\}$ of two vector fields on M is a "control dynamical system".

Recall that we assume X^1 and X^2 to be complete. Then one has, according to notations of II. 1 :

$$X^i_{t_1+t_2}(x) = X^i_{t_1} \circ X^i_{t_2}(x) \quad ; \quad \forall x \in M \; ; \; \forall t_1, t_2 \in \mathbb{R} \qquad i = 1, 2 \;,$$

$$X^i_o(x) = x$$

$$x \to X_t(x) \quad \text{is a diffeomorphism on } M \text{ for every } t \;,$$

which means that to every vector field is associated a group homomorphism of $\mathbb{R}$ into the set Diff(M) of all diffeomorphisms of M . Let us denote by :

$$X^i : \mathbb{R} \to \text{Diff}(M) \qquad i = 1, 2$$

this group homomorphism ; denote by X^{i+} the semi-group homomorphism obtained when one restricts to $\mathbb{R}^+$.

II. 2. 2. Notation : Let us denote by $G(\mathcal{D})$ and $G^+(\mathcal{D})$ respectively the free product of the two additive groups $\mathbb{R}$ and semi-groups $\mathbb{R}^+$:

$$G(\mathcal{D}) = \mathbb{R} * \mathbb{R}$$
$$G^+(\mathcal{D}) = \mathbb{R}^+ * \mathbb{R}^+ \;.$$

We denote by :

$$\mathcal{D} : G^+(\mathcal{D}) \to \text{Diff}(M)$$
$$\mathcal{D} : G(\mathcal{D}) \to \text{Diff}(M)$$

the two (semi) groups homomorphisms induced on $\mathbb{R}^+ * \mathbb{R}^+$ and $\mathbb{R} * \mathbb{R}$ respectively, by the group homomorphisms X^i $i=1,2$. Namely let s be an element of $\mathbb{R} * \mathbb{R}$. By definition, an element s of $\mathbb{R}^+ * \mathbb{R}^+$ is of the form :

$$(t_1 , i_1) (t_2 , i_2) \dots (t_j , i_j) \dots (t_r , i_r) ,$$

where :

$$r \in \mathbb{N} , \; t_j \in \mathbb{R}^+ \setminus \{0\} ; \; i_j \neq i_{j+1}$$

and the diffeomorphism $\mathcal{D}^+(s)$ is :

$$\mathcal{D}^+(s) = X_{t_1}^{i_1} \circ X_{t_2}^{i_2} \circ \dots \circ X_{t_j}^{i_j} \circ \dots \circ X_{t_r}^{i_r} .$$

On $G(\mathcal{D})$ you define $\mathcal{D}$ in the same way except that in $G(\mathcal{D})$ elements of t, of the sequence are not supposed to be positive.

II. 2. 3. <u>DEFINITION</u> : The (positive) orbit $G^{(+)}(\mathcal{D}).x$ of a point x under the action of the control dynamical system defined on M by $\mathcal{D}(X^1, X^2)$ is the set of points of the form :

$G^{(+)}(\mathcal{D}).x := \{\mathcal{D}(s)(x) \; ; \; s \in G^+(\mathcal{D})\}$

or :

$G^+(\mathcal{D}).x := \{X_{t_1}^{i_1} \circ X_{t_2}^{i_2} \circ \dots \circ X_{t_2}^{i_2}(x) ; \; (t_1, i_1)(t_2, i_2), \dots ,(t_r, i_r) \in G^+(\mathcal{D})\}$

$G(\mathcal{D}).x := \{\mathcal{D}(s)(x) \; ; \; s \in G(\mathcal{D})\}$

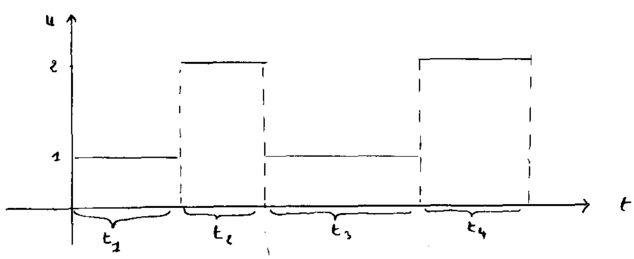
or :

$G(\mathcal{D}).x := \{X_{t_1}^{i_1} \circ X_{t_2}^{i_2} \circ \dots \circ X_{t_r}^{i_r}(x) ; \; (t_1, i_1)(t_2, i_2) \dots (t_r, i_r) \in G(\mathcal{D})\}.$

This definition and definition 2. 5 are justified by the following :

II. 2. 4. <u>Remark.</u> The set of right continuous piecewise constant inputs with values in the set $\{1, 2\}$ is naturally isomorphic to $\mathbb{R}^+ * \mathbb{R}^+$.

Take an input $\mathcal{U}(t)$; $0 \leqslant t \leqslant T$ like the one drawn below :



then to $\mathcal{U}$ associate the element of $\mathbb{R}^+ * \mathbb{R}^+$ defined by :

$$(t_4, 2)\ (t_3, 1)\ (t_2, 2)\ (t_1, 1)$$

(it is necessary to reverse the order because diffeomorphisms are composed on the left). This clearly defines an isomorphism between the semigroup of inputs (under concatenation) and $\mathbb{R}^+ * \mathbb{R}^+$. Let us denote by $\mathcal{U}$ the set of all inputs and define the "reachable set from x", $\mathcal{R}(x)$ as the set of values at time T $(0 \leq T < \infty)$ of solutions of differential equations :

$$\frac{dx}{dt} = f(x, \mathcal{U}(t)) \ ; \ \mathcal{U} \in U$$

with :

$$f(x, 1) = X^1(x) \ ; \ f(x, 2) = X^2(x) \ .$$

It is trivial to see that one has the equality :

$$(4) \qquad \mathcal{R}(x) = G^+(\mathcal{D}).\, x \ .$$

Thus the study of the controllability of a non linear system turns out to be the study of the positive orbit through x. One must notice that the relation $x < y \Leftrightarrow y \in G^+(\mathcal{D}).\, x$ defines an order on M and if one looks for the induced equivalence relation, $G(\mathcal{D}).\, x$, the orbit through x, is precisely the class of x. The physical meaning of this equivalence relation is the following one : let us say that two states x_1 and x_2 are comparable if there exists an experiment (i. e. an input) such that either x_2 is reachable from x_1 under the action of the input either x_1 is reachable from x_2 and that two states x_1 and x_2 are comparable if there exists a finite chain of comparable states from x_1 to x_2. Two states are comparable if one is in the positive orbit of the other one, and are related if they are in the orbit of the other. We hope that this is enough to justify the following :

II. 2. 5. <u>DEFINITION</u> : Let $(M, \mathcal{D})$ be a control dynamical system, one says that the system :

- has accessibility property iff $\forall x \in M$, $\mathrm{Int}\,(G^+(\mathcal{D}).\, x) \neq \emptyset$
- is controllable iff $\forall x \in M$, $G^+(\mathcal{D}).\, x = M$
- is orbitally minimal iff $\forall x \in M$, $G(\mathcal{D}).\, x = M$.

II. 2. 6. DEFINITION : An "observed" non linear control dynamical system is a triple :

$$\Sigma = (M, \mathcal{D}, \varphi)$$

where $(M, \mathcal{D})$ is a control system and φ is a smooth function on $\mathbb{R}$. An "observed initialized" non linear control dynamical system is a 4-triple $\Sigma = (M, \mathcal{D}, x_o, \varphi)$ where $(M, \mathcal{D}, \varphi)$ is an observed system and x_o just a fixed initial condition in M. One defines the associated input output mapping in the same way like in definition I. 1.1.

II. 2. 7. DEFINITION : An observed system is observable iff given two different states x_1 an x_2 in M there exists at least one input $\mathcal{U}(t)$, $0 \leq t \leq T$, or equivalently one element of the semi-group $G^+(\mathcal{D})$ such that :

$$\varphi(\mathcal{D}(s).\, x_1) \neq \varphi(\mathcal{D}(s).\, x_2)$$

(see II. 2. 2 for notation $\mathcal{D}(s)$) .

II. 3. Non linear system behaviour-results :

To state the results we need a technical definition.

II. 3. 1. DEFINITIONS : The "rank $r(\mathcal{D})(x)$" of a family of vector fields at point x is the dimension of the linear subspace of TM_x defined by the values at point x of the elements of the Lie algebra generated by $\mathcal{D}$.

The "observability rank $\theta . x$" of a family $\mathcal{D}$ of vector fields and a function $\varphi : M \to \mathbb{R}$ at point x is the dimension at x of the smallest linear space of one forms containing $d\varphi$ and closed under Lie differentiation by elements of $\mathcal{D}$.

II. 3. 2. THEOREM : i) The system $(M, \mathcal{D})$ has accessibility property if (and only if in analytical case) the rank $r(\mathcal{D})(x)$ is equal to m (dim of M) at each point.

ii) The system $(M,\mathcal{D})$ is orbitally minimal if (and only if in analytical case) the rank $r(\mathcal{D})(x)$ is equal to m (dim of M) at each poin

iii) The observed system $(M,\mathcal{D},\varphi)$ is observable if (and only if in analytical case) the observability rank $0.x$ is equal to m (dim of M) at each point.

iv) The orbit through x, $G(\mathcal{D}).x$ has always a natural manifold structure, of dimension equal to the rank $r(\mathcal{D}).x$ at x in the analytical case.

Proof. This theorem is a "résumé" of various results from different authors including HERMANN, HERMES, JURDJEVIC, KRENER, SUSSMANN ... Proof of points i), ii), iv) can be found in [19], proof of point iii) is in [11].

II. 3. 3. Comments.

1) There is no analogue of controllability results (I. 1. 3) for linear systems. This is because controllability is intimately associated to some "global rigid" property like linearity. Actually controllability and observability results generalize to systems defined on lie groups but we shall not report on it see [1], [2], [12].

2) By opposition to the linear case we see that controllability and observability are intimately mixed together in the definition of the observability rank, and it shows by the way that no reasonable "duality theory" between controllability and observability is to be expected in general.

3) The analyticity needed to obtain global results from rank knowledge is clear.

II. 4. Realization theory - results.

II. 4. 1. DEFINITION : A non anticipative input output mapping F is realizable if there exists an initialized observed system $(M,\mathcal{D},x_o,\varphi)$ whose associated input output mapping F_Σ is equal to F (see II. 2. 6 for def. of F_Σ).

II. 4. 2. DEFINITION : A minimal realization Σ of a realizable input output mapping is a realization $(M, \mathcal{D}, x_o, \varphi)$ which is both orbitally minimal and observable.

II. 4. 3. THEOREM : Assume that Σ and Σ' are two minimal realizations of a realizable input output mapping ; if the vector fields of $\mathcal{D}$ and $\mathcal{D}'$ are complete and analytic then Σ and Σ' are isomorphic (in the natural meaning).

Every realizable input output mapping has a minimal realization if it is realizable by a complete analytic system.

Proof : See [27] .

II. 4. 4. Comments.

1) Minimality cannot be defined in terms of dimension of M , for instance $(\mathbb{R}, \frac{\partial}{\partial x}, 0, \sin)$ and $(\mathbb{R} (\bmod 2\pi), \frac{\partial}{\partial x}, 0, \sin)$ are two realization of the same I. 0. map . $\mathbb{R}$ and S^1 have the same dimension but are not isomorphic.

2) It is a little bit to long to explain why completeness and analyticity are essential in II. 4. 3 we refer to [27] . We just give examples without any comments.

Example 1. $\Sigma = (\mathbb{R}, |x| \frac{\partial}{\partial x}, 0, \varphi(x) = \{ \begin{smallmatrix} 0 & x \leqslant 0 \\ x & x > 0 \end{smallmatrix} \})$. (One may replace $|x| \frac{\partial}{\partial x}$ and φ by smooth but nonanalytic vector field and function). To make this system observable one have to identify all negative points and thus the quotient is $[0, \infty]$ which is not a manifold. More wild examples can be found.

Example 2. $\Sigma = (\mathbb{R}^2, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, (0, 0), \varphi(x, y) = x+y^2)$, $\Sigma' = (\mathbb{R}^2 / \{-1, 0\}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, (0, 0), \varphi(x, y) = x + y^2)$ are both minimal realization of the same I. O. map, but the second one (because of the "hole" at $\{-1, 0\}$) is not complete and thus they are not isomorphic ($\mathbb{R}^2$ and $\mathbb{R}^2 / \{-1, 0\}$ are not homeomorphic).

II. 5. Classification.

So far it seems that there is no reasonable definition of equivalence of control dynamical system. If the family $\mathcal{D}$ reduces to 1 vector field then we have the standard definitions of equivalence of dynamical system via the phase portrait. Tentative has been made (see [7] and [8]) but where not very conclusive. Because of the lake of equivalence relation we have no clear idea of what is a structurally stable control system, because it has to be a system with a similar (equivalent) behaviour if one changes a little the datas. The only available result in this spirit is the next one.

II. 5.1. THEOREM : Let S denote the set of all systems Σ with the same underlying manifold M . Topologize S with natural topologies for smooth function spaces. Then there exists an open dense (i.e. generic) subset of S , say G , such that :

$$\Sigma \in G \Rightarrow \begin{array}{l} \Sigma \text{ has accessibility property} \\ \Sigma \text{ is orbitally minimal .} \end{array}$$

Proof. See [18], [25] .

Some results on "stability" of reachable set are in [4] .

Local classification (classification around some fixed point in M) via algebraic properties of the Lie algebra generated by $\mathcal{D}$ or classification on Lie groups are available in BROCKETT, HIRSHORN, HERMES, KRENER... but we shall not report on them.

II. 6. Identification.

We shall not report on identification (and non linear filtering) by lake of knowledge on the subjet. We just want to point out a question which seems (at least to me) important. In linear theory one replace, when possible, the deterministic input $\mathcal{U}$ by a "pseudo-white noise" generated by some technological apparatus and then analyse the stochastic output as if it where generated by a white noise. This is legitimate by convergence theorems. In the non linear case, in the presence of more than one input, such convergence theorems do not exist and one must be very suspicious about the results. One can see the book [21] and papers [6], [20], [28] on this question.

III. How realization theory can be used (if it can !).

In this last part we try to explain our feeling about usefulness of this theory in experimental sciences. Our feeling is that realization theory can hardly help to find "the good model" of some experimental situation but it can help to reject a model. More precisely, one has to analyse qualitative properties of the tentative model and then to reject it if these properties do not fit well with actual properties. It is a similar idea of that expressed by THOM is Siam Review Vol. 19, n°2 : "The statements that C. T. allows one to produce are of the following nature : "if, in the interval of time (t_0, t_1), the system exhibited some morphology (M_0^1), then one has to expect that in a further interval (t_1, t_2) it will exhibit some morphology (M_1^2)". Such a statement can never be considered as an absolutely certain prediction, such as the ones derived from physical laws. The future morphology (M_1^2) derives from (M_0^1) by an hypothesis about the simplicity of the underlying dynamics. If the prediction is realized, then there is nothing to be surprised about. If the prediction fails (that may happen) and a morphology M_{12} different from M_1^2 does appear, this is interesting, because it shows that our original assumptions were too simple, and some new element of complication has to be introduced into the picture. Paradoxically, one could say that C. T. is more interesting when it fails than when it is successful."

III. 1. Chemical kinetics.

There exist chemical reactions like the celebrated B. Z. Z. reaction (see [9]) which exhibit periodic outputs in presence of non periodic inputs. This can be explained only by some non linear differential equations.

It is assumed that two known complex products A and B when they react are decomposed into intermediates products $x_1\ x_2\ x_3 \dots x_m$. Some of them are observable the others are not and even the number m of intermediates is unknown. Chemist want to know them. As soon as $x_1 x_2 \dots x_m$

are known the kinetics of the reaction are rather well known as some systems of differential equations of the following type :

$$\frac{dx_i}{dt} = \sum_j a_{ij} x_j + \sum_{j,k} b_{ijk} x_j x_k + c_i \tag{5}$$

and then one can compute the input output behaviour of the model and compare to actual datas. This looks very much as realization problem. To be a little bit more specific assume that A and B are constant and known, x_1 is observed and is a periodic function. Try to make a model which respects the constraints : (5) and moreover where x_i (which is a concentration or chemical activity) is positive. There is a difference with our previous realization theory in that we ask explicitely the realization to be imbeded in $\mathbb{R}^n$ in a very specific way. By the way if it turns out that the manifold M of the minimal realization is, let us say for fun, a klein bottle, then it must be imbeded at least in $\mathbb{R}^4$, and we need at least four intermediate products to explain the reaction. It seems to us important to ask for imbeded realization, because usually people wants to explain phenomenas in terms of a certain number of variables (small number if possible) related by algebraic or differential equalities.

III. 2. Imbeded realization.

We define imbeded realization and minimal imbeded realization with respect to a class of admissible vector fields and admissible observations.

III. 2. 1. Notation.

We denote, for $n = 0, 1, \dots, \infty$, by :

$$\mathcal{A}_n \subset V(\mathbb{R}^n)$$

a subset (non empty !) of the set $V(\mathbb{R}^n)$ of all smooth (non everywhere defined) vector fields of $\mathbb{R}^n$. For instance $\mathcal{A}_n$ may be the set of linear vector fields, or non vanishing vector fields ... etc. In the same way we denote by O_n $n = 0, 1, \dots,$ a subset of the set of smooth mappings from $\mathbb{R}^n$ into $\mathbb{R}$.

III. 2. 2. DEFINITION : Let $\mathcal{A}_n$ $(n = 0, 1, \ldots, \infty)$ be the class of admissible vector fields and θ_n the class of admissible observation functions. Let t be a realizable input output mapping, an "admissible realization" Σ of F is a 5-tuple :

$$(\Sigma = (M, \mathcal{D}, x_0, \varphi), \mathbb{R}^n)$$

where Σ is a realization of F and moreover :

* M is an imbeded submanifold of $\mathbb{R}^n$.

** The vector fields X of $\mathcal{D}$ are restriction to M of vector fields in $\mathcal{A}_n$.

*** The function φ is the restriction to M of some function in $\mathbb{R}^n$.

II. 2. 3. DEFINITION : An admissible realization $(\Sigma = (M, \mathcal{D}, x_0, \varphi), \mathbb{R}^n)$ is "minimal" iff :

$$\Sigma = (M, \mathcal{D}, x_0, \varphi)$$

is minimal and n has the smallest possible value.

Now we state, as an example, a theorem (definitely trivial) for admissible realization with constant inputs and periodic output in the class of vector fields with no singularities and everywhere defined.

III. 2. 4. THEOREM : Let $\mathcal{A}_n$ denote the set of everywhere defined vector fields of $\mathbb{R}^n$ without any singularities (zeros). Let O_n be the set of all mappings :

$$\varphi : \mathbb{R}^n \to \mathbb{R}.$$

Let F be a constant input, periodic output mapping. Then F has admissible realization imbeded in $\mathbb{R}^3$ but none in $\mathbb{R}^2$.

Proof. Denote by : $t \to y(t)$, $0 \leqslant t \leqslant T$ the periodic output. Denote by $M = \{x, y, z,\ x^2 + y^2 = 1\ ;\ z = 0\}$. Denote by X the vector field :

$\frac{T}{2\pi}(-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y})+(x^2+y^2-1)\frac{\partial}{\partial z}$, let $\mathcal{D}=\{X\}$. Denote by φ any smooth mapping whose restriction to M is given by $\varphi(x,y,0)=y(\frac{T\theta}{2\pi}(x,y))$, where $\theta(x,y)$ is the polar angle of (x,y) in the plane. This is definitely an admissible imbeded realization in $\mathbb{R}^3$. Moreover $(M,\mathcal{D},0,\varphi)$ is a minimal realization. This realization cannot be imbeded in $\mathbb{R}^2$ because then the manifold M has to be a smooth jordan curve in the plane and $\mathcal{D}$ must be the restriction to M of some nowhere vanishing vector field everywhere defined in the plane. By Poincarre Bendixon theorem we know that this is impossible.

This theorem and its proof was given to illustrate our idea but no general results are known presently on "admissible minimal realization".

III. 2. 5. Example.

We apply Th. III. 2. 4. to an academic situation. We observe a population composed of two different type of fisches : x_1 the big ones, x_2 the small ones. We notice that the population of small fishes is a periodic function of time. If we try to explain this on the basis of a prey-predador phenomena ; the big fishe eat the small ones. We use a classical model for the dynamics :

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a\,x_1 + b\,x_1 x_2 \qquad a>0 \quad b>0 \\ \frac{dx_2}{dt} &= c\,x_2 + d\,x_1 x_2 \qquad c>0 \quad d<0 \end{aligned}\right\} \quad \begin{aligned} x_1 &> 0 \\ x_2 &> 0 \end{aligned}$$

Direct examination of this class of admissible vector fields shows that they have no critical points in the positive orthant and thus by Th. III. 2. 4. (or by direct use of Poincarre Bendixon theorem !!) we have a contradiction. One cannot explain the periodic phenomena on the basis of prey-predator action. This type of considerations has been used successefully (in a little bit more complicate setting) in chemical kinetics, see [9], [22] .

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