Position Space Renormalization Group

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Abstract

An outline is given of the position space renormalization group approach to phase transition and critical phenomena. The method is illustrated by simple examples of renormalization transformations for the two-dimensional Ising model.

Introduction

In this colloquium on group theoretical methods in physics, a lecture on the position space renormalization group approach to phase transitions and critical phenomena is somewhat displaced, since so far very little group theoretical methods are used in this field. It is the hope of the author that this contribution will help to wet the appetite of group theoreticians for this intruiging field of power laws and scaling behavior with seemingly arbitrary but yet universal exponents. Therefore it seems preferable, rather than trying to explain in a short paper a fairly heavy machinery which is well documented elsewhere[1)], to illustrate the method at the hand of simple examples for which the two dimensional Ising model is ideally suited.

The renormalization approach pretends to have given a clue to the understanding of the old problem of critical phenomena. In position space variety of the family of renormalization procedures is less powerful than e.g. the field theoretic method but has the advantage of being simpler in its set up and staying closer to the microscopic description of a system.

In order to appreciate the contribution of the renormalization approach we start with a brief summary of the basic features of the ferromagnetic phase transition in Ising type system.

Then the renormalization group is introduced and a simple realization of (approximate) renormalization transformations is constructed.

From the renormalization transformations the critical and phase transition properties can be deduced and the paper ends with a note explaining how symmetry plays an essential role in the determination of fixed points and their properties, which are the key ingredients of the renormalization approach.

1. Power laws and critical scaling behavior

Consider a lattice with N sites, each being occupied by a spin s_i having values $s_i = \pm 1$. Let the interaction energy of this system of N degrees of freedom be given by

$$H(s) = H\sum_i s_i + K\sum_{<ij>} s_i s_j \qquad (1)$$

where the first term represents a magnetic energy, the second term a nearest neighbor interaction energy with the sum running through all nearest neighbor pairs <ij> on the lattice (with periodic boundary conditions).

The problem is to determine the free energy

$$F = \log \mathrm{Tr} \exp H(s) \qquad (2)$$

where the Tr runs through all 2^N configurations of the N spins. The usual $-1/k_B T$ in the exponentials are incorporated in constants H,K of H(s) and in the free energy F. Therefore we will call H a field-like variable and K a temperature-like variable. The free energy per site f = F/N exhibits in the thermodynamic limit ($N\to\infty$) a singularity along the H = 0 axis for $K > K_c$ and lattice dimension $d \geq 2$. f(H,K) is for all K an even function of H. For $K \leq K_c$ the zero field magnetisation

$$m(K) = [\partial f(H,K)/\partial H]_{H=0} \qquad (3)$$

vanishes and for $K > K_c$ it has a finite value (the spontaneous magnetisation). Thus f(H,K) has for small H the behavior

$$\begin{aligned} K < K_c \qquad & f(H,K) = \tfrac{1}{2}\chi(K)H^2 + O(H^4) \\ K > K_c \qquad & f(H,K) = m(K)|H| + O(H^2) \end{aligned} \qquad (4)$$

and in between the curious critical behavior

$$K = K_c \qquad f(H,K_c) = A|H|^{1+1/\delta} + \dots \qquad (5)$$

with the critical exponent $\delta = 15$ for d = 2 and $\delta = 4.817$ (estimated[2)]) for d = 3. For H = 0, f(0,K) behaves near the critical coupling (or inverse temperature) K_c as

$$f(0,K) \sim |K-K_c|^{2-\alpha} \qquad (6)$$

with another critical exponent α (the specific heat exponent) with $\alpha = 0.0$ for $d = 2$ (implying a $(K-K_c)^2 \ln|K-K_c|$ behavior) and $\alpha = 0.11$ for $d = 3$.

The behavior around $H = 0$ and $K = K_c$ can be summarized in a scaling formula

$$f(H,K) \sim |K-K_c|^{2-\alpha} \Phi(x)$$
$$\text{with } x \sim |H|/|K-K_c|^{\Delta} \tag{7}$$

where the crossover exponent $\Delta = \delta(2-\alpha)/(1+\delta)$. The expression (7) contains all other power laws valid at K_c. For instance the behavior of the spontaneous magnetisation for $K > K_c$ follows by differentiation with respect to H as

$$m(K) \sim \dot{\Phi}(0)|K-K_c|^{\beta} \tag{8}$$

with

$$\beta = 2 - \alpha - \Delta = (2-\alpha)/(1+\delta) \tag{9}$$

The intriguing point about this curious behavior is that the basic two exponents α and δ and the function Φ are supposedly universal i.e. independent of

i) the structure of the lattice (triangular, square etc)

ii) the size of spin variables i.e. $s = \frac{1}{2}, 1, 0, \ldots$ as long as they are scalers.

iii) the character of the interaction: next nearest neighbor and higher interactions may be added to (1).

Besides the already mentioned dimensionality d of the lattice, also the dimensionality n of the spin variable is important for the character of the critical behavior. We have as realizations

$n = 1$	Ising model, gas-liquid
$n = 2$	X-Y model, λ-transition Helium
$n = 3$	Heisenberg model

while the limit $n = \infty$ yields the spherical model and $n = 0$ the self avoiding random walk problem.

In addition to the parameter d and n, the phase transition is dependent on the less easy to quantify notion of symmetry.

2. The renormalization transformation

By a renormalization transformation we want to map a hamiltonian H(s) for a system with N degrees of freedom onto a hamiltonian H'(s') for a system having a lesser number N' of degrees of freedom. The map is constructed such that the free energy of both systems are simply related. This can be achieved by introducing a coupling hamiltonian $H_I(s',s)$ with the property

$$\mathrm{Tr}' \exp H_I(s',s) = 1 \tag{10}$$

such that the free energy of the system H(s) with coupling $H_I(s',s)$ equals F

$$\log \mathrm{Tr}\,\mathrm{Tr}' \exp [H_I(s',s) + H(s)] = \log \mathrm{Tr} \exp H(s) = F \tag{11}$$

Now we define H'(s') as

$$G + H'(s') = \log \mathrm{Tr} \exp[H_I(s',s) + H(s)] \tag{12}$$

where G is determined by the restriction

$$\mathrm{Tr}' H'(s') = 0 \tag{13}$$

as it was also tacitly assumed that H(s) was traceless to start with. Due to (10) the free energy F' of H'(s')

$$F' = \log \mathrm{Tr}' \exp H'(s') \tag{14}$$

is related to F by

$$F = G + F' \tag{15}$$

Rather than considering the map (12) in abstract hamiltonian space we prefer to see (12) as a transformation in the parameter space of the interaction constants. Let us denote the set of interaction parameters (e.g. H,K in (1)) collectively by K. Then by (12) the K' of H(s) are function of K and so is g = G/N:

$$\begin{cases} K' = K'(K) \\ g = g(K) \end{cases} \tag{16}$$

The free energies F and F' are functions of N,K and N',K'. In the thermodynamic limitwe have for the free energy per site according to (15)

$$f(K) = g(K) + b^{-d} f(K'(K)) \tag{17}$$

where we have introduced the reduction factor

$$b^{d} = N/N' \tag{18}$$

such that b is a linear scale contracting factor.

Although (17) only establishes a relation between the free energies at different points K and K' in the parameter space it contains through iteration the solution for free energy. If we denote the sequence of points which are obtained from K by repeated use of (16) by K, K', ... $K^{(j)}$, ..., we may solve (17) as

$$f(K) = \sum_{j=0}^{\infty} b^{-dj} g(K^{(j)}) \tag{19}$$

provided we assume the boundary condition

$$\lim_{j\to\infty} b^{-dj} f(K^{(j+1)}) \to 0 \tag{20}$$

which is usually easy to check as $K^{(j)}$ moves in most cases to a trivial regime where the free energy is sufficiently bounded that (20) holds. By (19) one has expressed the free energy in terms of the renormalization transformation (RT) (16).

For the study of critical phenomena the RT (16) is only of value if it has two additional properties:

i) The RT exhibits a non trivial fixed point $K^{\star}$ i.e. a point with

$$K^{\star} = K'(K^{\star}) \tag{21}$$

ii) In the neighborhood of $K^{\star}$ the functions K'(K) and g(K) are regular.

Within this context one sees the aim of the renormalization approach: to explain the singularities in f(K) in terms of the regular function f(K) and K'(K) through the use of (17).

Even without calculation one can deduce certain properties from the regularity assumptions. From (17) one sees that if K is a point in which f(K) is singular than K' is also such a point. Thus points of similar singularity (phase transitions, critical points) form invariant subspaces of the RT (16). In each of the invariant subspace we may find fixed points. The one in the critical subspace is of most interest to us.

3. Examples

Consider a triangular lattice and put a new spins s_i' between three old spins

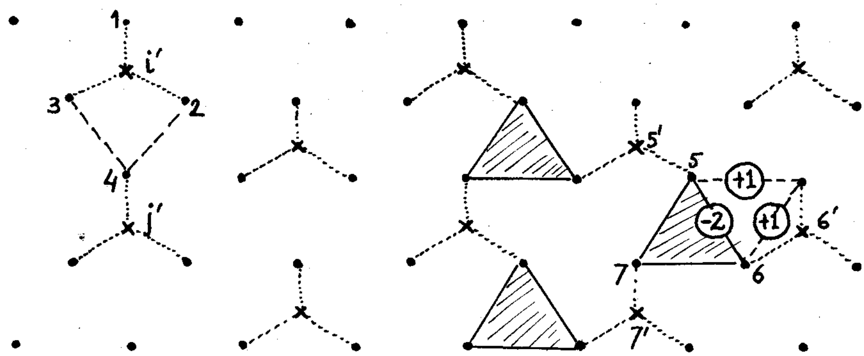


fig. 1. • sites of older spins ········ coupling between new and old spins (p)
x sites of new spins ---- near neighbor coupling (K).

s_1, s_2, s_3 in the way indicated in fig. 1. A coupling hamiltonian is introduced which reads for the i' spin

$$H_{i'}(s',s) = ps'_{i'}(s_1+s_2+s_3) - a - b\,(s_1s_2+s_2s_3+s_3s_1) \tag{22}$$

where a and b have to be chosen as

$$a = (\tfrac{1}{4})\log\,[\,2^4 \cosh 3p \cosh^3 p] \qquad b = (\tfrac{1}{4})\log\,[\cosh 3p/\cosh p] \tag{23}$$

such as to comply with condition (1). The total $H_I(s',s)$ is the sum over all triangles with new spins inside. The parameter p is arbitrary and will be decided upon later. The coupling $H_I(s',s)$ is such that if $H(s)$ has the triangular symmetry the new $H'(s')$ will have it too.

For this coupling we will construct approximations to the RT (16) which yield upper and lower bounds to the free energy.

For the lower bound we separate

$$H_I(s',s) + H(s) = H_o(s',s) + V(s) \tag{24}$$

where $H_o(s',s)$ contains besides $H_I(s',s)$ the part of H(s) which concerns the couplings inside a triangle. Thus for the triangle 123 we put

$$H(s_1+s_2+s_3) + K(s_1s_2+s_2s_3+s_3s_1)$$

into $H_o(s',s)$. Then V(s) contains only interactions between the triangles containing a new spin.

Now we have the inequality

$$\log \mathrm{Tr} \exp [H_o(s',s) + V(s)] \geq \log[\mathrm{Tr} \exp H_o(s',s)] + \langle V(s)\rangle_o \tag{25}$$

with $\langle V(s)\rangle_o$ defined by

$$\langle V(s)\rangle_o = [\mathrm{Tr}\, V(s) \exp H_o(s',s)] / [\mathrm{Tr} \exp H_o(s',s)] \tag{26}$$

(Making use of $\langle \exp V\rangle \geq \exp \langle V\rangle$). The evaluation of the two terms on the right hand side of (25) is simple since $H_o(s',s)$ is a sum of isolated triangle hamiltonians. One finds for H = 0

$$\begin{aligned} \log \mathrm{Tr} \exp H_o(s',s) &= G_o(K) = (N/3)\, [\exp 3K + 3 \exp - K] \\ \langle V(s)\rangle_o &= 2K\, q^2(K) \sum_{\langle i'j'\rangle} s_i'\, s_j' \end{aligned} \tag{27}$$

with q(K) given by

$$q(K) = [\mathrm{tgh}3p \exp 3K + \mathrm{tgh}p \exp -K] / [\exp 3K + 3 \exp -K] \tag{28}$$

For $H \neq 0$ the expressions are somewhat more complicated. One convinces oneself easily that a situation without a field H carries over into a situation without new field H' due to the invariance of the coupling $H_I(s',s)$ under a simulaneous spin flip of new and old spins.

Defining on the basis of (27) the approximate RT

$$\begin{cases} K_1'(K) = 2K\, q^2(K) \\ g_1(K) = (\frac{1}{3}) \log [\exp 3K + 3 \exp - K] \end{cases} \tag{29}$$

one has due to (25) that

$$G + H'(s') \geq G_1 + H_1'(s') \qquad (30)$$

where $G_1 = Ng_1$ and $H_1'(s')$ is a nearest neighbor hamiltonian with K_1' as coupling. The structure (29) of $K_1'(K)$ can be understood easily if one realizes that a typical term in $<V>_o$ is for the triangles i' and j'

$$K < s_3 s_4 >_o = K < s_2 s_4 >_o = K[< s_3 >_o + < s_2 >_o] < s_4 >_o = 2K\, q^2(K)\, s_i' s_j'$$

q(K) being the average alignement of new and old spins. Note that q(K) increases for every K with the strength of the coupling p. (30) tells us that the free energy

$$f(K) = g(K) + b^{-d} f(K'(K)) > g_1(K) + b^{-d} f(K_1'(K)) \qquad (31)$$

and thus by iteration that

$$f(K) \geq \sum_{j=0}^{\infty} b^{-dj}\, g_1(K_1^{(j)}(K)) \qquad (32)$$

Since $g_1(K)$ is monotonically increasing with K, the best bound is that which makes $K_1^{(j)}(K)$ maximally large and this happens for $p = \infty$. We note that for $p = \infty$ the relation (29) has three fixed points $K^\star = 0$, $K^\star \neq 0$ and $K^\star = \infty$.

The non-trivial fixed point is obtained for

$$q(K^\star) = 1/\sqrt{2} \qquad (33)$$

or $K^\star = (\frac{1}{4}) \log[(3 - \sqrt{2})/(\sqrt{2} - 1)]$

Obviously the $K_1'(K)$ and $g_1(K)$ as given by (29) and (28) with $p = \infty$, are analytic around the fixed points.

Next we sketch the construction of an upper bound[3)]. The starting point is to find a V(s) satisfying

$$\mathrm{Tr}\, V(s) \exp H(s) = 0 \qquad (34)$$

A realization of such a V(s) can be constructed for a nearest neighbor interaction by giving the bounds alternating weight factors as e.g. indicated in fig. 1. Then a second requirement is that

$$H_o(s',s) = H_i(s',s) + H(s) - V(s) \qquad (35)$$

is a managable hamiltonian i.e. a sum of non interacting fields. By the weights of fig. 1 one can constrain the $H_o(s',s)$ to the shaded triangles only. For the shaded triangle of sites 5,6,7 we obtain for the contribution to $H_o(s',s)$

$$H_{567}(s',s) = p(s_5's_5 + s_6's_6 + s_7's_7) + H(s_5 + s_6 + s_7) - a + \\ + (3K - b)\ (s_5s_6 + s_6s_7 + s_7s_5) \tag{36}$$

From (26) we derive

$$0 = \mathrm{Tr'Tr}\ V(s)\ \exp\ [H_I(s',s) + H(s)] = \mathrm{Tr'Tr}\ V(s)\ \exp\ [H_o(s',s) + V(s)] \tag{37}$$

This implies that (see (25))

$$F = \log\ \mathrm{Tr'Tr}\ \exp\ [H_o(s',s) + V(s)] \leq \log\ \mathrm{Tr'Tr}\ \exp\ H_o(s',s) \tag{38}$$

Then the approximate RT defined by

$$G_u + H_u'\ (s') = \log\ \mathrm{Tr}\ \exp\ H_o(s',s) \tag{39}$$

implies a free energy bound

$$F \leq G_u + F_u'$$

or
$$f(K) \leq g_u(K) + b^{-d}\ f(K_u'(K)) \tag{40}$$

By iteration the upper bound becomes

$$f(K) < \sum_{j=0}^{\infty} b^{-dj}\ g_u(K_u'(K)) \tag{41}$$

The actual calculation of $G_u(K)$ and $K_u'(K)$ from (39) and $H_o(s',s)$, given by a sum of contributions of the type (36), yields for H = 0:

$$\begin{cases} g_u(K) = (1\tfrac{1}{2})\ \log[Z_1(K)\ Z_2^3(K)] \\ K_u'(K) = (\tfrac{1}{4})\ \log[(Z_1(K)/Z_2(K))] \end{cases} \tag{42}$$

with

$$\begin{cases} Z_1(K) = \exp 9K + 3\exp - 3K \\ Z_2(K) = \exp (9K - 4b) + (2+ \exp 4b) \exp - 3K \end{cases} \tag{43}$$

The optimalization of the bound (41) through variation of p (which enters via b(p) defined in (23)) is in general different but in this case simple since p enters only through $Z_2(K)$ into the problem and the optimal bound result under the condition that $Z_2(K)$ is stationary i.e.

$$\partial Z_2(K)/\partial b = [-4 \exp (9K - 4b) + 4 \exp (4b - 3K)] \; \partial b/\partial K = 0$$

$$\text{or} \quad b = 3K/2 \tag{44}$$

which reads explicitly for p

$$p(K) = (\tfrac{1}{2}) \log [1 + \exp 6K + [(1 + \exp 6K)^2 - 4]^{\frac{1}{2}}]/2 \tag{45}$$

The optimal value for p or b simplifies (34) to

$$\begin{cases} g_u(K) = (1/12) \log (\exp 9K + 3 \exp - 3K)(2 \exp 9K + 2 \exp - 3K)^3 \\ K_u(K) = (\tfrac{1}{4}) \log (\exp 9K + 3 \exp - 3K)/(2 \exp 9K + 2 \exp - 3K) \end{cases} \tag{46}$$

One verifies that $K_u(K)$ again has fixed points at $K^\star = 0$, $K^\star \neq 0$ and $K^\star = \infty$. Now the non-trivial fixed point has the value $K^\star = 0.3798$.

4. Relation between fixed point and critical properties

In the previous section two approximate RT's were constructed (29) and (38) leading to a lower and upper bound for the free energy. More interesting than being a bound is the fact that the expressions lead to a singularity in the free energy. Strictly speaking such behavior should be deduced from the series (32) and (41) which indeed can be done elegantly but it needs the somewhat involved introduction of scaling fields[4]. Here we will follow the simpler and less ambitious route be assuming the existence of a singularity of the type

$$f_{sing}(K) \simeq A|K-K^{\star}|^{2-\alpha} \tag{47}$$

and investigating the possibilities for the exponent α. Near $K^{\star}$ the RT may be linearized

$$K' - K^{\star} = (\partial K'/\partial K)_{K^{\star}} (K - K^{\star}) + .. \tag{48}$$

Then inserting (47) and (48) in the basic relation (17) and remembering that g(K) is regular we find by comparing singular parts

$$A|K-K^{\star}|^{2-\alpha} = b^{-d} A|(\partial K'/\partial K)_{K^{\star}} (K-K^{\star}|^{2-\alpha} \tag{49}$$

Thus α has to fulfil

$$1 = b^{-d} \lambda_T^{2-\alpha} \tag{50}$$

with

$$\lambda_T = (\partial K'/\partial K)_{K^{\star}} \tag{51}$$

(50) tells that the exponent α is determined by the derivative λ_T at the fixed point. (The subscript T on λ_T refers to the temperature-like character of the K variable). In order that $2-\alpha$ be a positive exponent the value of λ_T should exceed unity. It is easy to confirm this for the two examples given, but it is more fundamental to derive it from the structure of the RT. Both (29) and (38) have the property that small K transform in even smaller K' and large K in even larger K'. The non-trivial fixed

point $K^\star \neq 0$ is the point where the transformation changes character. Points below $K^\star$ transform towards the $K^\star = 0$ fixed point and points above $K^\star$ to the $K^\star = \infty$ fixed point. λ_T gives the rate at which the stream is away from $K^\star$. A value $\lambda_T < 1$ would have implied a stream towards the $K^\star \neq 0$ fixed point. Computing α on the basis of (29) and (38) one finds via (50) the values $\alpha = -0.2671$ and $\alpha = -0.5148$ respectively (to be compared with the exact $\alpha = 0$).

This picture can be extended to the (H,K)-plane where the basic equation (17) may be written as

$$f(H,K) = g(H,K) + b^{-d} f(H'(H,K), K'(H,K)) \qquad (52)$$

One finds generally that near $K^\star$ positive fields H transform to stronger positive fields H' and similarly negative H to more negative H'. In analogy to (51) we write

$$\lambda_H = [\partial H'/\partial H]_{H=0,\ K=K^\star} \qquad (52)$$

The magnetic singularity, defined in (5) with exponent δ, can be determined again by comparing powers via

$$1 = b^{-d} \lambda_H^{\ 1+1/\delta} \qquad (54)$$

In order to express the independence of the exponents from the particular RT, Wegner[5] puts

$$\lambda_T = b^{y_T} \qquad \lambda_H = b^{y_H} \qquad (55)$$

such that α and δ are expressed in y_T and y_H as

$$\begin{cases} 2 - \alpha = d/y_T \\ 1 + 1/\delta = d/y_H \end{cases} \qquad (56)$$

Near $H = 0$ $K = K^\star$ the RT reduces up to linear terms to

$$\begin{aligned} K' - K^\star &= b^{y_T} (K-K^\star) + \dots \\ H' &= b^{y_H} H \qquad + \dots \end{aligned} \qquad (57)$$

Using these relations for the singular parts in (52) we have the equality

$$f_{sing}(H,K) = b^{-d} f_{sing}(b^{y_H}, K^{\star} + b^{y_T}(K-K^{\star})) \tag{58}$$

which implies the scaling behavior (7) with crossover exponent $\Delta = y_H/y_T$.

The next point is to show the appearance of a phase transition (spontaneous magnetisation) for $H = 0$ and $K > K^{\star}$. We differentiate (52) with respect to H and set $H = 0$ afterwards. $g(H,K)$ and $K'(H,K)$ are even functions of H and $H'(H,K)$ is an odd function of H. Introducing

$$\mu(K) = [\partial H'(H,K)/\partial H]_{H=0} \tag{59}$$

we obtain in this way for the (spontaneous) magnetisation

$$m(K) = 0 + b^{-d} m(K')\mu(K) \tag{60}$$

This recursive relation can be solved by iteration as

$$m(K) = [\prod_{j=0}^{\infty} b^{-d} \mu(K^{(j)})] m(K^{(\infty)}) \tag{61}$$

The function $\mu(K)$ can be shown[6] to vary from $\mu(\infty) = b^{-d}$ via $\mu(K^{\star}) = \lambda_H = b^{y_H} < b^d$ to a value $\mu(0)$ which is well below b^d. So for

$$\begin{aligned} K \leq K^{\star} \quad & m(K) = 0 \\ K > K^{\star} \quad & m(K) \neq 0 \end{aligned} \tag{62}$$

since for $K < K^{\star}$, $K^{(j)}$ approaches $K^{(\infty)} = 0$ and both the product in (61) and $m(0) = 0$; for $K = K^{\star}$ the product in (61) contains only factors $b^{-d+y_H} < 1$ and thus $m(K^{\star}) = 0$ whereas for $K > K^{\star}$, $K^{(j)}$ approaches $K^{(\infty)} = \infty$ and $m(\infty) = 1$ and the factors in the product approach 1 leaving the product finite. Thus (61) demonstrates the way a phase transition (62) appears in the renormalization group approach.

As a final point of this section we want to extend the picture given to a more general RT which takes us outside the (H,K)-plane (to which both examples are confined). A better approximation than (29) or (38) to the real RT will generate all kinds of couplings such as next nearest neighbor, 4-spin coupling, etc. The idea is that despite of this wealth of complications the basic features of the above given picture survives. In order to deal with a set of parameters we label them as K_α and (16) obtain the form

$$\begin{cases} K'_\alpha = K'_\alpha(K_\beta) \\ g = g(K_\beta) \end{cases} \tag{63}$$

The singularities of the free energy are controlled by the transformation linearized around the fixed point

$$K'_\alpha - K^\star_\alpha = \sum_\beta T_{\alpha\beta}(K_\beta - K^\star_\beta) + \dots \tag{64}$$

with $T_{\alpha\beta}$ given by

$$T_{\alpha\beta} = [\,\partial K'_\alpha(K_\beta)/\partial K_\beta]_{K^\star} \tag{65}$$

$T_{\alpha\beta}$ has eigenvalues b^{y_i} and (left) eigenvectors $\varphi_\alpha^{(i)}$

$$\sum_\alpha \varphi_\alpha^{(i)}\, T_{\alpha\beta} = b^{y_i}\varphi_\beta^{(i)} \tag{66}$$

With the eigenvectors (normal) coordinates (scaling fields) are constructed

$$\mu_i = \sum_\alpha \varphi_\alpha^{(i)}(K_\alpha - K^\star) + \dots \tag{67}$$

which transform as

$$u'_i = b^{y_i}\, u_i \tag{68}$$

The basic distinction in the eigenvalues is

$$\begin{cases} \lambda_i > 1,\ y_i > 0 & \text{relevant} \\ \lambda_i = 1,\ y_i = 0 & \text{marginal} \\ \lambda_i < 1,\ y_i < 0 & \text{irrelevant} \end{cases} \tag{69}$$

It means that, upon applying the RT, in case of the relevant eigenvalue λ_i the corresponding u_i increase, whereas they decrease for the irrelevant λ_i. Thus in the relevant directions the RT streams the points away from the fixed point and in the irrelevant directions towards the fixed point. In each direction the free energy, when expressed in the coordinates u_i, has a possible singularity of character $|u_i|^{d/y_i}$. On physical grounds of boundedness of the free energy one must conclude that only the relevant directions can have singularities. This is also born out by doing the mathematics more precisely in terms of the scaling fields[4).

A critical fixed point has two singular directions and therefore two relevant eigenvalues. The fixed point has all $u_i = 0$. The collection of points with only the relevant $u_i = 0$ is the subspace which streams towards the fixed point (the domain of attraction). Since the RT connects points of equal singularity in the free energy, the domain of attraction of a critical fixed point is the critical surface. Under an RT the character (exponent) of the singularity does not change but its amplitude may. Therefore the exponents are universal but the amplitude (e.g. A in (47)) may change along the critical surface.

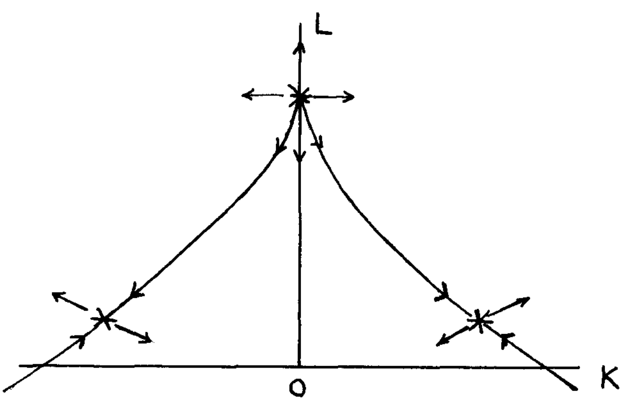
5. Symmetries

To a certain extend the features of an RT can be determined by symmetry arguments. The Ising model exhibits the up-down symmetry which can be used to help locate the fixed points. Let us consider a coupling $H_I(s',s)$ which is even in the combined old and new spin variables

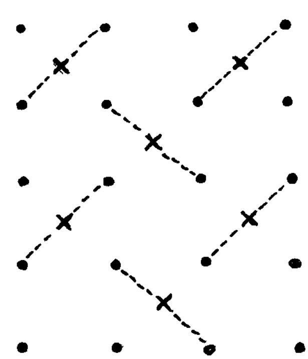
$$H_I(+s',+s) = H_I(-s',-s) \tag{70}$$

i.e. a simultaneous flip of old and new spins leaves H_I invariant. From the definition (12) it then follows immediately that the subspace of even H(s) is invariant. Fixed points thus occur in the even subspace (as was the case in the examples given) or in pairs which have the even interactions equal and the odd ones opposite.

For a fixed point in the even subspace (with only $K_\alpha \neq 0$ corresponding to even interactions) the matrix $T_{\alpha\beta}$ defined in (65) breaks up for the same reason in an even-even and an odd-odd part. So the eigenvalues can be distinguished in even and odd eigenvalues. The critical fixed point has one even relevant exponent and one odd relevant exponent. A change in temperature of the fixed point system means multiplying all $K^\star_\alpha \neq 0$ by a factor and thus the temperature couples to the even eigendirections. So we identify the relevant even eigenvalue as temperature-like. A change of the magnetic field (from zero) at the fixed point couples to the odd directions and the odd relevant eigenvalue is thus field-like. Symmetry helps also to explore further the even subspace. Let us consider the criticality in the space of a nearest (K) and next-nearest neighbor interaction (L) on a square lattice



a)



b)

Fig. 2. a) Critical line in (K,L)-plane
b) Coupling guaranteeing symmetry

By dividing the lattices into two sublattices as the black and white fields of a checkerboard and using the invariance of the free energy under a spin slip of one sublattice one finds

$$f(K,L) = f(-K,L) \tag{71}$$

Such a summetry can be preserved in an RT (even in approximations) if one designs such $H_I(s',s)$ that the new spins couple only to old spins of one sublattice. A possible choice is indicated in fig. 2b.

For positive K and L we find the usual ferromagnetic fixed point $K^\star, L^\star$, having a relevant temperature like variable with exponent y_T (=1 in an exact theory). In the (K,L)-plane we see one more irrelevant direction in which the critical line runs. Through the symmetry (71) we have exactly the same situation at $-K^\star, L^\star$ where an anti-ferromagnetic fixed point is located. The anti-ferromagnetic and ferromagnetic critical line meet each other at the L-axis (see fig. 2a). By symmetry a third fixed point has to appear there. In fact for K = 0 we have two independent sublattices. The RT breaks up at the L-axis into product of two independent RT one for each sublattice. In the fixed point on the L-axis one can construct eigenvalues by combination of the eigenvalues of the ferromagnetic fixed point. Amongst others an eigenvalue appears with exponent $y_{TT} = Y_T + y_T - d = 0$, which result from temperature-like deviations on both sublattices. This is a direction containing 4-spin couplings. A marginal exponent $y_{TT} = 0$ is a signal of a line of fixed points. In fact this is to be expected because a square lattice with next nearest neighbor L and 4-spin coupling Q corresponds to the symmetric 8-vertex model for which α is non-universal and depends on the ratio of L to Q. Such a non-universality should exhibit itself as a line of fixed points for which each point has its own exponents. An RT which automatically yields this fixed line has not been found. If so the marginality of y_{TT} would follow and therefore also the value $y_T = 1$ for a square lattice which would be a determination of the critical exponent by symmetry alone.

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