REMARKS ON THE ENERGY REPRESENTATION OF SOBOLEV-LIE GROUPS

by

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In the construction of quantum field models (with two or three dimensional space-time) Euclidean random fields have been an essential tool. These random fields have an intrinsic mathematical interest, being the natural extension to a many dimensional continuous parameter space of homogeneous symmetric Markov processes. The results to be reported here give a sort of non commutative extension of the concept of Euclidean random fields. Let us recall first the latter concept. A (generalized) random field is a mapping $x \to \xi(x)(\omega)$ from the parameter space $x \in R^d$ into distribution-valued random variables, i.e. for any sample point ω, $\xi(\cdot)(\omega)$ is a generalized function, so that for a real function φ in a suitable test function space $\langle \xi, \varphi \rangle \equiv \int \xi(x)(\cdot)\,\varphi(x)\,dx$ is a real valued random variable. $\xi(x)(\cdot)$ is a Euclidean random field if the finite dimensional distributions are invariant under the Euclidean group in R^d
A "non commutative extension" of this concept is concerned with the case where the parameter space (where x takes values, i.e. R^d) is replaced by some Riemannian manifold M and the state space (where $\langle \xi, \varphi \rangle (\omega)$ takes values, for fixed ω, i.e. R) is replaced by some Lie group G. This kind of extension is suggested by physical models (e.g. the "non linear σ-model", chiral models and gauge fields, and the theory of current algebras), as well as by the need of a theory of "non commutative distributions" (see e.g. [1]).
The simplest Euclidean random fields are the Gaussian ones (which correspond to free quantum fields). They are defined by the canonical Gaussian measure written formally as $\exp(-\frac{1}{2}[\alpha^2 \int \nabla\xi(x)^2\,dx + m^2 \int \xi(x)^2\,dx])$,
where α, m are non-negative parameters. For $\alpha = 0$ we have physically so called "ultralocal models", mathematically "white noise", both of which have "non commutative extensions" in terms of representations of the group of smooth mappings from a manifold M into a Lie group G (see e.g. [1]). Recently, Ismagilov [2], Vershik, Gelfand and Graev [3] and ourselves [4] have studied

a correspondent non commutative extension of the case $\alpha > 0$, $m = 0$.
These representations (also suggested by Parthasarathy and Schmidt [5] and Streater [6]) provide a non commutative extension of the Euclidean random fields. They are physically non trivial in as much as they include e.g. the non linear σ -model. The group G is taken to be compact (or more generally with compact Lie algebra) and the representation of the group of smooth mappings from M into G is given in terms of an "energy expression" which is the non commutative analogue of the one appearing in the Euclidean measure. For this reason it is natural to call the representation "energy representation". It is in fact a unitary representation of a metric group, called "Sobolev-Lie group" (the one obtained by closing the group of smooth mapping from M into G in the natural metric given by above "energy").
In [3] it is shown (in the case of a semisimple compact group G) that the representation is irreducible whenever $\dim M \geq 2$. In [4] it is shown that in the case $\dim M = 1$ it coincides with the representation given by left multiplication on the paths of the Brownian motion on the Lie group G . This, and further considerations in [4], indicates the sense in which the representation provides non commutative analogues of Euclidean random fields.

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