UNBOUNDED REPRESENTATIONS OF THE POINCARE AND GAUGE GROUPS

IN THE INDEFINITE METRIC QUANTIZATION OF THE ELECTROMAGNETIC FIELD

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1. Occurrence of unbounded representations.

The standard Gupta-Bleuler quantization of the free electromagnetic field [1] starts with a Fock space $\mathcal{F}$ over one-particle states belonging to $\mathcal{K}_1 \equiv L^2(C^+) \otimes \mathbb{C}^4$, that is, 4-component functions defined on the forward light cone C^+ and square integrable with respect to the Lorentz-invariant measure $d^3k/|\underline{k}|$. Besides its scalar product $(.|.)$, which is positive definite but Lorentz noninvariant, $\mathcal{F}$ is also equipped with an indefinite, Lorentz invariant sesquilinear form $<.|.>$. The two are linked by the relation $<.|.> = (.|\eta .>$, where $\eta = \eta^+ = \eta^{-1}$ is the so-called metric operator. The physical states are obtained by restriction to the closed subspace $\mathcal{F}'$ of vectors $|\psi> \in \mathcal{F}$ satisfying the Gupta condition $k^\mu a_\mu(k)|\psi> = 0$ ($a_\mu(k)$ are the annihilation operators), and quotient by the subspace $\mathcal{F}'' \subset \mathcal{F}'$ of vectors of zero norm : $\mathcal{F}^{phys} = \mathcal{F}'/\mathcal{F}''$.

In this set-up, it has been known for some time [2] that Poincaré as well as gauge transformations are represented in the large space $\mathcal{F}$ by unbounded operators. More specifically, Rideau shows, with help of suitable coherent states,that a Poincaré transformation (a,Λ), defined by $a_\mu(k) \to e^{i\Lambda k.a}(\Lambda^{-1})_\mu^\nu a_\nu(\Lambda k)$, is implemented in $\mathcal{F}$ by an operator $U(a,\Lambda)$, unitary with respect to $<.|.>$ and unbounded in the norm $\|\psi\|^2 \equiv (\psi|\psi)$. Similarly a gauge transformation $a_\mu(k) \mapsto a_\mu(k) + k_\mu \phi(k)$ (with $|\underline{k}|\phi \in L^2(C^+)$) is implemented by an operator $U(\phi)$ with the same properties. Of course the restriction of $U(a,\Lambda)$, resp. $U(\phi)$, to the physical states is unitary (in the Hilbert sense), resp. trivial.

Why does this happen ? A naive answer would be to question the use of the $(.|.)$-norm for defining (un)boundedness of an operator. Indeed, in the Gupta-Bleuler philosophy, the basic quantity for the physical description is $<.|.>$, rather that $(.|.)$. So let us study the space $\mathcal{F}$ equipped with $<.|.>$.

2. Mathematical structure of $\mathfrak{F}$ equipped with $\langle .|. \rangle$.

We consider first $f \in \mathcal{X}_1 \equiv L^2(C^+) \otimes \mathbb{C}^4$. Writing explicitly the two inner products, we get :

$$(f|f) \equiv \|f_0\|^2 + \sum_{j=1}^{3} \|f_j\|^2 = (f^-|f^-) + (f^+|f^+)$$

$$\langle f|f \rangle \equiv -\|f_0\|^2 + \sum_{j=1}^{3} \|f_j\|^2 = -(f^-|f^-) + (f^+|f^+).$$

Thus $\mathcal{X}_1$ is the direct sum of two Hilbert spaces $\mathcal{X}_1^+$, $\mathcal{X}_1^-$, such that the (nondegenerate) indefinite form $\langle .|. \rangle$ is positive on $\mathcal{X}_1^+$, negative on $\mathcal{X}_1^-$. Moreover, there exists a metric operator $\eta_1 = \eta_1^+ = \eta_1^{-1}$ such that $(.|.) = \langle .|\eta_1 . \rangle$ is positive definite. Obviously $\mathfrak{F}$ has exactly the same structure. Such spaces are called <u>Krein spaces</u> or <u>J-spaces</u> and have been thoroughly analyzed in the literature [3],[4].

Let $\mathcal{K}$, $\langle .|. \rangle$, be a Krein space with a metric operator J (there exists infinitely many of these, but they all give equivalent norms). With respect to the nondegenerate form $\langle .|. \rangle$, $(\mathcal{K},\mathcal{K})$ is a dual pair [5]. As such it possesses canonical topologies, in particular the Mackey topology $\tau(\mathcal{K},\mathcal{K})$, which is the finest locally convex topology such that $\mathcal{K}$ is its own dual. In addition, as for any Krein space, the Mackey topology of $\mathcal{K}$ is precisely the (norm) topology defined by any metric operator J, i.e. $\|f\|_J^2 = \langle f|Jf \rangle$. Let now A be an operator on $\mathcal{K}$. From the last result, A is continuous for the Mackey topology iff it is continuous for every J-norm. Returning now to $\mathfrak{F}$, we see that the $(.|.)$-norm defines precisely the Mackey topology on $\mathfrak{F}_{\langle .|. \rangle}$, and thus the natural notion of continuity for operators. In other words, $U(a,\Lambda)$, $U(\phi)$ are genuine discontinuous (i.e. unbounded) operators on $\mathfrak{F}_{\langle .|. \rangle}$.

Let again $A = A^*$ be an operator on the Krein space $\mathcal{K}$, self-adjoint with respect to $\langle .|. \rangle$. The spectrum of A is not necessarily real, but only symmetric to the real axis [4]. The spectrum is real, however, iff A commutes with some metric operator. In that case only, A generates a bounded one-parameter group $\{e^{iAt}\}_{t \in \mathbb{R}}$. If we consider now the Lorentz generators in $\mathcal{X}_1$ and $\mathfrak{F}$, it turns out that infinitesimal rotations do commute with η, but infinitesimal boosts do not : hence a one-parameter subgroup of boosts $t \to \Lambda_t$ is represented by an unbounded group $U(0,\Lambda_t)$, whereas rotations are always represented by bounded operators $U(0,R)$.

In conclusion, the phenomenon of unbounded representations is well understood in a correct treatment of the indefinite metric, but it is a genuine pathology that cannot be avoided within $\mathfrak{F}_{\langle .|. \rangle}$.

3. Beyond Fock space.

Regularizing an unbounded operator A on $\mathcal{F}$ is a familiar problem. What is needed is a dense, invariant domain $\mathcal{V}^{\#}$ and a topology on $\mathcal{V}^{\#}$, finer than the norm topology and such that A $: \mathcal{V}^{\#} \to \mathcal{V}^{\#}$ is continuous. Then by transposition we get a triplet structure $\mathcal{V}^{\#} \subset \mathcal{F} \subset \mathcal{V}$, where $\mathcal{V}$ is the dual of $\mathcal{V}^{\#}$, and a continuous action ${}^{t}A : \mathcal{V} \to \mathcal{V}$. Doing this for U(a,Λ),U(φ) results in a continuous representation of the respective groups in $\mathcal{V}$, with a restriction to $\mathcal{F}$ which is not norm-continuous. Various formal schemes can be used, such as rigged or nested Hilbert spaces, partial inner product spaces.

A first solution regularizes the Poincaré generators as well. Looking first at $\mathcal{K}_1 = L^2(C^+) \otimes \mathbb{C}^4$, the explicit form of those generators as differential operators suggests $\mathcal{S} \otimes \mathbb{C}^4$ as a suitable domain ($\mathcal{S}$ is the Schwartz space on C^+). As for the Gupta condition, it is easily seen that $\mathcal{K}_1' = L^2 \otimes E$, where E is a suitable 3-dimensional subspace of $\mathbb{C}^4$ and $\mathcal{K}_1'' = L^2 \otimes E''$ with $E'' \subset E'$ (i.e. the nonpositivity of the theory is governed by the $\mathbb{C}^4$ part). Thus $\mathcal{K}_1^{phys} = L^2 \otimes \hat{E}$, where $\hat{E} \equiv E'/E''$. The projection $P_1 = \mathcal{K}_1 \to \mathcal{K}_1'$, although not orthogonal with respect to $\langle . | . \rangle$, is Mackey continuous, and thus preserves the mathematical structure, and the same is true for $P_1'' : \mathcal{K}_1' \to \mathcal{K}_1''$. Finally we get $\mathcal{S} \otimes \hat{E} \subset L^2 \otimes \hat{E} \subset \mathcal{S}' \otimes \hat{E}$, regularizing simultaneously the Poincaré generators, U(a,Λ) and U(φ). Passing to the full Fock space, a natural domain would consist of coherent states of the form $|f\rangle \equiv \exp a_{\mu}^{+}(f^{\mu})|0\rangle$, with $f^{\mu} \in \mathcal{S} \otimes \hat{E}$. However, these have to be truncated, since only finite sums can be used, lest we loose continuity of the generators. This leads us to a domain $\mathcal{V}^{\#}$, which is the multicomponent analogue to the usual Borchers algebra.

An alternative solution, which however does not regularize the Poincaré generators, consists in replacing above $\mathcal{S}$ by bounded functions of compact support, $\mathcal{S}'$ by locally integrable functions. In either case, we get continuous representations in $\mathcal{V}$ and $\mathcal{V}^{\#}$, with the usual reducibility properties. This result is sufficient for application of some generalized spectral theorems of the SNAG type to suitable abelian subgroups [6].

[1] See e.g. F. Strocchi and A.S. Wightman, J. Math. Phys. 15,2198-2224 (1974).

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