ON THE CONSTRUCTION OF GRADED LIE ALGEBRAS

by

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The purpose of this contribution is to initiate a classification of graded Lie algebras (GLA) by dimension, for use in the future as a source of examples which exhibit some of the similarities and differences between ordinary Lie algebras (LA) and GLA.

A GLA $L = L_0 \oplus L_1$ consists of an even part L_0, which is a LA, and an odd part L_1, which in particular is an L_0-module. There is a bilinear bracket operation which satisfies conditions (1) - (7) below. We say that L and L' are graded isomorphic (or equivalent) if there are isomorphisms $L_0 \leftrightarrow L_0'$ and $L_1 \leftrightarrow L_1'$ which preserve the bracket. We can ask the question: given a LA L_0 and an L_0-module M, how many GLA $L = L_0 \oplus L_1$ can we construct where L_1 and M are isomorphic as L_0-modules? Answering this question is the basis for the classification scheme.

It is convenient to distinguish two types of GLA: we say that L is trivial if $[L_1, L_1] = \{0\}$; otherwise, L is non-trivial. We note that a non-trivial GLA can be trivialized simply by putting to zero all anticommutators. In general we advocate classifying trivial GLA and then attempting to de-trivialize them.

We say that L is an (m, n) algebra and has dimension $m \oplus n$ if $\dim L_0$ (resp. L_1) is m (resp. n). We only consider $m + n \leqslant 3$. The elements of L_0 (resp. L_1) are denoted by Latin letters (resp. Greek letters) taken from the beginning of the alphabet. Then the commutativity and the Jacobi relations for L are

$$[a, b] = -[b, a], \qquad (1)$$
$$[a, \alpha] = -[\alpha, a], \qquad (2)$$
$$[\alpha, \beta] = [\beta, \alpha], \qquad (3)$$

for all $a, b \in L_0, \alpha, \beta \in L_1$, and

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0, \qquad (4)$$
$$[[a, b], \alpha] + [[b, \alpha], a] + [[\alpha, a], b] = 0, \qquad (5)$$
$$[[a, \alpha], \beta] + [[\alpha, \beta], a] - [[\beta, a], \alpha] = 0, \qquad (6)$$
$$[[\alpha, \beta], \gamma] + [[\beta, \gamma], \alpha] + [[\gamma, \alpha], \beta] = 0, \qquad (7)$$

for all $a, b, c, \in L_0$, $\alpha, \beta, \gamma \in L_1$.

We first consider the trivial algebras. The (m, 0) algebras, $m \leq 3$, are LA and have been classified. The (0, n) algebra is the direct sum of n copies of the (0, 1) algebra defined by the anticommutator $[\alpha, \alpha] = 0$. This leaves us the task of classifying the (1, 1), (1, 2) and (2, 1) indecomposable trivial algebras.

(1, 1): the bracket relation between basis elements is $[a, \alpha] = p\,\alpha$ Either $p = 0$, which gives a decomposable algebra, or, if $p \neq 0$, we can scale to give $p = 1$.

(1, 2): The action of a on the basis α, β is defined by a real 2 x 2 matrix which can be taken in one of the following forms: (1) $\begin{pmatrix} p & 0 \\ 0 & s \end{pmatrix}$; (2) $\begin{pmatrix} p & 1 \\ 0 & p \end{pmatrix}$; (3) $\begin{pmatrix} p & q \\ -q & p \end{pmatrix}$, where $q \neq 0$ and $p/q \geq 0$. In case (1) we can assume $|p| \geq |s| > 0$. Then by scaling we can take $p = 1$ and $0 < |s| \leq 1$. In case (2), either $p = 0$ or we can scale to give $p = 1$. In case (3) we can scale to give $q = 1$ and $p \geq 0$.

(2, 1): There are two choices for L_0 : either the decomposable Abelian algebra or the indecomposable algebra with non-trivial relation $[a, b] = b$. Suppose $[a, \alpha] = p\alpha$, $[b, \alpha] = q\alpha$. When L_0 is Abelian we can reduce p or q to zero, which decomposes L. When L_0 is non-Abelian the relation (5) forces $q = 0$.

We now find the non-trivial algebras. There are no (m, 0) or (0, n) non-trivial GLA. We can quickly dispose of the (1, n) algebras. If α, β are any two basis elements of L_1 we can write $[\alpha, \beta] = S_{\alpha\beta}\, a$, where S is a real, symmetric matrix. By a linear transformation we can take S in diagonal form: $S_{\alpha\beta} = \delta_{\alpha\beta} S_\alpha$. Either all of the S_α are zero, in which case L is trivial, or at least one, S_α say, is non-zero. In this case, put $\alpha = \beta = \gamma$ in (7) to give $3\,[[\alpha, \alpha], \alpha] = 0$. This implies $[a, \alpha] = 0$. Now, for $\gamma \neq \alpha$, put $\alpha = \beta$ in (7). As $\gamma \neq \alpha$ and $S_\alpha \neq 0$ this condition leads to $[a, \gamma] = 0$. It follows that $\gamma \neq \alpha$ decouples unless $S_\gamma \neq 0$. Hence the only way to obtain an indecomposable algebra is to have $S_\alpha \neq 0$ and $[a, \alpha] = 0$ for all basis elements α. The α can be scaled to ensure $S_\alpha = \pm 1$. Finally, possibly permuting the α and changing the sign of a leads to the $1 + n/2$ or $1 + (n - 1)/2$ inequivalent indecomposable non-trivial (1, n) algebras.

We can also discuss (m, 1) algebras in some generality. We can assume a basis $\{a_i\}$ for L_0, in which either (a) $[a_i, \alpha] = 0$ for all i or (b) $[a_1, \alpha] = \alpha$ and $[a_i, \alpha] = 0$ for all $i > 1$. The relations (6) and (7) give $[[\alpha, \alpha], a_i] = 2[[\alpha, a_i], \alpha]$, for all i, and $3[[\alpha, \alpha], \alpha] = 0$. In case (a) we deduce that $[\alpha, \alpha]$ lies in the centre of L_0. It is not hard to see that indecomposability leads to the rejection of this case. In case (b) $I = \{a \in L_0: [a, \alpha] = 0\}$ forms an ideal of codimension one in L_0 containing $[\alpha, \alpha] = b$ in its centre. Evidently we can write $[a_1, b] = 2b$ which implies that L_0 is non-Abelian. For m = 2 we can write $[a, b] = b$, $[a, \alpha] = \frac{1}{2}\alpha$, $[\alpha, \alpha] = b$ where $a = \frac{1}{2}a_1$.

From this analysis we find that the number of families of equivalence classes of indecomposable real GLA, which are not LA, in dimensions one, two and three, are 1, 2 and 11, respectively. The corresponding numbers for ordinary LA are 1, 1 and 9. These numbers of course depend on our unspecified definition of a family of equivalence classes. We have, for example, for reasons which will be given in a future publication, separated, for the trivial (1, 2) algebras in case (1), the values of s, $0 < |s| < 1$, $s = \pm 1$.

In a future publication we will extend the above classification to dimension four, and give tabluations of derived series, radical, Killing form, etc.