EXCEPTIONAL PARAFERMIONS IN A HILBERT SPACE OVER AN ASSOCIATIVE ALGEBRA

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Every vector of a Hilbert space admitting multiplication by octonions generates a linear manifold over the reals of at most 128 dimensions, and the basis for this manifold can provide a faithful representation of C_7, the Clifford algebra of order seven(1). Multiplication defined as an equivalence relation in a (minimal one-sided ideal recovers the structure of the non-associative octonion algebra, and the basic properties of Hilbert spaces with octonion multipliers can therefore be studied in the framework of Hilbert spaces over <u>associative</u> algebras(2). The set of algebraically closed linear manifolds of Hilbert spaces of this type has a structure consistent with that of the calculus of propositions characterizing quantum mechanics, and is isomorphic to a Hilbert space over a field in which there are superselection rules(3).

Let $e_1, e_2, \ldots e_7$ be the generating elements of the associative real Clifford algebra C_7, satisfying $\{e_a, e_b\} = -\delta_{ab}$, $e_a^* = -e_a$, $(e_a e_b)^* = e_b e_a$. A minimal projection in the algebra is

$$P_o^+ = \frac{1}{16}(1 - e_1e_2e_3)(1 - e_5e_1e_6)(1 - e_6e_2e_4)(1 + e_1e_2e_3e_4e_5e_6e_7) , \qquad (1)$$

where the $+$ superscript refers to the last sign. A complete basis is provided by $\rho_{ij}^{\pm} = e_i P_o^{\pm} e_j^*$ $(i = 0, \ldots 7, e_o = 1)$, which satisfies $\rho_{ij}^{\sigma} \rho_{k\ell}^{\sigma'} = \delta_{\sigma\sigma'} \delta_{jk} \rho_{i\ell}^{\sigma}$ ($e_1e_2 \ldots e_7$ is in the center). The equivalence relation $P_o^+ ab = P_o^+(ab)_o^+$, where $(a)_o^+ = \Sigma\, e_i a_i$, a_i real, defines the multiplication law for octonions in the form used by Gürsey and Günaydin(4).

The right linear scalar product $(f,g) = (g,f)^*$ has values in C_7, $(f,f) \geq 0$ and $\| f \|^2 = \mathrm{tr}(f,f)$, where the trace is defined in a representation of C_7. Two algebraically closed closed linear manifolds, M_1 and M_2, are orthogonal if $\mathrm{tr}((f_1,f_2)a) = 0$ for all $\underline{a} \in C_7$, for every $f_1 \in M_1$, $f_2 \in M_2$. If this condition is weakened to include only $\underline{a} \in ₵(1,e_7)$, the Abelian subalgebra spanned over the reals by 1 and e_7, one obtains linear manifolds closed over $₵(1,e_7)$ corresponding to the proposal of Günaydin and Gürsey(4). In this case, it suffices that $\mathrm{tr}(f,g) = \mathrm{tr}((f,g)e_7) = 0$. Using the algebraic identity (we restrict ourselves henceforth to $H^+ = H \sum_i \rho_{ii}^+)$ $f = \sum_{ij} f_{ij}\, e_{ij}$, these bilinear forms are, respectively,

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$$\sum_{k,i} (f_{ki}, g_{ki}) \; ; \; \sum \{(f_{ko}, g_{k7}) - (f_{k7}, g_{ko}) + (f_{k1}, g_{k4}) - (f_{k4}, g_{k1}) + (f_{k2}, g_{k5}) - (f_{k5}, g_{k2}) + (f_{k3}, g_{k6}) - (f_{k6}, g_{k3})\} . \tag{2}$$

For each value of the index k, these forms are invariant, respectively, under SO(8) and Sp(8). The symmetry of the full scalar product (for each k) is therefore $SO(8) \cap Sp(8) = U(4)$. Indeed, let us call

$$\psi_o^k = f_{ko} - f_{k7} e_7$$
$$\psi_\alpha^k = - f_{k\alpha} - f_{k\ \alpha+3}\ e_7 \tag{3}$$

so that

$$f = \sum_k e_k P_o (\psi_o^k + \sum_{\alpha=1}^{3} \psi_\alpha^k e_\alpha) . \tag{4}$$

Then,

$$(f,g)_c = tr(f,g) + e_7\ tr((f,g)e_7) = \sum_k \{(\psi_o^k, \chi_o^k) + \sum_\alpha (\chi_\alpha^k, \psi_\alpha^k)\} , \tag{5}$$

where χ corresponds to g. For each k, (5) is the scalar product given by Günaydin[4]. Observables may be defined to be linear over $\mathbb{C}(1,e_7)$, i.e., $A(fa) = (Af)a$, $\underline{a} \in \mathbb{C}(1,e_7)$. The most general operator <u>linear</u> over $\mathbb{C}(1,e_7)$ satisfies[5]

$$Af = \sum_{k,j} e_k P_o \{A_{oo}^{kj} \psi_o^j + A_{o\alpha}^{kj} \psi_\alpha^{j*} + \sum_\alpha (A_{\alpha o}^{kj} \psi_o^{j*} + \sum_\beta A_{\alpha\beta}^{kj} \psi_\beta^j) e_\alpha\} , \tag{6}$$

explicitly exhibiting superselection rules when H is viewed as a family of Hilbert spaces over $\mathbb{C}(1,e_7)$ (note that A may act antilinearly on this family). With the help of the form of (6), we may construct the tensor product

$$gTf = \sum_j e_j P_o \{(\chi_o^j + \chi_\alpha^j e_\alpha) \times (T_{oo} \psi_o^j + T_{o\beta} \psi_\beta^{j*}) + \sum_\alpha (\chi_\alpha^j + \chi_o^j e_\alpha) \times (T^*_{\alpha o} \psi_o^j + T^*_{\alpha\beta} \psi_\beta^{j*})\} , \tag{7}$$

in which T acts linearly with a numerical matrix, and g acts linearly with χ_α^j occurring through an action analogous to the second and third terms of (6), and the first and fourth carry χ_o^j. Clearly, $gT(fa) = (gTf)a$, $a \in \mathbb{C}(1,e_7)$. For $\hat{T}_{o\alpha} = \hat{T}_{\alpha o} = 0$, $\hat{T}_{\alpha\beta} = \hat{T}_{oo}\delta\alpha\beta = e_7 \delta\alpha\beta$, (7) takes on the form of the tensor product given by Günaydin[4]. In this case, there exists an operator A_{12} such that $A_{12}(g\hat{T}f) = (A_1 g)\hat{T}(A_2 f)$, provided that A_1 and A_2 do not connect to $\psi_\alpha j^*$, $\psi_o j^*$.

When these conjugate components are involved, the existence of A_{12} implies that $T_{\alpha o} = 0 \leftrightarrow T_{\alpha\beta} = 0$ and $T_{o\alpha} = 0 \leftrightarrow T_{oo} = 0$. If we wish to impose the condition $(ga)\check{T}f = (g\check{T}f)a$, then we must take $\check{T}_{\alpha o} = \check{T}_{\alpha\beta} = 0$, so that only the first term of the right hand side of (7) occurs in $g\check{T}f$.

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