THE SPINNING ELECTRON

by

Jens Peder Dahl - Department of Chemical Physics,
Technical University of Denmark, DK-2800 Lyngby,
Denmark

The idea that the electron is a spinning top was introduced by Uhlenbeck and Goudsmit [1] fifty years ago and has been touched upon at numerous occasions ever since. Dirac [2] found "a great deal of truth in the spinning electron model, at least as a first approximation", but he did not make any attempts to interpret the "other dynamical variables" required "besides the co-ordinates and momenta of the electron". Instead, he created a purely mathematical model of the electron, in which these variables are represented by 4×4 matrices.

Several authors have felt the need of some type of interpretation of the internal variables in Dirac's theory, and have explored the quantum theory of rotating systems with this in mind [3,4]. Yet, no simple alternative to Dirac's mathematical model of the electron has emerged from these investigations.

The alternative does exist, however, as it has been shown in a recent paper by the author [5]. It is in fact nothing but the elementary 3-dimensional rotor governed by relativistic quantum mechanics. The dynamics of the rotor is in all respects identical with the dynamics of a Dirac particle, and hence it characterizes such a particle as being neither more nor less than a particle for which it is possible to talk about an orientation in space.

Let the position of the rotor in space and time be given by the coordinates (x_1,x_2,x_3,ict), and the orientation of the internal axis system $(\underline{e}_1,\underline{e}_2,\underline{e}_3)$ by the Euler angles (α,β,γ). The spin operators (s_1,s_2,s_3) which generate rotations of the internal system, will then have the form

$$\begin{cases} s_1 = i\hbar\left(\sin\alpha\frac{\partial}{\partial\beta} + \cot\beta\cos\alpha\frac{\partial}{\partial\alpha} - \frac{\cos\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right), \\ s_2 = i\hbar\left(-\cos\alpha\frac{\partial}{\partial\beta} + \cot\beta\sin\alpha\frac{\partial}{\partial\alpha} - \frac{\sin\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right), \\ s_3 = -i\hbar\frac{\partial}{\partial\alpha}. \end{cases}$$

Together with the operators $\zeta_i = \underline{s}\cdot\underline{e}_i$ $(i=1,2,3)$, with which they commute, they form an O(4) algebra. We note that

$$\zeta_3 = -i\hbar\frac{\partial}{\partial\gamma},$$

and that $s^2 = s_1^2+s_2^2+s_3^2 = \zeta_1^2+\zeta_2^2+\zeta_3^2$. The common eigenfunctions of s^2, s_3 and ζ_3 are denoted D^s_{mn}, with $s=0,\frac{1}{2},1,\ldots$ The quantum numbers $m = -s,-s+1,\ldots,s$ and $n = -s,-s+1,\ldots,s$ refer to the operators s_3 and ζ_3, respectively. The functions $D^s_{mn}(\alpha,\beta,\gamma)$ span, for a given s, an invariant function space Ω_s of dimension $(2s+1)^2$.

A detailed study of the conditions under which it is possible to construct a local relativistic dynamics in the so-called instant form [6] leads to the result, that the probability amplitude $\Psi(x_1,x_2,x_3,\alpha,\beta,\gamma,t)$ for the rotor must be built over the function space $\Omega_{\frac{1}{2}}$ exclusively, i.e.

$$\Psi = \sum_{i=1}^{4} \psi_i(x_1,x_2,x_3,t)\ \Theta_i(\alpha,\beta,\gamma) ,$$

where

$$\begin{cases} \Theta_1 = D^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}} = N \cos\frac{\beta}{2}\, e^{i\alpha/2}\, e^{i\gamma/2} , \\ \Theta_2 = D^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}} = N \sin\frac{\beta}{2}\, e^{-i\alpha/2}\, e^{i\gamma/2} , \\ \Theta_3 = D^{\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}} = -N \sin\frac{\beta}{2}\, e^{i\alpha/2}\, e^{-i\gamma/2} , \\ \Theta_4 = D^{\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}} = N \cos\frac{\beta}{2}\, e^{-i\alpha/2}\, e^{-i\gamma/2} , \end{cases}$$

and $N = (8\pi^2)^{-\frac{1}{2}}$.

The generators of $\mathcal{IL}_0$, the proper, orthochronous, inhomogeneous Lorentz group are found to be represented by

$$p_i = -i\hbar\frac{\partial}{\partial x_i} , \quad H = m_o c^2\zeta_3' + c\zeta_1'\, \underline{s}'\cdot\underline{p} ,$$

$$J_i = \sum_{j,k} \varepsilon_{ijk}\, x_j p_k + s_i , \quad K_i = \frac{i}{c}\, x_i H + \frac{\hbar}{2}\, \zeta_1' s_i' ,$$

with $i,j,k = 1,2,3$ and $s_i' = \frac{2}{\hbar}\, s_i$, $\zeta_i' = \frac{2}{\hbar}\, \zeta_i$. The equation of motion

$$(m_o c^2\zeta_3' + c\zeta_1'\underline{s}'\cdot\underline{p})\Psi = i\hbar\, \frac{\partial\Psi}{\partial t}$$

becomes identical with the Dirac equation, when it is transformed to a matrix representation by taking the inner product with Θ_1, Θ_2, Θ_3, Θ_4 in turn.

Very simple expressions are found for the improper operations, which together with the operations of $\mathcal{IL}_0$ form the full inhomogeneous Lorentz group $\mathcal{IL}$, viz.

Space inversion P : $x_i \rightarrow -x_i$ plus $C_2(\underline{e}_3)$,

Time inversion T' : $t \rightarrow -t$ plus $C_2(\underline{e}_2)$,

Strong inversion I: $x_i \rightarrow -x_i$, $t \rightarrow -t$ plus $C_2(\underline{e}_1)$,

with $C_2(\underline{e}_i)$ denoting a rotation through π about the i'th internal axis. All operations of $\mathcal{TL}$ are thus represented by linear operators.

Particle-antiparticle conjugation C is identical with complete complex conjugation and is antilinear. Wigner's time reversal operation T is equal to CT' and is thus composed of two more elementary operations.

The strong inversion operation is identical with the well-known PCT-operation. This operation is thus identified as a genuine operation of the group $\mathcal{TL}$. The operation C is not a member of this group, and the designation PCT is thus somewhat misleading.

The complete invariance group of the 3-dimensional rotor, and hence of a Dirac particle, is $\{E, C\} \times \mathcal{TL}$, with E being the identity operation.

References:

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