A group theoretical derivation of the minimal coupling in elementary quantum mechanics

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This paper is intended to give a contribution to the process of remodelling quantum mechanics by group theoretical tools [1]. Traditionally the minimal coupling in q.m. is "derived" by correspondence with classical mechanics [2]. In the course of the remodelling process a derivation by a pseudo-invariance argument has already been given, which makes use of instantaneous transformations [3;4] (see also p. 117 of [1]). Although this may be the best argument that is available for arbitrary electromagnetic fields (that break space-time symmetry) it is in fact a "waste" of invariance in those cases where a special field is present that preserves a part of this symmetry. Important special cases are the uniform fields. In this paper the (Schrödinger resp. Klein-Gordon) equation of a charged particle in an external uniform parallel e.m. field is derived from the (Galilei resp. Poincaré) symmetry group G of the field. Our point is that the minimal coupling is obtained at once in the equation (without a detour via the free particle equation) and that this follows from exact rather than pseudo-invariance.

The crucial physical imput is the presupposition that in elementary q.m. the wave functions transform "locally" (up to a phase factor) under G:

(1) $$(U(g)\psi)(g\circ x) = \exp\{i\theta(g;x)\}\ \psi(x).$$

This condition originates from the interpretation of $\psi(x)$ as probability-amplitude-density and it is formulated without reference to any particular kind of interaction. Nevertheless, within the framework of elementary q.m. it implies the principle of minimal coupling (for uniform fields).

The unitary operators U(g) form a projective representation of G

(2) $$U(g')U(g) = \exp\{i\xi(g',g)\}U(g'g).$$

Here ξ is a group exponent, which can be expressed in terms of the phase function θ by substitution of (1) into (2):

(3) $$\xi(g',g) = \theta(g';g\circ x) + \theta(g;x) - \theta(g'g;x).$$

Due to the freedom of a phase factor in the operators U(g) in (1) we may choose θ such that $\theta(g;x_o) = 0$, where x_o is the origin of space-time. If we denote by h_x the translation from the origin x_o to the space-time event x then the substitution of (g,h_x,x_o) for (g',g,x) in (3) gives $\xi(g,h_x) = \theta(g;x)$. Hence, the projective representations in configuration space can always be brought into the form

(4) $$(U(g)\psi)(g\circ x) = \exp\{i\xi(g,h_x)\}\ \psi(x).$$

The symmetry group G of a uniform e.m. field with both electric and magnetic vectors parallel to the z-axis is a 6-parameter Lie group generated by the translations in space-time, the rotations around the z-axis and the (Galilei or Poincaré) boosts along the z-axis. (We do not consider inversions).

The group will be parametrised as follows

(5) $$g = \exp\{ibH - i\vec{a}\cdot\vec{P}\}\exp\{i\chi K_z - i\phi J_z\}$$

where $\chi = v$ for Galilei and $\chi = \text{arctanh}(v)$ for Poincaré. The projective representations of G can be found from the ordinary representations of group extensions of G or, alternatively, the group exponents ξ of G can be found by exponentiation of the extensions of the Lie algebra of G [5]. The non-vanishing commutators of the extended Lie algebra are

(6) $$\begin{cases} [H,P_z] = -i\varepsilon\ ,\ [H,K_z] = -iP_z, & [P_z,K_z] = \begin{cases} -i\mu \text{ for Galilei} \\ -iH \text{ for Poincaré} \end{cases} \\ [P_x,P_y] = -i\beta\ ,\ [J_z,P_x] = iP_y\ ,\ [J_z,P_y] = -iP_x & \end{cases}$$

where ε,β and μ are real constants. (In principle there are more non-vanishing commutators for the extended Lie algebra, but these do not give rise to extensions of the Lie group G as G is not simply connected. This is analogous to the two-dimensional Galilei group; see page 240 of reference [5]).

The exponents of G are $\xi = \varepsilon\xi_1 + \beta\xi_2$ ($+\mu\xi_o$ only for Galilei) where

(7) $$\begin{cases} \xi_o(g',g) = \frac{1}{2}v'^2 b + v'a_z \quad \text{(only for Galilei)} \\ \xi_1(g',g) = \begin{cases} \frac{1}{2}(a_z'b - b'a_z - b'v'b) & \text{for Galilei} \\ \frac{1}{2}(a_z'b - b'a_z)\text{ch}\chi' + \frac{1}{2}(a_z'a_z - b'b)\text{sh}\chi' & \text{for Poincaré} \end{cases} \\ \xi_2(g',g) = \frac{1}{2}(\vec{a}' \times R(\phi')\vec{a})_z\ . \end{cases}$$

The infinitesimal generators in configuration space, calculated by differentiation of (4) after substitution of (7) for ξ, operate on $\psi(x)$ as follows

(8) $$\begin{cases} H = i\partial_t - \frac{1}{2}\varepsilon z\ ,\ P_z = -i\partial_z - \frac{1}{2}\varepsilon t,\ K_z = \begin{cases} it\partial_z + \mu z & \text{(Gal)} \\ it\partial_z + iz\partial_t & \text{(Poin)} \end{cases} \\ P_x = -i\partial_x - \frac{1}{2}\beta y,\ P_y = -i\partial_y + \frac{1}{2}\beta x,\ J_z = -ix\partial_y + iy\partial_x. \end{cases}$$

It can be checked that the operators $c_{/\!/}$ and $c_\perp$ defined by

(9) $$c_{/\!/} = \begin{cases} 2\mu H - P_z^2 + 2\varepsilon K_z & \text{(Galilei)} \\ H^2 - P_z^2 + 2\varepsilon K_z & \text{(Poincaré)} \end{cases}$$

(10) $$c_\perp = P_x^2 + P_y^2 - 2\beta J_z$$

commute with all generators of the Lie algebra (6). Hence, in an irreducible representation $c_{/\!/}$ and $c_\perp$ are (real) multiples of the unit operator. Combination of (9) and (10) gives (for $\mu \neq 0$)

(11) $$\begin{cases} H - \frac{1}{2\mu}(\vec{P}^2 - 2\varepsilon K_z - 2\beta J_z) = \frac{1}{2\mu}(c_{/\!/} - c_\perp) & \text{for Galilei} \\ H^2 - (\vec{P}^2 - 2\varepsilon K_z - 2\beta J_z) = c_{/\!/} - c_\perp & \text{for Poincaré} \end{cases}$$

Substitution in (11) of the expressions (8) gives after some calculation

(12) $$\begin{cases} i\partial_t + \frac{1}{2}\varepsilon z - \frac{1}{2\mu}\{(-i\partial_x+\frac{1}{2}\beta y)^2+(-i\partial_y-\frac{1}{2}\beta x)^2+(-i\partial_z+\frac{1}{2}\varepsilon t)^2\} = \frac{c_{/\!/} - c_\perp}{2\mu} & \text{(Gal)} \\ (i\partial_t + \frac{1}{2}\varepsilon z)^2 - \{\quad " \qquad " \qquad " \quad\} = c_{/\!/} - c_\perp & \text{(Poin)}\ . \end{cases}$$

It is clear now how we should interprete the constants $\varepsilon,\beta,\mu,c_{/\!/}$ and $c_\perp$ in these equations. We put $\varepsilon = eE$ and $\beta = eB$; for Galilei we put $\mu = m$ (mass) and $\frac{1}{2\mu}(c_{/\!/} - c_\perp) = \mathcal{V}$ (internal energy); for Poincaré we put $c_{/\!/} - c_\perp = m^2$(squared mass).

Then we obtain the equation of motion for a particle with mass m and charge e (and internal energy ϑ for Galilei) in the e.m. field with $\vec{E}$ and $\vec{B}$ parallel to the z-axis:

$$(13)\quad \begin{cases} i\partial_t - eA_o - \frac{1}{2m}(-i\vec{\nabla} - e\vec{A})^2 = \vartheta & \text{(Schrödinger)} \\ (i\partial_t - eA_o)^2 - (-i\vec{\nabla} - e\vec{A})^2 = m^2 & \text{(Klein-Gordon)} \end{cases}$$

Indeed the e.m. field appears in these equations by minimal coupling to its potential

$$(14)\quad A_o = -\tfrac{1}{2}Ez\ ,\quad \vec{A} = (-\tfrac{1}{2}By,\ \tfrac{1}{2}Bx\ ,\ -\tfrac{1}{2}Et)\ .$$

The particular gauge of the potential (here the "symmetric gauge") is due to our conventions, especially our choice of exponents. Another (but equivalent) expression for the exponents (7) will lead to the equations (13) with the potential in a different gauge. Together with the equations of motion we get from (10) the eigenvalue equation for the discrete Landau (energy) levels,

$$(15)\quad (-i\partial_x - eA_x)^2 + (-i\partial_y - eA_y)^2 = c_\perp$$

with spectrum $c_\perp = (2n+1)|eB| \quad (n = 0,1,2,\ldots)$.

Discussion. The combination of two basic principles, viz. the superposition principle (Hilbert space) and the invariance principle (symmetry group), which already for a free particle resulted in a satisfactory derivation of the equations of motion [1], has been applied here to a particle in a uniform e.m. field.We have dealt with this physical system in the external field approximation (as usual in elementary q.m.) because we considered the field as given once and for all, as if it were a characteristic feature of a (Galilean or Minkowski) universe containing the same set of events as the empty universe but having a smaller group of space-time symmetries. We used explicitely the principle of locality of the transformation of the wave functions (also usual in elementary q.m.). Equation (1) gives the most general form for U(g) such that $|\psi(x)|^2$ transforms as a scalar function, i.e. such that $|U(g)\ \psi(g\circ x)|^2 = |\psi(x)|^2$. In our derivation of the equations of motion for a charged particle in an external uniform parallel e.m. field we never used the "trick" of the substitution $i\partial \rightarrow i\partial - eA$. This derivation, which is meant to be in the spirit of reference [1], results from a more comprehensive investigation on projective representations in quantum mechanics [6;7].

Literature

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