THE MAXIMAL ABELIAN SUBGROUPS OF THE CONFORMAL GROUP OF SPACE-TIME*

by

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The problem of finding all maximal abelian subgroups of a given Lie group G is of particular interest. Mathematically this is directly related to the problem of classifying all possible types of Lie algebras. In physical applications a classification of all maximal abelian subalgebras of a given Lie algebra L provides a classification of all possible complete sets of additive quantum numbers, characterizing a physical system invariant under the group $\langle \exp L \rangle$. This can also provide a classification of different dynamical systems, associated with a certain dynamical group, help to find all coordinate systems in which a given partial differential equation allows the separation of variables, and have many further applications.

The problem of finding all conjugacy classes of Cartan subalgebras of real and complex semisimple Lie algebras, has been solved[1,2]. That of finding all abelian subalgebras of a semisimple Lie algebra, including those that contain nilpotent elements (in finite dimensional representations) is more difficult. All maximal abelian subalgebras of maximal dimension have been constructed by Maltsev[3] for arbitrary complex simple Lie algebras. While a considerable amount of literature exists on commutative matrices (even in book form[4] with numerous references to earlier work), no systematic theory of all abelian subalgebras of a general semisimple Lie algebra is available. The aim of this example and a forthcoming article about the general case is to fill this gap.

Let us introduce the usual physical generators of the conformal group of space-time, namely the rotations L_i, Lorentz boosts K_i, dilation D, translations P_μ and special conformal transformations $C_\mu (i=1,2,3,\ \mu=0,1,2,3)$.

Work is in progress on a complete classification of all subalgebras of the conformal Lie algebra. So far we know all subalgebras of sim(3,1), opt(3,1), o(3,2) and o(4,1) (for notations see Fig. 1). On Fig. 1 we present some of the subalgebras of o(4,2), namely:
(1) All 12 maximal abelian subalgebras; (2) All three maximal solvable subalgebras; (3) Two of the nine maximal subalgebras, namely sim(3,1) and opt(3,1). We also indicate there mutual inclusions. On Fig. 2 we present similar diagrams for the de Sitter algebras o(4,1) and o(3,2). In this case we only indicate the maximal solvable and

maximal abelian subalgebras (classified under the appropriate de Sitter groups) and their mutual inclusions.

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* Work supported in part by the National Research Council of Canada and by a NATO research grant.

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FIGURE CAPTIONS

Fig. 1. The SO (4,2) conjugacy classes of maximal abelian subalgebras of o(4,2), opt(3,1) and sim(3,1). Dashed and thick boxes indicate maximal solvable and maximal abelian subalgebras, respectively (a generator X_i represents X_1, X_2 and X_3). A dashed thick box indicates an algebra that is simultaneously maximal solvable and maximal abelian, i.e. a compact Cartan subalgebra.

Fig. 2. Maximal abelian subalgebras of o(3,2) and o(4,1). (The conventions are the same as in Fig. 1.)

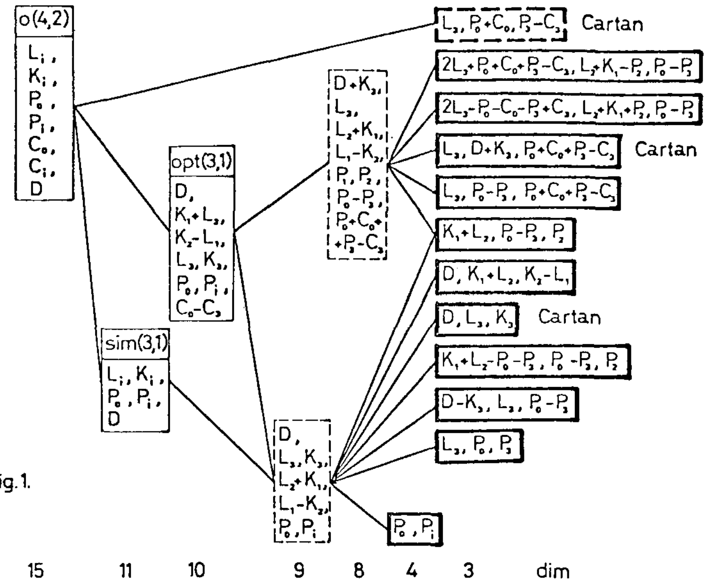

Fig. 1.

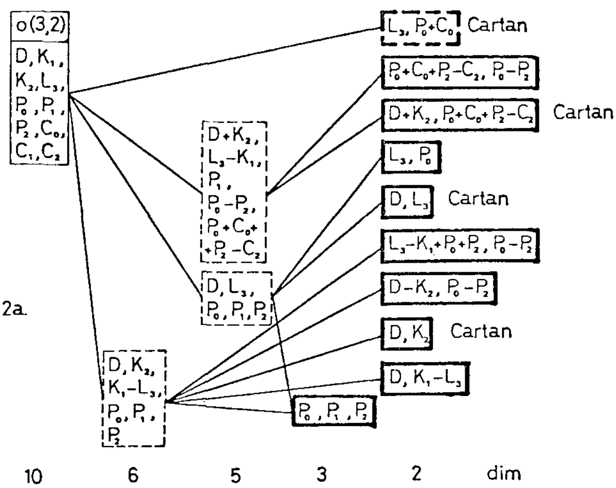

Fig. 2a.

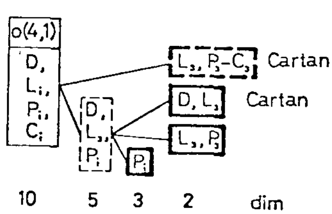

Fig. 2b.