Harmonic Analysis on Graded (or Super) Lie Groups

by Bertram Kostant
Department of Mathematics, MIT, Cambridge, Mass., USA

There is much more to graded (or super) Lie theory than just writing down indices, coordinate changes and the classification of the simple graded Lie algebras. Drawing from experience in ordinary Lie theory, graded Lie groups are likely to be a useful object only insofar as one can develop a corresponding theory of harmonic analysis. This, of course, means representation theory. If G is an ordinary Lie group, then for the most part, the representations of G are to be found on spaces of functions on G under the action of left translation. Thus, for example, if $G = \mathbb{R}$ is the real line, the one dimensional representations of $\mathbb{R}$ are found by left translation of the functions $e^{i\lambda x}$. In general, one is lead to considering functions on homogeneous spaces G/H, or more generally, vector bundles over homogeneous spaces, i.e. induced representations.

We have been particularly concerned with the so-called orbit method for constructing the irreducible representations. This involves geometric quantization and line bundles over symplectic homogeneous spaces G/H which arise as those orbits of the coadjoint action which satisfy an integrability condition - a Bohr-Sommerfeld type condition. The method has been quite successful in yielding the representations of compact groups, solvable Lie groups, and the generic representations of semi-simple Lie groups. In fact, recent results show that the general reduction theory in harmonic analysis eventually comes down to looking at coadjoint orbits. The main objective of the paper [1] is to develop this theory for graded Lie groups.

A graded Lie group is a special case of a graded manifold. A graded manifold is a pair (X,A) where X is a usual manifold and $A(X)$ is a graded commutative algebra satisfying certain axioms - one of which implies that X is the space of maximal ideals of $A(X)$. The ring $A(X)$ plays the role of the "functions" on X. Certain authors regard $C^{\infty}(X)$, the space of ordinary C^{∞} functions on X, as a part of $A(X)$. We find this to be an unsatisfactory condition - one which severely restricts the morphisms of graded manifolds and makes $A(X)$ nothing more than the section of a fiber bundle over X. In our theory, $C^{\infty}(X)$ is a quotient of $A(X)$ and not a subobject. The usual notation of points, tangent vectors, higher derivatives at points, is replaced in the graded case, by $A(X)^{*}$, the linear functionals on $A(X)$ which vanish on ideals of finite codimension. One has $X \subseteq A(X)^{*}$. Also the tangent space $T_p(X) = (T_p(X))_0 \oplus T_p(X)_1$, with its even and odd components, at any $p \in X$, is included in $A(X)^{*}$. The space $A(X)^{*}$ is a coalgebra and corresponds to the distributions of finite support on X. The graded manifold (G,A) is a graded Lie group if $A(G)^{*}$ has an algebra structure as well as a coalgebra structure, and they are related so as to be Hopf algebra. The tangent space $g = g_0 \oplus g_1$ at the identity is a graded Lie algebra (GLA) and (any GLA may so arise). The group G is then a usual Lie group having the even part g_0 as its Lie algebra. If R(G) is the group algebra of G then

$$A(G)^* = \mathbb{R}(G) \divideontimes U(g)$$

where $\divideontimes$ denotes smash product and U(g) is the universal enveloping algebra of the graded Lie algebra g. This representation theory of (G,A) is then the algebra representation theory of $A(G)^*$, subject to certain continuity conditions.

One of the main theorems in [1] is the existence of homogeneous spaces (G/H,A/B) where (H,B) is a graded Lie subgroup of (G,A) and the existence of induced representations of (G,A) by characters on (H,B). This is in particular carried out for the coadjoint orbits where (G/H,A/B) carries a graded symplectic structure and one has the machinery of the Hamilton-Jacobi theory in the graded context. One also has an integrality criteria (Bohr-Sommerfeld) for orbits as in the usual case (see [2] for a new interpretation of this condition in the usual case). Furthermore, one also has prequantization and it is shown that the spin representation is a special case of prequantization.

It seems likely that the orbit method in the graded case will be as successful as in the usual case for developing harmonic analysis of a graded Lie group. One interesting indication of this is that Kac's results on the classification of finite dimensional irreducible representations of a solvable graded Lie group are indeed predicted by the orbit method.

References

1. Kostant, Bertram, Graded Manifolds, Graded Lie Theory, and Prequantization, "Lecture Notes in Mathematics," vol. 570, Springer-Verlag, 1975, pp. 177-306.
2. Cathelineau, ., On an Integrality Theorem of Kostant, Ecole Normale Superieure, 10 (1977), 73-86.