The Exceptional Groups as Candidates for Supersymmetry

T. S. Santhanam

Physikalisches Institut, Universität Würzburg

and

Matscience, Madras, India

Many years ago Gamba[1] pointed out some peculiarities of the eight dimensional space. In this space, the vector $\underline{x}$ and the bispinors denoted by φ and ψ are of equal dimension. Besides the following quantities invariant under rotations

$$F = \underline{x}^2 = x_1^2 + x_2^2 + \cdots + x_8^2 , \tag{1}$$

$$\Phi = \varphi^T B \varphi = \varphi_1^2 + \varphi_2^2 + \cdots + \varphi_8^2 , \tag{2}$$

$$\Psi = \psi^T B \psi = \psi_1^2 + \psi_2^2 + \cdots + \psi_8^2 \tag{3}$$

(where the matrix B transforms the vector $\underline{x}$ into its transpose and $B^2 = I$) are equivalent. Thus, it is only a matter of convention to define which is the vector and which are the spinors of first and second kind. This beautiful property is usually referred to as the principle of "triality". In fact, the trilinear form

$$K = \varphi^T B \underline{x} \psi \tag{4}$$

is left invariant under rotations. Writing $\underline{x}$, φ and ψ as octonions

$$\underline{x} = x_1 e_1 + x_2 e_2 + \cdots + x_8 e_8 \tag{5}$$

$$\varphi = \varphi_1 e_1 + \varphi_2 e_2 + \cdots + \varphi_8 e_8 \tag{6}$$

$$\psi = \psi_1 e_1 + \psi_2 e_2 + \cdots + \psi_8 e_8 \tag{7}$$

one can write the invariants (1), (2) and (3) as

$$F = n(\underline{x}), \tag{8}$$

$$\Phi = n(\varphi), \tag{9}$$

$$\Psi = n(\psi), \tag{10}$$

and

$$-K = \bar{\varphi} \cdot (x\,\psi) \tag{11}$$

where the inner product of two octonions A and B is defined to be

$$A \cdot B = \frac{1}{2}\left(\bar{A}B + \bar{B}A\right) \tag{12}$$

and n(A) is the norm

$$n(A) = \bar{A}A = A\bar{A} = a_1^2 + a_2^2 + \dots + a_8^2 \tag{13}$$

Here $\bar{A}$ is the conjugate octonion. The set of all octonion Hermitian matrices M of the type

$$M = \begin{pmatrix} p & A & \bar{B} \\ \bar{A} & q & C \\ B & \bar{C} & r \end{pmatrix} \tag{14}$$

where p, q and r are real numbers, $\underline{A}$, $\underline{B}$, $\underline{C}$ are octonions ($\bar{A}$, $\bar{B}$, $\bar{C}$ being their adjoints) form the exceptional Jordan algebra. Multiplication of two elements of this algebra is defined as

$$MN = NM = \frac{1}{2}\left(M \times N + N \times M\right) \tag{15}$$

where M x N is the usual matrix product.Multiplication is therefore commutative, but not associative. One can show that the associator defined by

$$\{L, M, N\} = L\,(MN) - (LM)\,N \tag{16}$$

is indeed analogous to the commutator of the ordinary matrix algebra. Under eight dimensional rotations the octonions or the eight dimensional vector and spinors of first and second kind transform differently. <u>But, there is a subgroup of rotations in eight dimensions with fourteen parameters that transform eight</u>

dimensional vectors and the spinors in the same way. This group is the exceptional group G_2.

Recently F. Gliozzi et al.[2] have shown the following connection between dual spinor model and supersymmetry. In the space of ten dimensions D = 10, the Neveu-Schwarz[3] model involves oscillators with D - 2 = 8 Lorenz indices.

In Ramond model[4] there is the spinor with dim $2^5 = 32$. But, if one imposes the Majorana and Weyl conditions which decrease each by half the number of components then the degeneracy is eight for both fermions and bosons. A Yang-Mills theory in this space has been shown to be supersymmetric provided the fermions and Majorana-Weyl.Majorana conditions imposes reality condition while Weyl condition imposes that the spinor is an eigenfunction of γ_5.

What we observe is therefore that the Majorana-Weyl conditions on a dual spinor in D = 10 make the spinor and vector equivalent. On the other hand we have seen that in eight dimensions, restriction to the subgroup G_2 again makes the eight dimensional vector and spinors of first and second kind equivalent. This would imply that G_2 leaves bosons and fermions on equal footing and hence is a candidate for supersymmetry.

References

1) A. Gamba, J. Math. Phys. 8, 775 (1967)

2) F. Gliozzi, J. Scherk and D. Olive, CERN preprint TH-2253

3) A. Neveu and J. H. Schwarz, Nucl. Phys. B 31, 86 (1971)

4) P. Ramond, Phys. Rev. D 3, 2415 (1971)