COMPLETENESS OF NETWORKS OF STATES

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It has long been known that the Heisenberg-Weyl (H-W) group of operators generates, from any chosen initial state, a system of states that spans the whole state space. This is an expression of the completeness of the set of position and momentum operators, which are the generators of the H-W group. A special case is when the initial state is the ground state of a harmonic oscillator; the system here consists of the coherent states (minimal wave-packets) of quantum optics. Although overcomplete, this system has nevertheless useful properties as a basis set for the state space[1]. A development of these ideas has been concerned with systems of states, generated by infinite discrete subgroups of the H-W group associated with lattices of points in the phase space. For one dimension, the discrete group operators are $e^{ib\hat{x}}$ and $e^{ia\hat{p}/\hbar}$, where $\hat{x}$ and $\hat{p}$ are the position and momentum operators and b and a are real numbers (lattice constants). The lattice in question consists therefore of points $(na, m\hbar b)$ in phase space. The generators commute, (the group is abelian) only when $ab = 2\pi$ x integer, i.e. the lattice unit cell area is a multiple of h. The corresponding system of states generated is a 'network' of functions $|\phi_{mn}\rangle$:

$$\langle x|\phi_{mn}\rangle = e^{-imb\hat{x}}\, e^{-ina\hat{p}/\hbar}\, \langle x|\phi\rangle$$

with initial state $|\phi\rangle \equiv |\phi_{00}\rangle$. The questions to be asked are: do these networks of states span the state space, as for the systems generated by the full H-W group? and what are their properties as basis sets? Progress has recently been made on these questions by several authors[2,3]. One emergent fact is the importance of the lattice unit cell area. For coherent states ($|\phi\rangle$ the harmonic oscillator ground state) it has been shown that the network is incomplete when $ab > 2\pi$, and infinitely overcomplete when $ab < 2\pi$. At the critical value $ab = 2\pi$ the coherent state network is overcomplete by just one state; this particular system, of one minimal wave-packet in a phase space area of h, was first stated to be complete by von Neumann, but the final proof is more recent. From now on we discuss networks for the critical area <u>only</u> (thus $b = 2\pi/a$), but generalise to an arbitrary initial $|\phi\rangle$. These are of practical interest, since certain networks of functions in two-dimensional (real) space which have been used for studying Landau levels in a crystal, can be mapped onto the above networks defined on phase space[4].

A criterion for completeness for a network with arbitrary $|\phi\rangle$ has been found[5], by use of the kq representation[6]. Here, a state $|f\rangle$ is represented by a square-integrable function $\langle kq|f\rangle$ defined on the kq cell $-\pi/a < k \leq \pi/a$; $-a/2 < q \leq a/2$ with periodicity $\langle k + \frac{2\pi}{a}\, q|f\rangle = \langle kq|f\rangle$; $\langle k,q+a|f\rangle = e^{ika}\, \langle kq|f\rangle$.

The real number a is arbitrary, but if we identify it with the network constant a

the network functions have the simple form

$$\langle kq|\phi_{mn}\rangle = e^{-ikna}\ e^{2\pi iqm/a}\ \langle kq|\phi\rangle$$

The network is complete, by definition, if any state orthogonal to the entire system is the zero state. From Fourier analysis it follows that for such a state $|f\rangle$ we have $\langle kq|f\rangle\langle kq|\phi\rangle = 0$ a.e. (almost everywhere), implying $|f\rangle = 0$ and completeness if and only if $\langle kq|\phi\rangle$ is non-zero a.e.[5]. In many important cases $\langle kq|\phi\rangle$ has only isolated zeros and completeness is proved. But to make use of the $|\phi_{mn}\rangle$ as a basis we must (a) find a minimally complete subset $\{|\phi_{mn}\rangle\}'$, where the prime denotes possible exclusion of some members of the network, (b) construct the biorthogonal set $\{|\tilde{\phi}_{mn}\rangle\}'$ to the reduced set and (c) verify the expansion property

$$|f\rangle \sim \Sigma'\ |\phi_{mn}\rangle\langle\tilde{\phi}_{mn}|f\rangle$$

The sign $\sim$ means the nature of the equivalence is to be specified. These properties depend strongly on the form of the function $\langle kq|\phi\rangle$, particularly on its zeros.

For an important class of $|\phi\rangle$, which includes the coherent state case, $\langle kq|\phi\rangle$ is smooth (continuously differentiable to all orders) and has only isolated zeros. We assume we are dealing with this class, and furthermore that there is at least one zero. Then we can show that the number of states to be removed for minimality depends on the number and order of the zeros, and that the expansion exists as a distribution on the space of smooth functions but not in norm convergence. Take the case where $\langle kq|\phi\rangle$ has one simple zero; this includes the coherent state case. Then, removing any one state leaves a minimally complete set; remove $|\phi_{00}\rangle$ and the biorthogonal functions are

$$\langle kq|\tilde{\phi}_{m}\rangle = (e^{-ikna}e^{2\pi iqm/a} - 1)/2\pi\langle\phi|kq\rangle \quad ; \quad (m,n) \neq (0,0)$$

Losing no essential generality, we have supposed the zero at $k = 0$, $q = 0$. The expansion can then be proved as an equality of distributions, in the kq representation

$$\langle kq|f\rangle \sim [\Sigma'\ \langle\tilde{\phi}_{mn}|f\rangle\ e^{-ikna}\ e^{2\pi iqm/a}]\ \langle kq|\phi\rangle \qquad \circledast$$

for any $|f\rangle$. We are here working in the space of distributions of the form

$$T(kq)\langle kq|\phi\rangle \ ; \ T(kq) \equiv \Sigma a_{mn}\ e^{-ikna}\ e^{2\pi iqm/a} \qquad \text{(no prime)}$$

where T is a periodic distribution (the a_{mn} are of 'slow increase')[7]. For $|f\rangle = |\phi\rangle$ for example, one has $\langle\tilde{\phi}_{mn}|\phi\rangle = -1$ and $T = 1 - 2\pi\ \delta(k)\delta(q)$.

Clearly the convergence is not in the norm for $|f\rangle = |\phi\rangle$ (which, we remark, is the excluded network function). The distributions in the expansion all have $a_{00} = 0$; there is a non-uniqueness of the coefficients in so far as we can add to the R.H.S. of $\circledast$ any $\bar{T}$ for which

$$\bar{T}\ (kq)\langle kq|\phi\rangle \sim 0 \qquad ; \qquad \bar{a}_{00} = 0$$

Because of the zero in $\langle kq|\phi\rangle$, there are always non-trivial solutions to this equation, which are derivatives of $\delta(k)\ \delta(q)$[7]. The expansion is therefore never unique. We can view these results differently by observing that any solution of the last equation without the restriction $\bar{a}_{00} = 0$ yields a linear relation among the $|\phi_{mn}\rangle$. The solution $\bar{T}\ (kq) = \delta(k)\ \delta(q)$ gives $\Sigma|\phi_{mn}\rangle \sim 0$ (no prime), which is of course the expansion for $|\phi\rangle$ rearranged; the other solutions correspond to the non-uniqueness mentioned above.

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