THE DIAGRAM LATTICE AS STRUCTURAL

PRINCIPLE IN MATHEMATICS

Adalbert Kerber

Lehrstuhl D für Mathematik
Rhein.-Westf. Techn. Hochschule
51 Aachen, Germany

It is the aim of this lecture and the following one presented by E. Ruch to draw the attention of the audience to a certain partial order on the set P(n) of partitions of a given natural number n.

This partial order in fact establishes a lattice structure on P(n) which turns out to be the underlying combinatorial structure of the representation theory of the symmetric group S_n. This partial order can be described in terms of double-cosets of certain subgroups of S_n, in terms of intertwining numbers of specific representations of S_n which are induced from such subgroups, as well as it can be expressed in terms of numbers of 0-1-matrices with prescribed row and column sums and in terms of properties of Young-tableaus.

These various ways of describing the partial order in question open the way to recognize this lattice structure on P(n) as basic for various applications. Applications in mathematics are in particular the representation theory of S_n and of related groups like wreath products $G \wr S_n$, and certain existence theorems in combinatorics. This will be described here, while applications to sciences which are closely related will be given by E. Ruch.

1. The diagram lattice

Let n denote a natural number, i.e. $n \in N := \{1,2.3,...\}$. A partition of n is a finite sequence

$$\alpha = (\alpha_1,...,\alpha_h)$$

with the following properties:

$$\text{(i)} \ \forall\, 1 \leq i \leq h \ (\alpha_i \in N)$$

$$\text{(ii)} \ \forall\, 1 \leq i < h \ (\alpha_i \geq \alpha_{i+1}),$$

$$\text{(iii)} \ \Sigma_1^h \alpha_i = n .$$

We shall sometimes abbreviate this by simply writing

$$\alpha \vdash n.$$

The partitions of n=6 are for example:

(6), $(5,1)$, $(4,2)$, $(4,1^2):=(4,1,1)$, $(3^2):=(3,3)$, $(3,2,1)$,
$(3,1^3):=(3,1,1,1)$, $(2^3):=(2,2,2)$, $(2^2,1^2):=(2,2,1,1)$,
$(2,1^4):=(2,1,1,1,1)$, $(1^6):=(1,1,1,1,1,1)$.

A partition α of n can be visualized by the corresponding Young-diagram $[\alpha]$, which consists of n nodes in h rows and α_1 columns. The i-th row of the diagram consists of α_i nodes, and all the rows start in the same column:

$$[\alpha] := \begin{array}{llll} \times & \times \cdots\cdots\cdots\cdots \times & & \alpha_1 \text{ nodes} \\ \times & \times \cdots\cdots\cdots \times & & \alpha_2 \text{ nodes} \\ \cdots\cdots\cdots\cdots\cdots & & & \cdots\cdots \\ \times & \times \cdots \times & & \alpha_h \text{ nodes} . \end{array}$$

Because of $\alpha_i \geq \alpha_{i+1}$ and as all the rows start in the same column, the lengths α_i', $1 \leq i \leq h' = \alpha_1$, of the columns also form a partition of n, which we denote by α':

$$\alpha' := (\alpha_1',...,\alpha_{h'}').$$

It is called the partition associated with α. Its Young-diagram $[\alpha']$ is obtained from $[\alpha]$ by simply reflecting $[\alpha]$ in the main diagonal, e.g.

$$[\alpha] := [3,2,1^2] = \begin{array}{lll} \times & \times & \times \\ \times & \times & \\ \times & & \\ \times & & \end{array} \quad \text{yields} \quad [\alpha'] = \begin{array}{llll} \times & \times & \times & \times \\ \times & \times & & \\ \times & & & \end{array} = [4,2,1].$$

We denote by P(n) the set of all the partitions of n:

$$P(n) := \{\alpha \mid \alpha \vdash n\}.$$

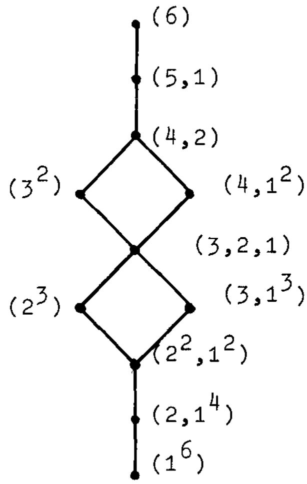
The dominance order "$\trianglelefteq$" is defined on P(n) with the aid of the partial sums

$$\sigma_i^\alpha := \Sigma_1^i \alpha_\nu \ , \ 1 \leq i \leq h.$$

We put, if $\alpha = (\alpha_1, \ldots, \alpha_h) \vdash n$, $\beta = (\beta_1, \ldots, \beta_k) \vdash n$:

$$\alpha \trianglelefteq \beta :\Leftrightarrow \forall\, 1 \leq i \leq \min\{h,k\}\ (\sigma_i^\alpha \leq \sigma_i^\beta).$$

The smallest n, where "$\trianglelefteq$" is not a total order, is n=6. The order diagram of $(P(6), \trianglelefteq)$ looks as follows:



It is important to characterize the situation when $\alpha \trianglelefteq \beta$ and there is no $\gamma \vdash n$ such that $\alpha \triangleleft \gamma \triangleleft \beta$, which we abbreviate by

$$\alpha \blacktriangleleft \beta.$$

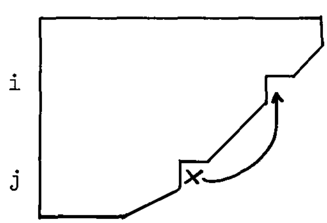
The following lemma which characterizes this situation is easy to prove:

<u>1.1 Lemma</u>: $\alpha \blacktriangleleft \beta$ holds if and only if there exist i and j such that

(i) $i < j$, and $\beta_i = \alpha_i + 1$, and $\beta_j = \alpha_j - 1$, while for all $r \neq i,j$ we have $\beta_r = \alpha_r$,

(ii) either $i = j-1$ or $\alpha_i = \alpha_j$.

In other words: $\alpha \blacktriangleleft \beta$ holds if and only if $[\beta]$ is obtained from $[\alpha]$ by raising a node upwards from the end of the j-th row to the end of the i-th row, and this step is as small as possible:



It is not difficult to show that 1.1 implies

1.2 Lemma: $\forall\ \alpha,\beta \vdash n\ (\alpha \trianglelefteq \beta \iff \beta' \trianglelefteq \alpha')$.

Using the partial sums again, we can define an infimum $\alpha \wedge \beta$ and a supremum $\alpha \vee \beta$ of two partitions α and β of n as follows (cf. ref. 1):

(i) $\alpha \wedge \beta := \gamma$, where $\sigma_i^\gamma := \min\{\sigma_i^\alpha, \sigma_i^\beta\}$, $1 \leq i \leq \max\{h,k\}$,

(ii) $\alpha \vee \beta := \delta$, where $\sigma_i^{\delta'} := \min\{\sigma_i^{\alpha'}, \sigma_i^{\beta'}\}$, $1 \leq i < \max\{h',k'\}$.

It was shown in ref. 1 that the following holds:

1.3 Theorem: $(P(n), \trianglelefteq, \wedge, \vee)$ is a lattice.

We call this lattice the diagram lattice since the name "partition lattice" might be misleading, it is already a standard name for a different lattice structure.

This lattice is examined in ref. 2, where it is shown that the Moebius function on this lattice takes values 0, ± 1 only.

Partitions and the dominance order were hitherto used mainly in connection with graphs and the question, which partitions form the edge degree sequence of a graph (cf. ref. 3, chapter 6). But we shall not stress this fact here, since we are above all interested in characterizations of the dominance order in terms of group theory and representation theory, in order to get a better insight into this combinatorial structure $(P(n), \trianglelefteq)$ as well as into its applications.

It may be mentioned that the discovery of this structure being the underlying combinatorial structure of a great part of representation theory of the symmetric groups is quite recent, although it is quite obvious already from the proofs used in the classical approaches.

2. Young-subgroups of symmetric groups

We would like now to characterize partitions α and β of n, which satisfy

$$\alpha \trianglelefteq \beta.$$

It is our aim to give four characterizations of this fact, one of them is a representation-theoretical one, another one is group-theoretical, and the last two of them are combinatorial characterizations.

In order to do this we introduce a specific class of subgroups of the symmetric group S_n.

We consider the symmetric group S_n, acting on the set

$$\underline{n} := \{1,\dots,n\}.$$

If $\gamma = (\gamma_1,\dots,\gamma_r) \vdash n$, then we can form partitions of the set $\underline{n}$ into pairwise disjoint subsets $\underline{n}_i^\gamma$ of order γ_i, $1 \leq i \leq r$, i.e.

$$\underline{n} = \bigcup_{i=1}^{r} \underline{n}_i^\gamma,\ \forall\ i \neq j\ (\underline{n}_i^\gamma \cap \underline{n}_j^\gamma = \emptyset),\ \forall\ i\ (|\underline{n}_i^\gamma| = \gamma_i)\ .$$

Let now S_i^γ denote the subgroup of S_n which consists of the $\gamma_i!$ elements leaving each element of $\underline{n} \setminus \underline{n}_i^\gamma$ fixed, $1 \leq i \leq r$. We can form the product S_γ of all these subgroups:

$$S_\gamma := \prod_{i=1}^{r} S_i^\gamma\ ,$$

which is obviously isomorphic to the direct product

$$\times_{i=1}^{r} S_{\gamma_i}$$

of the symmetric groups S_{γ_i}. S_γ is called a Young-subgroup corresponding to γ.

The classical development of the representation theory of S_n starts off with an examination of certain representations of S_n which are induced from specific one-dimensional representations of such Young-subgroups. In order to describe this, we denote by IS_γ the identity representation of S_γ, where each $\pi \in S_\gamma$ is mapped onto the one-rowed matrix (1). By AS_γ we denote the alternating representation of S_γ, where $\pi \in S_\gamma$ is mapped onto $(\mathrm{sgn}\pi)$, $\mathrm{sgn}\pi = \pm 1$ being the sign of the permutation π.

The representations of S_n induced from IS_γ and AS_γ are denoted by

$$IS_\gamma \uparrow S_n, \text{ and } AS_\gamma \uparrow S_n.$$

If now α and β are partitions of n, we would like to evaluate the intertwining numbers (i.e. the inner products of the corresponding characters)

$$(IS_\alpha \uparrow S_n, IS_\beta \uparrow S_n)$$

and

$$(IS_\alpha \uparrow S_n, AS_\beta \uparrow S_n).$$

In order to do this, we apply Mackey's intertwining number theorem (cf. ref 4,(44.5)), which gives

<u>2.1</u> $$(IS_\alpha \uparrow S_n, IS_\beta \uparrow S_n) = \sum_{S_\alpha \pi S_\beta} (I(S_\alpha \cap \pi S_\beta \pi^{-1}), I(S_\alpha \cap \pi S_\beta \pi^{-1})),$$

if the sum is taken over the complete system of double-cosets $S_\alpha \pi S_\beta$ of S_α and S_β in S_n. Furthermore we obtain

<u>2.2</u> $$(IS_\alpha \uparrow S_n, AS_\beta \uparrow S_n) = \sum_{S_\alpha \pi S_\beta} (I(S_\alpha \cap \pi S_\beta \pi^{-1}), A(S_\alpha \cap \pi S_\beta \pi^{-1})),$$

if again the sum is taken over the complete system of double-cosets.

Since the intersection $S_\alpha \cap \pi S_\beta \pi^{-1}$ is a direct product of symmetric groups and as both $I(S_\alpha \cap \pi S_\beta \pi^{-1})$ and $A(S_\alpha \cap \pi S_\beta \pi^{-1})$, which are the identity representation and the alternating representation of this intersection, are irreducible, we have always

$$(I(S_\alpha \cap \pi S_\beta \pi^{-1}), I(S_\alpha \cap \pi S_\beta \pi^{-1})) = 1,$$

while

$$(I(S_\alpha \cap \pi S_\beta \pi^{-1}), A(S_\alpha \cap \pi S_\beta \pi^{-1})) = \begin{cases} 1, & \text{if } S_\alpha \cap \pi S_\beta \pi^{-1} = \{1\} \\ 0, & \text{otherwise.} \end{cases}$$

Hence 2.1 shows that $(IS_\alpha \uparrow S_n, IS_\beta \uparrow S_n)$ is equal to the number of double-cosets $S_\alpha \pi S_\beta$, while $(IS_\alpha \uparrow S_n, AS_\beta \uparrow S_n)$ is equal to the number of double-cosets with <u>trivial-intersection-property</u>

2.3 $$S_\alpha \cap \pi S_\beta \pi^{-1} = \{1\}.$$

This leads us to a closer examination of double-cosets of Young-subgroups. Here we have a result of A.J. Coleman (ref. 5) at hand:

<u>2.4 Theorem</u>: If $\alpha = (\alpha_1, \ldots, \alpha_h)$ and $\beta = (\beta_1, \ldots, \beta_k)$ are partitions of n with corresponding Young-subgroups S_α and S_β, then $\rho \varepsilon S_n$ is contained in $S_\alpha \pi S_\beta$ if and only if for $1 \leq i \leq h$ and $1 \leq j \leq k$

$$|\underline{n}_i^\alpha \cap \pi[\underline{n}_j^\beta]| = |\underline{n}_i^\alpha \cap \rho[\underline{n}_j^\beta]|.$$

This theorem shows that the double-coset $S_\alpha \pi S_\beta$ is characterized by the numbers $z_{ij} := |\underline{n}_i^\alpha \cap \pi[\underline{n}_j^\beta]|$, which we may put together into the hxk-matrix

$$(z_{ij}) := (|\underline{n}_i^\alpha \cap \pi[\underline{n}_j^\beta]|).$$

We now obtain from 2.4:

<u>2.5 Theorem</u>: The mapping

$$f: S_\alpha \pi S_\beta \mapsto (|\underline{n}_i^\alpha \cap \pi[\underline{n}_j^\beta]|)$$

establishes a one-to-one correspondence between the set of double-cosets $S_\alpha \pi S_\beta$ and the set of hxk-matrices (z_{ij}) with nonnegative integral entries z_{ij} and prescribed row sums $\alpha_i = \Sigma_j z_{ij}$ and prescribed column sums $\Sigma_i z_{ij} = \beta_j$. The restriction of f to the set of double-cosets $S_\alpha \pi S_\beta$ with trivial-intersection-property 2.3 in particular establishes a one-to-one correspondence between the set of these double-cosets and the set of hxk-0-1-matrices with prescribed row sums α_i and prescribed column sums β_j.

This theorem together with 2.1 and 2.2 demonstrates the equivalence of the problems of evaluating $(IS_\alpha \uparrow S_n, IS_\beta \uparrow S_n)$ (or $(IS_\alpha \uparrow S_n, AS_\beta \uparrow S_n)$ resp.), counting the number of double-cosets $S_\alpha \pi S_\beta$ (or those with trivial-intersection-property, resp.), telling the number of hxk-matrices with nonnegative integral entries (or hxk-0-1-matrices, resp.) with prescribed row sums α_i and column sums β_j.

In order to connect this result with the diagram lattice of the preceding section we can use any one of the following two theorems (cf. ref. 1 and ref. 5):

<u>2.6 Theorem of Ruch/Schönhofer</u>: If S_α and S_β are Young-subgroups of S_n which correspond to partitions α and β of n, then the intertwining number $(IS_\alpha \uparrow S_n, AS_\beta \uparrow S_n)$ is nonzero if and only if $\alpha \trianglelefteq \beta'$.

<u>2.7 Theorem of Gale/Ryser</u>: If $\alpha = (\alpha_1,\ldots,\alpha_h)$ and $\beta = (\beta_1,\ldots,\beta_k)$ are partitions of n, then there exist 0-1-matrices with row sums α_i and column sums β_j if and only if $\alpha \trianglelefteq \beta'$.

The considerations made above show clearly that these two theorems are equivalent, although they sound quite different. The links between them are Mackey's intertwining number theorem and Coleman's characterization of the double-cosets of Young-subgroups.

The Gale/Ryser theorem is one of the most important existence theorems in combinatorics. It serves in particular for proofs of the existence of incidence structures. A typical and easy example is a necessary and sufficient condition for the existence of tactical configurations with prescribed parameters v,b,r and k. (A tactical configuration with parameters v,b,r and k is a triple (V,B,I) consisting of a set V of vertices, a set B of blocks, and an incidence relation $I \subseteq V\times B$ such that $|V| = v$, $|B| = b$, and where each $v \in V$ is incident with exactly r blocks, while each $b \in B$ is incident with exactly k vertices, so that in particular $v\cdot r = b\cdot k$ holds.)

The incidence matrix of such a tactical configuration is a vxb-0-1-matrix with row sums all equal to r and column sums all equal to k. The theorem of Gale and Ryser says that such a matrix (and hence also a tactical configuration with parameters v,b,r and k) exists if and only if we have $\alpha \trianglelefteq \beta'$, where $\alpha := (r^v)$ and $\beta := (k^b)$, i.e. if and only if

$$(r^v) \trianglelefteq (k^b)' = (b^k),$$

or equivalently (cf. 1.2):

$$(k^b) \trianglelefteq (r^k)' = (v^r).$$

Hence by the Gale/Ryser theorem a tactical configuration with parameters v,b,r and k exists if and only if $v\cdot r = b\cdot k$ and $r \leq b$, or equivalently if and only if $v\cdot r = b\cdot k$ and $k \leq v$.

There are many other existence theorems in combinatorics for the proof of which the Gale/Ryser theorem is the main tool (cf. e.g. ref. 6/7).

The Ruch/Schönhofer theorem gives a deeper insight into the decomposition of the induced characters $IS_\alpha \uparrow S_n$ which we introduced at the beginning of this section.

A first consequence of this theorem is

$$(IS_\alpha \uparrow S_n, AS_{\alpha'} \uparrow S_n) > 0.$$

We obtain in fact more than that, namely

<u>2.8</u> $$(IS_\alpha \uparrow S_n, AS_{\alpha'} \uparrow S_n) = 1,$$

if we notice that there is exactly one 0-1-matrix with row sums α_i and column sums α'_j and apply the equality of the intertwining number and the number of such 0-1-matrices. 2.7 means that these two induced representations have a uniquely determined irreducible constituent in common, which they both contain with multiplicity 1. We denote this constituent by $[\alpha]$ so that we obtain by a slight abuse of the notation:

2.9 $$[\alpha] := IS_\alpha \uparrow S_n \cap AS_{\alpha'} \uparrow S_n .$$

(We notice that $[\alpha]$ does depend only on the partition α of n and neither on the partition of $\underline{n}$ which gives S_α nor on the partition of $\underline{n}$ which gives $S_{\alpha'}$, for all Young-subgroups S_γ which correspond to a given $\gamma \vdash n$ are conjugate subgroups of S_n.)

In order to show that the system

2.10 $$\{[\alpha] \mid \alpha \vdash n\}$$

consists of pairwise inequivalent representations, so that it is a complete system since it is of maximal order, one proves the following

2.11 $$(IS_\alpha \uparrow S_n, [\beta]) > 0 \Rightarrow \alpha \trianglelefteq \beta .$$

This is clear from the theorem of Ruch/Schönhofer, for 2.9 shows that $(IS_\alpha \uparrow S_n, [\beta]) > 0$ implies $(IS_\alpha \uparrow S_n, AS_{\beta'} \uparrow S_n) > 0$, so that by 2.6 we obtain $\alpha \trianglelefteq \beta$ as it is stated.

In order to complete the proof of the fact that 2.10 is a complete system, it remains to show that if $[\alpha]$ and $[\beta]$ are equivalent, then $\alpha = \beta$. But this is easy to verify, for in the case when $[\alpha]$ is equivalent to $[\beta]$ we have

$$1 = (IS_\alpha \uparrow S_n, [\alpha]) = (IS_\alpha \uparrow S_n, [\beta]) = (IS_\beta \uparrow S_n, [\beta]) = (IS_\beta \uparrow S_n, [\alpha]),$$

so that again by the Ruch/Schönhofer theorem both

$$\alpha \trianglelefteq \beta \text{ and } \beta \trianglelefteq \alpha ,$$

and hence $\alpha = \beta$.

We now introduce the natural <u>lexicographic order</u> "$\leq$" on the set P(n) of all the partitions α of n by putting

$$\alpha < \beta :\Leftrightarrow \exists i \, (\alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i).$$

This is obviously a total order, furthermore it is clear that

2.12 $$\forall \alpha, \beta \vdash n \, (\alpha \trianglelefteq \beta \Rightarrow \alpha \leq \beta).$$

We put the multiplicities

$$m_{\alpha\beta} := (IS_\alpha \uparrow S_n, [\beta])$$

together into a matrix M_n, the rows and columns of which belong to the partitions of n with respect to the lexicographic order:

$$M_n := \begin{array}{c} [1^n] \; \ldots \; [\beta] \; \ldots \; [n] \\ \left(\begin{array}{ccc} & \vdots & \\ \ldots & m_{\alpha\beta} & \ldots \\ & \vdots & \end{array} \right) \end{array} \begin{array}{l} IS_{(1^n)} \uparrow S_n \\ \vdots \\ IS_\alpha \uparrow S_n \\ \vdots \\ IS_{(n)} \uparrow S_n \end{array}$$

The Ruch/Schönhofer theorem tells us that M_n is an upper triangular matrix with 1's along the main diagonal (cf. 2.8/9/11):

2.13
$$M_n = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix},$$

and it says that $m_{\alpha\beta} \neq 0$ implies $\alpha \trianglelefteq \beta$. It is this fact which we have in mind saying that the diagram lattice is the underlying combinatorial structure of a great part of the representation theory of the symmetric group. (Later than in ref. 1 this was also noticed by ref. 8,9 and 10.)

The classical approaches use a weaker argument, they only use the triangularity of M_n together with the fact that there are 1's along the main diagonal. They prove this partial result along an examination of idempotents.

This approach leads us to Young-tableaus and we would like to show that a closer examination yields a characterization of $\alpha \trianglelefteq \beta$ in terms of Young-tableaus. We give a short description of this since this characterization of the partial order turns out to be useful for a better understanding of various applications in sciences (cf. ref. 1).

A Young-tableau t^α with diagram $[\alpha]$ arises from $[\alpha]$ by replacing the nodes "×" of the diagram by the elements $i \in \underline{n} = \{1,\dots,n\}$. Replacing the nodes by these elements in their natural order, we obtain for example

$$t_1^\alpha := \begin{array}{llll} 1 & \dots\dots\dots\dots & & \alpha_1 \\ \alpha_1+1 & \alpha_1+2 & \dots & \alpha_1+\alpha_2 \\ \dots\dots\dots & & & \\ \dots\dots\ n & & & \end{array} .$$

The rows and columns of t_1^α establish partitions of the set $\underline{n}$. Let H_α^1 and V_α^1 denote the corresponding Young-subgroups, the horizontal and the vertical group of t_1^α.

The group algebra QS_n of S_n over the field Q of rational numbers contains the elements

$$\mathcal{H}_\alpha^1 := \sum_{\pi \in H_\alpha^1} \pi \quad , \text{ and } \quad \mathcal{V}_\alpha^1 := \sum_{\rho \in V_\alpha^1} \operatorname{sgn}\rho \cdot \rho \ .$$

These elements are essentially idempotent, the generated left ideals afford the representations $IS_\alpha \uparrow S_n$ and $AS_{\alpha'} \uparrow S_n$.

The classical argument showing that M_n is triangular now runs as follows. It is shown that for $\alpha > \beta$ we have

2.14 $$\mathcal{H}^1_\alpha QS_n \mathcal{V}^1_\beta = 0,$$

which implies by general representation theory

$$(IH^1_\alpha \uparrow S_n, AV^1_\beta \uparrow S_n) = (IS_\alpha \uparrow S_n, AS_{\beta'} \uparrow S_n) = 0.$$

In order to prove 2.14 we need only to show that for each $\pi \in S_n$ we have

2.15 $$\mathcal{H}^1_\alpha \pi \mathcal{V}^1_\beta \pi^{-1} = 0.$$

$\pi \mathcal{V}^1_\beta \pi^{-1}$ is the vertical group of the tableau πt^β_1, defined by

$$\text{if } t^\beta_1 = \begin{matrix} \cdots\cdots\cdots \\ \ldots i \ldots \\ \cdots\cdots \end{matrix} \quad , \text{ then } \pi t^\beta_1 := \begin{matrix} \cdots\cdots\cdots\cdots \\ \ldots \pi(i) \ldots \\ \cdots\cdots\cdots \end{matrix}$$

If now $\alpha > \beta$, then there exist two elements of $\underline{n}$, say i and j, which occur in t^α_1 in the same row and in t^β_1 in the same column, so that in particular $2\mathcal{H}^1_\alpha = \mathcal{H}^1_\alpha(1 + (ij))$, and $2\pi \mathcal{V}^1_\beta \pi^{-1} = (1 - (ij))\pi \mathcal{V}^1_\beta \pi^{-1}$, hence

$$\mathcal{H}^1_\alpha \pi \mathcal{V}^1_\beta \pi^{-1} = \tfrac{1}{4}\mathcal{H}^1_\alpha(1 + (ij))(1 - (ij))\pi \mathcal{V}^1_\beta \pi^{-1} = 0,$$

and we are done.

We notice that the main step was an application of the fact that if $\alpha > \beta$, then for any two t^α and t^β there exist two elements $i,j \in \underline{n}$, which occur in t^α in the same row and in t^β in the same column. This leads us to the following definition. If t^α is a tableau with diagram $[\alpha]$, β another partition of n, then we denote by

$$D(t^\alpha, \beta)$$

the set of all the Young-tableaus t^β, where any two i and j of $\underline{n}$, which occur in t^α in the same row, occur in t^β in different columns. Correspondingly we denote by

$$D(\alpha, t^\beta)$$

the set of all the Young-tableaus t^α, where any two i and j of $\underline{n}$, which occur in t^β in the same column, occur in t^α in different rows.

It is trivial that $D(t^\alpha,\beta) \neq \emptyset$ is equivalent to $D(\alpha,t^\beta) \neq \emptyset$. Less trivial is, that this is also equivalent to $\alpha \trianglelefteq \beta$. It is easy to show that $D(t^\alpha,\beta) \neq \emptyset$ implies $\alpha \trianglelefteq \beta$, the other direction is shown by describing an algorithm which yields a $t^\beta \in D(t^\alpha,\beta)$.

This allows us to sum up as follows:

<u>2.10 Theorem</u>: If $\alpha = (\alpha_1,\ldots,\alpha_h)$ and $\beta = (\beta_1,\ldots,\beta_k)$ are partitions of n, S_α and S_β corresponding Young-subgroups and t^α and t^β Young-tableaus with diagrams $[\alpha]$ and $[\beta]$, respectively, then the following properties are equivalent:

(i) $\alpha \trianglelefteq \beta'$,

(ii) $\beta \trianglelefteq \alpha'$,

(iii) There exist double-cosets $S_\alpha \pi S_\beta$ which satisfy

$$S_\alpha \cap \pi S_\beta \pi^{-1} = \{1\} .$$

(iv) There exist 0-1-matrices with row sums $\alpha_1, \ldots, \alpha_h$ and column sums $\beta_1, \ldots, \beta_k$.

(v) $D(t^\alpha, \beta') \neq \emptyset$,

(vi) $D(\alpha, t^{\beta'}) \neq \emptyset$.

The equivalence (i) $\Leftrightarrow$ (ii) is more or less trivial by 1.1, the equivalence (iii) $\Leftrightarrow$ (iv) was shown above, trivial is also (v) $\Leftrightarrow$ (vi) as well as (iv) $\Rightarrow$ (i) and (v) $\Rightarrow$ (i).

In order to prove the rest, one may use Ryser's algorithmic proof of (i) $\Rightarrow$ (iv) and conclude showing (i) $\Rightarrow$ (v) by describing an algorithm which allows to construct from t^α a $t^\beta \in D(t^\alpha, \beta')$.

3. A generalization

We saw that Young-subgroups S_γ , γ a partition of n, play an important role in the representation theory of the symmetric group S_n, and that some of the properties of the induced representations $IS_\gamma \uparrow S_n$ reflect the structure of the diagram lattice $(P(n),\trianglelefteq)$. These properties show up again in the applications, say in the theory of chirality of molecules (cf. ref. 1, 11), where symmetric groups and certain generalizations are considered.

These generalizations which we have in mind are the so-called hyperoctahedral groups $S_2 \sim S_n$, a special case of the wreath product $G\sim H$ of a group G with a subgroup H of S_n. This group $S_2\sim S_n$ shows up in the theory of chirality when a skeleton with n numbered sites is considered to which ligands may be attached, this group then is the group of all combinations of ligand permutations with site reflections (ref. 11, p. 19).

The wreath product $G\sim H$ consists of all the ordered pairs

$$(f;\pi),$$

where f is a mapping from $\underline{n}$ into G, and $\pi \in H$. For two such mappings $f,g:\underline{n} \to G$ and a $\pi \in H$ we define mappings fg, f^{-1}, g_π and e from $\underline{n}$ into G by putting for each $i \in \underline{n}$:

$$fg(i) := f(i)g(i),\ f^{-1}(i) := f(i)^{-1},\ g_\pi(i) := g(\pi^{-1}(i)),\ e(i):=1_G.$$

Then $G\sim H$ forms a group subject to the multiplication

$$(f;\pi)(g;\rho) := (fg_\pi;\pi\rho).$$

This group has the following normal subgroup:

$$G^* := \{(f;1_{S_n}) \mid f:\underline{n} \to G\}\ ,$$

which is isomorphic to the n-fold direct product $\otimes^{\underline{n}} G$ of G with itself. We call G^* the base group of $G\sim H$.

G^* has a complement isomorphic to H:

$$H' := \{(e;\pi) \mid \pi \in H\}\ .$$

Since G^* is a normal subgroup of $G\sim H$, we can very nicely apply Clifford's theory of representations of groups with normal subgroups in order to develop the representation theory of $G\sim H$ once the representation theory of G is known to a certain extent. We need a short description of how this works (for more details cf. ref. 12, 13).

Let F denote a representation of G over the complex field C with representation space V. If n denotes a natural number, then we obtain an ordinary representation of $G\sim S_n$ with representation space the tensor power

$$\overset{n}{\otimes} V := V \otimes_C V \otimes_C \cdots \otimes_C V,$$

n factors, by simply putting

$$(f;\pi)(v_1 \otimes \ldots \otimes v_n) := f(1)v_{\pi^{-1}(1)} \otimes \ldots \otimes f(n)v_{\pi^{-1}(n)},$$

for each $v_i \in V$.

We denote this representation by

$$\overset{\widetilde{n}}{\#} F,$$

since it extends the n-fold outer tensor power $\overset{n}{\#} F$ of F with itself, which is a representation of the base group G^* of $G\sim S_n$.

If furthermore D is a representation of H, then D', defined by

$$D'((f;\pi)) := D(\pi),$$

yields also a representation of $G\sim H$, a third one is therefore the inner tensor product

$$\overset{\widetilde{n}}{\#} F \otimes D'.$$

It follows from considerations in ref. 12 that if G is a finite group, each ordinary irreducible representation of $G\sim H$ is of the form

<u>3.1</u> $$(R \otimes S') \uparrow G\sim H,$$

where R is the restriction of an outer tensor product of the form

$$R = \overset{h}{\underset{i=1}{\#}} \left(\overset{\widetilde{\alpha_i}}{\#} F_i \right),$$

F_i suitable ordinary irreducible representations of G, which is a representation of $G\sim S_\alpha \leq G\sim S_n$, $\alpha \vdash n$, the restriction is to the subgroup

$$G\sim(H \cap S_\alpha).$$

S is an ordinary irreducible representation of the intersection $H \cap S_\alpha$ of H with the Young-subgroup S_α.

Hence Young-subgroups occur in the representation theory of wreath products $G\sim S_n$ as inertia factors, i.e. as little groups of the first kind in the terminology of ref. 14.

A special case is the wreath product $S_2\sim S_n$, the so-called <u>hyperoctahedral group</u> which was mentioned above. In this case the only irreducible representations of $G=S_2$ are $[2]$ and $[1^2]$, so that each ordinary irreducible representation of $S_2\sim S_n$ is of the form

3.2 $$\left[(\overset{\widetilde{r}}{\#}[2] \overset{\widetilde{s}}{\#}[1^2]) \otimes ([\alpha] \# [\beta])'\right] \uparrow S_2 \sim S_n$$

$$= \left[(\overset{\widetilde{r}}{\#}[2] \otimes [\alpha]') \# (\overset{\widetilde{s}}{\#}[1^2] \otimes [\beta]')\right] \uparrow S_2 \sim S_n,$$

where $r+s=n$ and $\alpha \vdash$, $\beta \vdash s$.

More generally for $S_m \sim S_n$: Let $\alpha^1, \ldots, \alpha^k$ be partitions of m and assume $\alpha^i \leq \alpha^j$, for all $i \leq j$, then a complete system of ordinary irreducible representations of $S_m \sim S_n$ is given by the set of representations of form

3.3 $$\left[(\overset{\widetilde{n_1}}{\#} [\alpha^1] \otimes [\beta^1]') \# \ldots \# (\overset{\widetilde{n_k}}{\#} [\alpha^k] \otimes [\beta^k]')\right] \uparrow S_m \sim S_n ,$$

where $n_i = n$ and $\beta^i \vdash n_i$.

The question arises which of the properties of the Young-subgroups S_γ and their representations which are relevant for the representation theory of S_n the corresponding subgroups $G \sim S_\gamma$ of $G \sim S_n$ and their representations do also possess.

The most important properties of the S_γ and the representations $IS_\gamma \uparrow S_n$ were described by the matrix $M_n = (m_{\alpha\beta})$. If

$$\xi^\alpha_\beta$$

denotes the value of the character of $IS_\alpha \uparrow S_n$ on the class of elements with cycle-partition $\beta \vdash n$, and if

$$\zeta^\alpha_\beta$$

denotes the vlaue of the character of $[\alpha]$ on the same class, then by definition of M_n we have for the matrices

$$X_n := (\zeta^\alpha_\beta) \quad \text{and} \quad \Xi_n := (\xi^\alpha_\beta)$$

(X_n is the character table of S_n!) the following equation:

3.4 $$X_n = M_n^{-1} \cdot \Xi_n .$$

Since M_n has determinant 1, M_n^{-1} is a matrix over the ring Z of the rational integers. Hence by 3.4 each character ζ^α of S_n is a Z-linear combination of the characters ξ^γ which are the characters of transitive permutation representations.

Thus the character ring of S_n possesses a Z-basis consisting of transitive permutation characters.

This is known since the time of Frobenius. That the same is true for hyperoctahedral groups was shown in ref. 15, ref. 16. The corresponding results for the most general case $G \sim H$ (ref. 17) read as follows:

3.5 Theorem: Let G denote a finite group and H a subgroup of S_n.

(i) If the characters of G and the characters of all the intersections $H \cap S_\gamma$ of H with Young-subgroups S_γ of S_n are Z-linear combinations of permutation characters, then the same holds for the characters of $G \sim H$.

(ii) If the character ring of G has a Z-basis of transitive permutation characters, then the same holds for the character rings of the groups $G \sim S_n$.

The proof is quite long so that we intend to sketch only those aspects of it, which are of general interest.

The basis for all character theoretical considerations concerning wreath products is a result on the character of the representation $\tilde{\#}^{n} F$ which will be described next.

If $\pi \in S_n$ has the following decomposition into pairwise disjoint cyclic factors:

$$\pi = \prod_{\nu=1}^{c(\pi)} (j_\nu \; \pi(j_\nu) \; \ldots \; \pi^{l_\nu -1}(j_\nu)),$$

then we associate with each cyclic factor $(j_\nu \; \ldots \; \pi^{l_\nu -1}(j_\nu))$ and the mapping $f:\underline{n} \to G$ the cycle-product $g_\nu(f;\pi)$ defined by

3.6
$$g_\nu(f;\pi) := f(j_\nu)f(\pi^{-1}(j_\nu))\ldots f(\pi^{-l_\nu+1}(j_\nu))$$
$$= f\, f_\pi \ldots f_{\pi^{l_\nu -1}}(j_\nu).$$

A direct calculation shows that $\tilde{\#}^{n} F$ has the following character:

3.7
$$\chi^{\tilde{\#}^{n} F}((f;\pi)) = \prod_{\nu=1}^{c(\pi)} \chi^F(g_\nu(f;\pi)) \; .$$

Since by 3.1 each ordinary irreducible representation of $G \sim H$ is induced from a product of such representations, it is more or less a matter of the multinomial theorem that 3.5 (i) is true. The proof of (ii) is more complicated.

In a similar way other properties carry over from the representations of G to the representations of $G \sim H$. We need only to wath whether these properties are invariant under formation of tensor powers, extension to the inertia group and induction as well as that they hold for the representations of the intersections $H \cap S_\gamma$ also. Such properties are for example the reality of characters, reality of representations, monomiality of representations etc. (cf. ref. 18).

Character tables of wreath products $S_m \sim S_n$, mn $\leq$ 15, can be found in ref. 19,20).

It is obvious that the representation theory of the wreath products $S_m \sim S_n$ already is considerably more complicated than that of S_n. While an ordinary irreducible representation of S_n is characterized by a single partition, an ordinary irreducible representation of $S_m \sim S_n$ needs for its characterization a $|P(n)|$-tuple of pertitions. It should be clear from this already that a lot of work still needs to be done until the representation theory of $S_m \sim S_n$ and more generally that of $G \sim H$ is as lucid as is nowadays the (ordinary) representation theory of S_n.

We would like therefore to conclude this section with a problem which is raised by applications of the representation theory of such wreath products.

It was mentioned above that in the theory of chirality the hyperoctahedral groups $S_2 \sim S_n$ play an important role. A question which is asked in this context (cf. ref. 11) is concerned with representations induced from the following subgroup of $S_2 \sim S_n$:

3.8 $$\mathrm{diag}S_2^{*} \cdot S_n' = \{(f;\pi) \mid f:\underline{n} \to S_2 \text{ constant}, \ \pi \in S_n\}.$$

The question asked is: What are the irreducible representations of this subgroup 3.8 of $S_2 \sim S_n$ and how can we obtain the decomposition of the representation induced into $S_2 \sim S_n$ by such an irreducible representation of 3.8?

The first part is easy to answer. Since 3.8 is isomorphic to the direct product:

$$\mathrm{diag}S_2^{*} \cdot S_n' \simeq S_2 \times S_n,$$

we obtain that each ordinary irreducible representation of this subgroup is either of the form

3.9 $$[\gamma]^{+},$$

where $\gamma \vdash n$, and for each $(f;\pi) \in \mathrm{diag}S_2 \cdot S_n'$ we have

3.10 $$[\gamma]^{+}((f;\pi)) := [\gamma](\pi),$$

or the representation is of the form

3.11 $$[\gamma]^{-},$$

where again $\gamma \vdash n$,but for $(f;\pi) \in \mathrm{diag}S_2^{*} \cdot S_n'$ we have now

3.12 $$[\gamma]^{-}((f;\pi)) := \mathrm{sgn} \prod_{i=1}^{n} f(i) \cdot [\gamma](\pi).$$

We would like to determine the decompositions of the induced representations

$$[\gamma]^{+} \uparrow S_2 \sim S_n \quad \text{and} \quad [\gamma]^{-} \uparrow S_2 \sim S_n.$$

Denoting the irreducible representation 3.2 of $S_2 \sim S_n$ by

$$[\alpha;\beta],$$

we have to determine those α and β which satisfy

$$([\gamma]^{\pm} \uparrow S_2 \sim S_n, [\alpha;\beta]) = ([\alpha;\beta] \downarrow \mathrm{diag}S_2^{*} \cdot S_n', [\gamma]^{\pm}) \neq 0.$$

If we denote by $[\alpha][\beta]$ the representation induced from $[\alpha] \# [\beta]$ into S_n, then it is not difficult to see that for $(f;\pi) \in \mathrm{diag}S_2^{*} \cdot S_n'$ we have

$$[\alpha;\beta]((f;\pi)) = \mathrm{sgn} \prod_{r+1}^{n} f(i) \cdot [\alpha][\beta](\pi) .$$

Hence if s is odd (even), then this restriction contains $[\gamma]^{-}$ (contains $[\gamma]^{+}$) if and only if

3.13 $$([\alpha][\beta], [\gamma]) \neq 0.$$

This together with the so-called Littlewood-Richardson-rule (cf. ref. 12,4.51) yields the desired result which is important for the application to chirality (cf. ref. 21,§2B). It furthermore shows that this can be generalized easily to groups $G \sim S_n$, G being abelian.

A numerical example is

$$[3,1^2]^{-} \uparrow S_2 \sim S_5 = [3,1;1] + [2,1^2;1] + [1^2;3] + [1^2;2,1] + [2;2,1] + [2;1^3] + [3,1^2],$$

while

$$[3,1^2]^{+} \uparrow S_2 \sim S_5 = [1;3,1] + [1;2,1^2] + [3;1^2] + [2,1;1^2] + [2,1;2] + [1^3;2] + [3,1^2].$$

References

1. E. Ruch/A. Schönhofer: Theorie der Chiralitätsfunktionen. Theor. Chim. Acta 19 (1970), 225-287.
2. T. Brylawski: The lattice of integer partitions. Discrete Math. 6 (1973), 201-209.
3. C. Berge:Graphs and Hypergraphs. North-Holland Publis-hing Company 1973.
4. C.W. Curtis/I. Reiner: Representation theory of finite groups and associative algebras. Interscience Publishers, New York 1962.
5. A.J. Coleman: Induced representations with applications to S_n and GL(n). Lecture Notes prepared by C.J. Bradley. Queen's Papers in Pure and Applied Mathematics, no. 4. Queen's University, Kingston, Ontario, 1966.
6. H.J. Ryser: Combinatorial Mathematics. Wiley, New York 1963.
7. H.-R. Halder/W. Heise: Einführung in die Kombinatorik. Hanser, München, 1976.
8. E. Snapper:Group characters and nonnegative integral matrices. J. Algebra 19 (1971), 520-535.
9. R.A. Liebler/M.R. Vitale: Ordering the partition characters of the symmetric group. J. Algebra 25 (1973), 487-489.
10. S. J. Mayer: On the irreducible characters of the symmetric group. Advances in Math. 15 (1975), 127-132.
11. A. Mead: Symmetry and Chirality. Topics in Current Chemistry, vol. 49. Springer, Berlin, 1974.
12. A. Kerber: Representations of Permutation Groups I. Lecture Notes in Math., vol. 240, Springer, Berlin, 1971.
13. A. Kerber: Representations of Permutation Groups II. Lecture Notes in Math., vol. 495, Springer, Berlin, 1975.
14. J.S. Lomont: Applications of finite groups. Academic Press, New York 1959.
15. D. Kinch/L. Geissinger: Representations of the hyperoctahedral groups.(in preparation)
16. S.J. Mayer: On the characters of Weyl groups of type C. J. Algebra 33 (1975), 59-67.
17. A. Kerber/J. Tappe: On permutation characters of wreath products. Discrete Math. 15 (1976), 151-161.
18. A. Kerber: Zur Theorie der M-Gruppen. Math. Z. 115 (1970), 4-6.
19. F. Sänger: Einige Charakterentafeln von Symmetrien symmetrischer Gruppen. Mitt. math. Sem. Univ. Giessen 98 (1973), 21-38.
20. B. Gretschel/A. Hilge: Berechnung der Charakterentafeln von Symmetrien symmetrischer Gruppen. Diplomarbeiten, Giessen 1973.
21. A. Mead/E. Ruch/ A. Schönhofer: Theory of Chirality Functions, Generalized for Molecules with chiral Ligands. Theor. Chim. Acta 29 (1973), 269-304.