Group Contractions and Infrared Effect in Theories with Spontaneous Breakdown of Symmetry

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In a spontaneously broken symmetry theory, one central problem is the relation between the invariance of the theory at the dynamical level (i.e. the level of the equations of the basic Heisenberg fields), and the symmetry properties at the phenomenological level. Indeed, a very interesting feature of such theories is the dynamical rearrangement fo the basic symmetry into a different symmetry at observational level, which manifests itself in the observable ordered states of the physical system. Well known examples are the cristal and the ferromagnet where the translational and spin-rotational invariance of the basic dynamics are replaced by the lattice and the spin-polarized symmetry. This two-level description, namely the dynamical level description in terms of basic Heisenberg fields $\psi(x)$ and the phenomenological level description in terms of physical free fields $\phi(x)$ (quasiparticle fields, in many-body terminology) is a crucial feature of Quantum Field Theory. To face the problem of the mapping between these two levels, we write

$$<a|\psi(x)|b> = <a|F(\phi(x))|b> , \qquad (1)$$

where $F(\phi)$ is a functional determined by the dynamics, $|a>$ and $|b>$ are states of the Fock space for physical particles, i.e. wave-packet states. To find a solution of the dynamics in terms of free in- (out-) fields $\phi(x)$ means to carry out a linearization procedure by which one passes from the non-linear Heisenberg equations to the linear free fields equations. In other words, a solution of the theory is found when the mapping (1) is determined. Due to the non-linearity of the dynamics, the mapping (1) is expected to be non-linear. As a consequence, one expects to observe different symmetry structures at the dynamical and the phenomenological level of the same original invariance of the theory [1]. Many examples of dynamical rearrangement of symmetry have been studied, see ref. [2] and refs. quoted therein. Consider, e.g., a ferromagnet [3]. The Lagrangian, which is made of the electron (Heisenberg) doublet field $\psi(x)$, is invariant under the SU(2)-spin-rotation group

$$\psi(x) \to e^{i\theta_i \lambda_i} \psi(x) , \qquad i = 1,2,3, \qquad (2)$$

with θ_i real parameters and $\lambda_i = \sigma_i / 2$, σ_i the Pauli matrices. By using functional integration methods, one can show that the transformations (2) are induced by the following transformations of the in-fields (quasiparticles) of the theory, namely the quasielectron doublet $\phi(x)$ and the magnon $B(x)$ (Goldstone boson):

$$B(x) \to B(x) + i\theta_1(M/2)^{\frac{1}{2}} f(x), \quad \phi(x) \to \phi(x), \quad \text{for } \theta_2 = \theta_3 = 0; \qquad (3)$$

$$B(x) \to B(x) - \theta_2(M/2)^{\frac{1}{2}} f(x), \quad \phi(x) \to \phi(x), \quad \text{for } \theta_1 = \theta_3 = 0; \qquad (4)$$

$$B(x) \to e^{-i\theta_3} B(x), \quad \phi(x) \to e^{i\theta_3\lambda_3} \phi(x), \quad \text{for } \theta_1 = \theta_2 = 0; \qquad (5)$$

for $f(x) \to 1$, with $f(x)$ a square-integrable function which is a solution of the mangon equation; M is the magnetization: $M = \langle S_3 \rangle$. The generators of the free fields transformations (3) - (5) satisfy the commutation relations

$$[S_1, S_2] = \text{const. } I, \quad [S_3, S_1] = i\, S_2; \quad [S_3, S_2] = -i\, S_1; \qquad (6)$$

The original SU(2) symmetry algebra is thus rearranged into the E(2) symmetry algebra. The need for $f(x)$ in (3) and (4) is due to the well known fact that the boson transformation $B \to B$ + const. cannot be induced by any unitary operator; thus it must be regarded as the limit for $f \to 1$ of (3) and (4). Due to the presence of $f(x)$, the space integrations are insensitive to locally infinitesimal terms of order of $1/V$ with the volume $V \to \infty$. In other words, since the limit $f(x) \to 1$ acts as an infrared cut-off for the magnon fields, infrared effects from the magnons do not contribute to S_i. On the other hand, they do contribute when integrated on the whole system thus recovering the original SU(2) algebra. This infrared effect is then the origin of the difference between dynamical and phenomenological symmetry structures (note that since quasiparticles are related to observable energy levels, the E(2) symmetry is related with observable results). In conclusion, we are in presence of a group contraction mechanism [4]. (We are not considering here gauge theories. The rearrangement in Abelian gauge theories is studied in ref. 5; the present results apply to this case too. We conjecture the same is true also for non-Abelian gauge theories.) Let us consider the case of SO(n). In this case we have a basis consisting of $\frac{1}{2}n(n-1)$ elements a_{ij}, $i<j$; in the canonical matricial representation of the SO(n) algebra the elements a_{ij} correspond to the matrices having all the entries equal to zero except for the entries ij and ji which are 1 and -1 respectively. The commutation relations are

$$\begin{aligned} &[a_{ij}, a_{mn}] = 0, \quad i,j \neq m\,n; \quad [a_{ij}, a_{jm}] = a_{im} \\ &[a_{ij}, a_{im}] = -a_{im}, \quad j < m; \quad [a_{ij}, a_{mj}] = -a_{im}, \quad i < m. \end{aligned} \qquad (7)$$

Let us consider the Euclidean algebra E(n) having a basis consisting of $\frac{1}{2}(n+1)\,n$ elements $\bar{a}_{ij}$, $i < j$, satisfying the following commutation relations

$$[\bar{a}_{ij}, \bar{a}_{mn}] = 0, \; i,j \neq m,n; \; [\bar{a}_{ij}, \bar{a}_{jm}] = \bar{a}_{im}; \; [\bar{a}_{ij}, \bar{a}_{mi}] = -\bar{a}_{im}, \; i<m;$$

$$[\bar{a}_{ij},\bar{a}_{im}] = -\bar{a}_{jm}, \quad j < m,\ i \neq 1; \quad [\bar{a}_{1j},\bar{a}_{1m}] = 0 \ .$$

We have the following

Theorem: The algebra E(n-1) is a contraction of the algebra SO(n) .

Proof: In SO(n) let us consider for each real number V the following basis

$$a^V_{ij} = a_{ij}, \quad i > 1; \qquad a^V_{1j} = \frac{1}{\sqrt{V}}\ a_{1j}. \tag{9}$$

The $V \to \infty$ limit of the commutators among the a^V_{ij}'s then leads to the desired contraction of SO(n) to E(n-1) . In connection with the previous example of the ferromagnet, note that the cases of SO(3) and SU(2) coincide by the isomorphism of these two algebras. To express the generators of symmetry transformations in terms of free fields is thus exactly equivalent to the limiting procedure $1/V \to 0$ by which contractions are obtained in the previous theorem. The mapping (1) among Heisenberg and asymptotic fields is indeed performed by considering matrix elements between wave packet states, which are insensitive to locally infinitesimal effects of order of magnitude 1/V with the volume $V \to \infty$. Same results can be obtained by using projective geometry arguments [2]. Suppose the solutions of the free field equation, say in a $n \times n$ matricial form, form a n-dimensional linear space V^n . Let $\tilde{P}^{n-1}$ be the (n-1)-dimensional projective space of V^n , i.e. the space whose elements are the lines for the origin of V^n . In the linearization procedure by which one goes from the dynamical equation to the free field equation, one passes from a non-linear to a linear realization of the basic invariance group, say G . This linear realization of G , say G^{in} , maps $\tilde{P}^{n-1}$ into $\tilde{P}^{n-1}$. Due to the spontaneous breaking condition, the allowed transformations in G^{in} are those which leave the hyperplane at infinity in $\tilde{P}^{n-1}$ invariant. Inhomogeneous (boson) transformations are thus allowed and G^{in} turns out to be a contraction of G . Finally let us observe that invariance of S-matrix under observable symmetry transformations leads to many low-energy theorems, as Dyson's low-energy theorem for magnons. Note also that boson transformations can induce macroscopic phenomena controlled by classical equations. These classical phenomena are created by condensation of Goldstone bosons: when a large number of bosons is condensed, observable symmetry patterns appear in ordered states, the quantum fluctuations become very small and the system behaves as a classical one. We thus see the central rolè played by the contraction operation in the passage to macroscopic physics: The basic symemtry group is dynamically rearranged to a contraction at observational level; in this way, Abelian (boson) transformations are introduced, which regulàte classical macroscopic phenomena through boson condensation.

References

[1] H. Umezawa, Nuovo Cimento 40 (1965) 450; R. N. Sen and H. Umezawa, Nuovo Cimento 50 (1967) 53

[2] C. De Concini and G. Vitiello, Nucl. Phys. B116 (1976) 141; Relation between projective geometry and group contraction in spontaneously broken symmetry theories, preprint 1977

[3] M. N. Shah, H. Umezawa and G. Vitiello, Phys. Rev. B10 (1974) 4724

[4] I. E. Segal, Duke Math. J. 18 (1951) 221; E. Inönü and E. P. Wigner, Proc. Nat. Acad. Sci. US 39 (1953) 510; E. Weimar, Nuovo Cimento 15B (1973) 245

[5] H. Matsumoto, N. J. Papastamatiou, H. Umezawa and G. Vitiello, Nucl. Phys. B97 (1975) 61