Instantons in Lattice Models with Discrete Symmetries

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We use the relation between the Ising model in R^d and the euclidean quantum field theory given by the lagrangian

$$L(\varphi) = -\frac{1}{2}\sum_{i=1}^{d}\left(\frac{\partial\varphi}{\partial x_i}\right)^2 - \frac{\lambda}{4}\left(\varphi^2 - \frac{\mu^2}{\lambda}\right)^2, \quad \mu,\lambda > 0. \tag{1}$$

The Ising model is the limit of (1) when the coupling constants tend to infinity $\mu \to \infty$, $\mu^2/\lambda = 1$. The effect of the limit is the restriction of φ to two discrete values denoted by $\sigma = \pm 1$ in each point of the space. The field equations of (1) reduce to the algebraic equation

$$\sigma - \sigma^3 = 0, \tag{2}$$

and the Hamiltonian is given by

$$H = -J \sum_{nn} \sigma_i \sigma_j, \quad J > 0. \tag{3}$$

In the theory given by (1) when one uses functional integration techniques to quantize the theory one introduces fields φ-called instantons [1] - which minimize the classical action

$$A[\varphi] = \int d^d x\, L(\varphi) \tag{4}$$

This fields are first of all solutions of the field equations, but there may be fields contributing which are not solutions of the field equations and which have to be constructed.
It has been proven by Coleman [2] that within the class of euclidean field theories with polynomial interaction and at least one minimum in the potential the solutions of the field equations with minimal action are spherical symmetric $\varphi = \varphi(|\vec{x}|)$ in $d > 2$ if they exist at all.
The connection between the solutions of (1) and (3) is best seen in R^1 :

$$\frac{d^2}{dx^2}\varphi(x) = -\mu^2\varphi + \lambda\varphi^3 \tag{5}$$

has the solution of finite action

$$\varphi(x) = \frac{\mu^2}{\lambda} \text{th} \left\{ \frac{\mu}{\sqrt{2}} (x-x_o) \right\}, \forall x_o \in R^1, \tag{6}$$

$$\lim_{\substack{\mu\to\infty \\ \mu^2/\lambda=1}} \varphi(x) = \sigma(x) = \text{sgn}\,(x-x_o) \xrightarrow[\text{lattice}]{} \sigma_i = \text{sgn}\,(i-i_o) \quad i,i_o \in Z \tag{7}$$

The sign of σ indicates the orientation of the spin. The higher states where σ changes sign n times are

$$\sigma(x) = \prod_{k=1}^{n} \text{sgn}(x-x_k) \xrightarrow{\text{lattice}} \sigma_i = \prod_{k=1}^{n} \text{sgn}(i-i_k)\ . \tag{8}$$

Solution (8) is no longer a limit of a solution of Eq. (5),whereas it plays an important role in the dynamics of the one dimensional Ising model. Eq. (8) is an equilibrium state of the system, being a solution of the ultralocal field equation (2). All solution (8) with n fixed, n is the number of kinks, have the same action and may be classified into instanton sectors.

In R^2 we don't know the analytic solution of (1), only that it may be spherical symmetric $\varphi = \varphi(|\vec{x}|)$ by extention of Colemans theorem. In the Ising model the solutions are closed contours [3] which separate σ's with opposite sign. A spherical symmetric solution is

$$\sigma(x) = \text{sgn}\,(|\vec{x}-\vec{x}_o|-r),\ \vec{x},\vec{x}_o \in R^2. \tag{9}$$

The main quantity attached to a contour of length l is the number $\upsilon(l)$ of all possible contours of this length within a domain V. In the old Peierls argument an upper limit of $\upsilon(l)$ was given by taking into account all shapes [4]:

$$\upsilon(l) \leq V\ 3^l \tag{10}$$

Now we consider a spherical area V_R of radius R, and a spherical contour of radius r with the notation $\upsilon(2\pi r)\to\upsilon(r)$.
The partition function and the magnetization density are given by:

$$Z = \sum_{1 \leq r \leq R} \upsilon(r) \exp\left(\frac{J}{kT} 2\pi R - \frac{2J}{kT} 2\pi r\right), \tag{11}$$

and

$$M(T) = 1 - \frac{2}{R^2} \frac{1}{Z} \sum_{1 \leq r \leq R} r^2 \upsilon(r) \exp\left(\frac{J}{kT} 2\pi R - \frac{2J}{kT} 2\pi r\right). \tag{12}$$

The condition for the existence of a phase transition at a certain critical temperature T_c (zero magnetic field) [$M(T < T_c) \leq 1$ and $M(T >> T_c) = o$] is

$$\upsilon(r) \geq r \exp\left(\frac{2J}{kT} 2\pi r\right). \tag{13}$$

This condition is satisfied by the Peierls contours (10). In the case of a solution (9) we get a nonexponential growth of $\upsilon(r)$:

$$\upsilon(r) = \pi\,(R^2-r^2)\,\alpha^{2\pi r},\ 1 < \alpha < 2\ . \qquad (14)$$

A good approximation of α is $\sqrt{2}$. Therefore spherically symmetric solution cannot generate phase transitions in the 2-dimensional Ising model. We generalize (9) to a "multicenter spherical symmetric" solutions

$$\sigma(x) = \prod_{i=1}^{n} \mathrm{sgn}\,(|\vec{x}-\vec{x}_i| - \frac{r}{n}),\ n \leq [r], \qquad (15)$$

$x, x_i \in V_R$. The number of configurations with a total length $2\pi r$ is

$$\upsilon(r) = \sum_{n=1}^{[r]} \upsilon_n(r) = \pi\,R^2\, r(1 - \frac{r\ln r}{R^2})\,\alpha^{2\pi r}. \qquad (16)$$

In the limit $R \to \infty$ we get

$$\upsilon(r) = \pi R^2 r\,\alpha^{2\pi r} \qquad (17)$$

which may satisfy condition (13) for an appropriate temperature. We get the relation

$$\upsilon_{\mathrm{Peierls}} \geq \upsilon_{\mathrm{multicenter}} \geq r\,\exp(\frac{2J}{kT}\,2\pi r) \geq \upsilon_{\mathrm{monocenter}} \qquad (18)$$

The conclusion is that the configurations (15) are sufficient to generate phase transitions. The construction (15) out of (8) may be used in euclidean QFT to construct higher instantons, once the basic instantons are derived from the field equations.

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References:

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