Semigroups and Effective Structures of Automata

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Extended Abtract

In physical realizations of complex systems the internal feedback flow of information (e.g. carried by electric signals) often gives rise to serious synchronization and reliability problems. Clearly, the maintainability of a system increases with decreasing interdependence of system components. On the other hand, the effect of signal delays due to propagation times decreases. Therefore, it is convenient - if not mandatory - to design complex systems in such a way that their internal feedback of information is minimal. For a class of finite state machines (automata) this goal is reached as soon as an algorithm is given by which each machine of this class can be decomposed into a cascade |1| of smaller machines each of which no longer decomposable in a meaningful way; cf. Fig. 1.

<u>A general Decomposition Schema.</u> Let $A = (X, Q, \delta)$ be a (semi)-automaton |1| and $T(A)$ its transition monoid; $T(A)$ is a monoid acting on the state set Q of A. Let $q, q' \varepsilon Q$; a partition π of Q is a $T(A)$-partition iff $q \equiv q' \pmod{\pi}$ and $t \varepsilon T(A)$ then $t(q) \equiv t(q') \pmod{\pi}$. A coordinate system |2| for A is an ordered pair $(S_Y, \{\beta_B\}_{B \varepsilon \pi})$ where S_Y is an action by a semigroup S on a set Y and π is a $T(A)$-partition. For each $B \varepsilon \pi$, $\beta_B : B \to Y$ is an injection and for each $t \varepsilon T(A)$, $B \varepsilon \pi$ there is a $s_B^t \varepsilon S$ such that for all $q \varepsilon B : s_B^t(\beta_B(q)) = \beta_C(t(q))$ where $C \supseteq t(B)$. Note, that Y may be choosen to be the maximal block of π.

Suppose that S acts faithful on Y and all β_B are bijections. Then $T(A)$ is isomorphic to a subsemigroup of the wreath product of S by $T(A)^{\pi}$, where $T(A)^{\pi}$ is the induced action of $T(A)$ on π |2|. Such a coordinate system gives rise to a decomposition of A into a feedbackfree <u>cascade</u> of two automata A_1 and A_2 |3|; the transition monoids are $T(A)^{\pi}$ and S and the state sets π and Y, respectively. The decomposition can then be iterated with A_i.

<u>Effective Decompositions.</u> We require for a decomposition of A into a cascade $A_1 \circledast A_2$ to be effective that (i) $|Q_i||T(A_i)| < |Q||T(A)|$ and (ii) $T(A_i)$ is a factor of $T(A)$, $i=1,2$. That is with respect to the

state set or to the order of the semigroup both components of the cascade are strictly smaller than the original automaton. Note, that V is a factor |4| of a semigroup S iff there is a subsemigroup U of S and a congruence relation ρ on U such that V is isomorphic to $^U/_\rho$. A coordinate system $C = (S_Y, \{\beta_B\}_{B \in \pi})$ for A is effective iff β_B is a bijection, $S \neq T(A)$ a subsemigroup of T(A) faithful on Y, and π a nontrivial T(A)-partition. Clearly, an effective coordinate system for A gives rise to an effective decomposition D_C of A. (However, the inverse may not hold). As a measurement for the effectiveness of an iterated decomposition $D_A = A_1 \ominus A_2 \ominus \ldots \ominus A_n$ of A we take $E(D_A) = \sum_{i=1}^{n} |Q_i||T(A_i)|$. Thus, $\min_{D_A} \{E(D_A)\}$ measures the complexity of A; examples are given in |5|.

Decomposition of PR-automata. In the usual decomposition theories permutation-reset (PR) automata are not directly decomposed, but are first covered by group like and identity reset automata. The group like automata are then decomposed by the wellknown Jordan-Hölder theorem into group like automata with simple groups |1|. This prodedure is, in general, not effective. However, in |6| it is shown that for PR-automata with transitive permutations there exist effective decompositions leading to more economic realizations. Example: Let $Q^i = \{1,2,...,i\}$. Automaton $A = (X, Q^5, \delta)$ with $T(A) = S(5)$ (the symmetric group on Q^5) is usually covered by the group like automaton $A = (S(5), S(5), \cdot)$ with $\cdot$ the group multiplication, and this automaton is decomposed (D_1) into $B = (C_2, C_2, \cdot)$ and $C = (A(5), A(5), \cdot)$. Our direct decomposition (D_2) yields $A_1 \ominus A_2$, with state sets $\pi = Q^2$ and $Y = Q^5$ and transformation monoids C_2 and A(5), respectively. Thus, instead of 120 states we need only 10, $E(D_1) = 3604$ and $E(D_2) = 304$. For more details the reader is referred to |7|.

Some further results. (1) Let G be a finite group and H a subgroup of G; $n = |G| > |H| = m > 1$. The coordinate system $C = (H_H, \{\beta_B\}_{B \in \{G:H\}})$ for the group like automaton $\hat{A} = (G,G,\cdot)$ is effective where $\beta_{Hg} : hg \to h$ ($h \in H$, $g \in G$) and the action of H is the right multiplication; $E(D_C) = m^2 + \frac{n^2}{mr}$, where $r = |H_G|$ and $H_G = \bigcap_{g \in G} g^{-1} Hg$. Hence, $(\frac{n}{m})^2 \leq E(D_C) - m^2 \leq (\frac{n}{m})^2 (\frac{n-m}{m})!$ Quite generally, suppose that $T(A) = G$ is a group acting transitively on Q. Let G_q be the isotropy group at an arbitrary state q of A. The subgroup lattice $L_q(A) = \{H | G_q \leq H \leq G\}$ is lattice isomorphic to the lattice of all G-partitions π. Hence, $L_q(A)$ is very useful in looking for effective decompositions and coordinate systems of

A. For example, if $G_q < H < G$ then the coordinate system $(S_Y, \{\beta_B\}_{B \in \{G:H\}})$ leads to an effective decomposition of A, where $S = {}^H/_{(H \cap G_q)_H}$, $Y = \{H : H \cap G_q\}$ and $|\pi| = |G : H| < |G : G_q| = |Q|$.

(2) Let T be the transition monoid of automaton A acting on Q and U(T) the set of all invertible elements of T, $u = |U(T)|$, $v = |Q|$, $t = |T|$. U(T) is a subgroup of T. Suppose, $u < v$, $1 < u < t$, then automaton A can effectively be decomposed into $A_1 \ominus A_2$ where A_1 has transition monoid $T\backslash U(T) \cup 1_Q$ acting on Q and $A_2 = (U(T), U(T), \cdot)$. Hence, $E(D) = u(u - v) + v(t + 1)$, |2, 6, 8, 9, 10|.

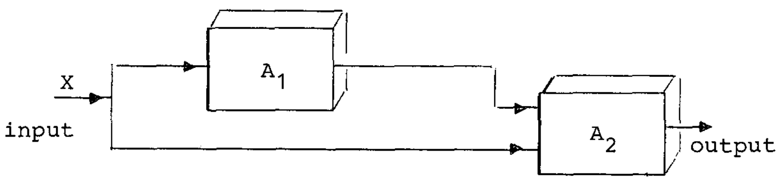

Fig. 1 $A_1 \ominus A_2$, feedbackfree decomposition of A (cascade)

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