Studies of some physically-relevant representation groups

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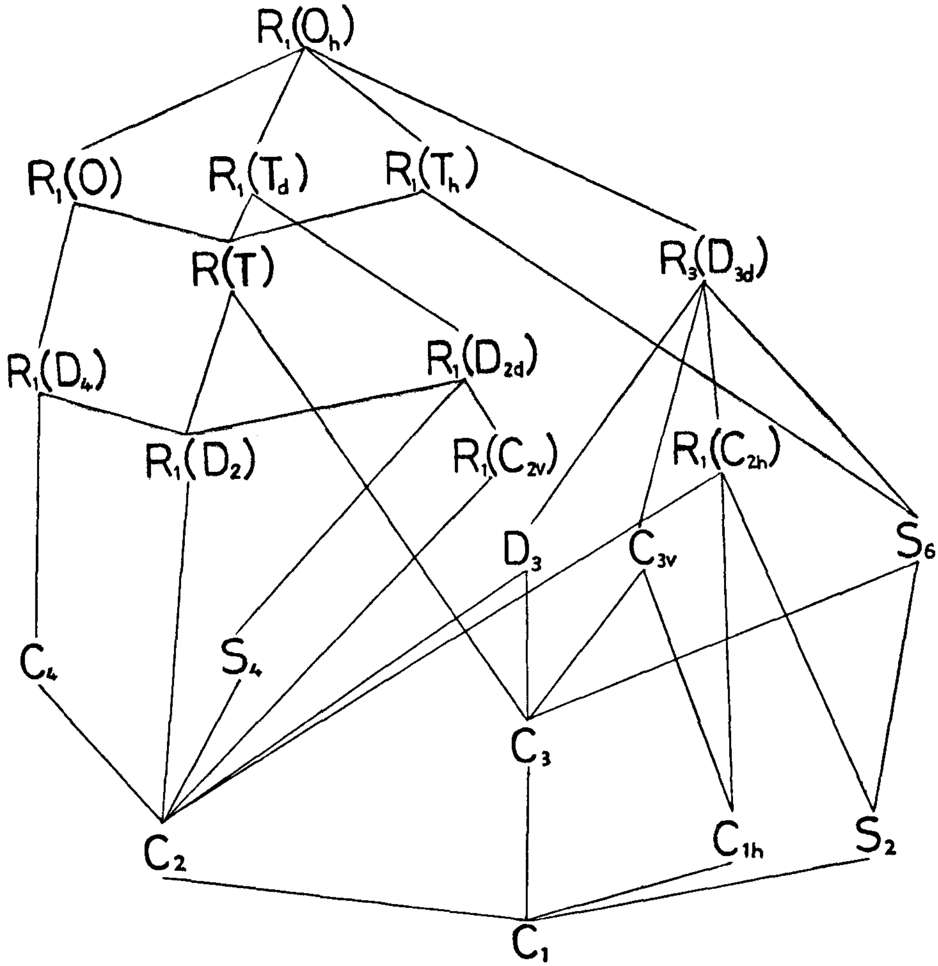
In 1904, Schur [1] defined a representation group, R, of a group, G, as a group having an invariant subgroup, M, such that the factor group R/M is isomorphic to G. When a group M could be found which had elements other than the identity, the restriction was added that M should be of minimum non-trivial order. For a given group, G, the abstract group M is unique and is known as the Schur's multiplicator of G. The multiplicator is restricted to lie in both the centre and the commutator subgroup of the representation group.

From the representation-theoretical point of view the canonical epimorphism between the cosets of R/M and the elements of G provides a relationship between the representation of R with those of G. For this purpose the irreducible representations of R are classified according to the irreducible representation of M obtained on subduction. (Since M lies in the centre of R the subduction process yields from a given irreducible representation of R a single irreducible representation of M with a frequency equal to the degeneracy of the representation of R). The representations of R which yield the totally-symmetric representation of M on descent can be put into correspondence with the ordinary (or vector) representations of G. Those yielding non-totally-symmetric representations of M correspond to the projective (or ray) representations of G. Projective representations may be classified according to the irreducible representations of M: this corresponds to the ω-classification of Backhouse and Bradley [2].

In general, a group G can have more than one representation group. Since these are abstract groups, they must be non-isomorphic and hence each different representation group yields a different set of projective representations of G. If projective representations are to be used in formulating physical problems one must be sure that the use of an alternative set does not lead to alternative physical predictions. Although this can be shown to be the case, it is proper to ask whether certain choices of representation groups are more convenient than others. In the case of the point groups, whether crystallographic or not, the number of different representation groups never exceeds four for a given point group. Certain subgroup-supergroup relationships exist between the representation groups of different point groups. In the following diagram the subgroup-supergroup relationships are indicated by tie-lines for all representation groups which are subgroups of $R_1(O_h)$, one of the four representation groups of the octahedral group O_h ($\equiv$ m3m). (A representation group such as $R(D_3)$ which is



isomorphic with the point group has been indicated simply as D_3. In the case of R(T), no subscript is used since T only has one representation group). The diagram indicates that studies involving a trigonal distortion of the octahedron may be facilitated by using the pair of representation groups $R_1(O_h)$ and $R_3(D_{3d})$ rather than some arbitrarily chosen sets of projective representations. Inspection of similar diagrams, a complete set of which may be found on pp. 124-131 of a thesis [3] (unpublished elsewhere) shows that alternative pairs of representation groups are $R_2(O_h)$ with $R_2(D_{3d})$ and $R_3(O_h)$ with $R_1(D_{3d})$ and that $R_4(O_h)$ has no subgroup which is also a representation group of D_{3d}.

The use of representation groups as a source of projective representations rather than the currently-available tables has the advantage that one can manipulate the representations as ordinary representations of a genuine group. This enables symmetrized powers to be found unambiguously as well as the establishment of the multiplication rules. Detailed study [4] also showed that some of the currently-available tables contain inaccuracies which would lead to erroneous physical results. These occur mainly in the phase factor (i.e. +1, -1, +i or -i) of certain characters: we have been able to check that they could not derive from the character table of a genuine group.

[1] Schur, I., J. reine u. angew. Math., 120, 20 (1904).

[2] Backhouse, N.B. and Bradley, C.J., Quarterly J. Maths. Oxford (2), 21, 203 (1970).

[3] Green, K.F., "Representation Theory of Finite Groups", Ph.D. Thesis, University of Kent at Canterbury (1976).

[4] Boyle, L.L. and Green, K.F., Phil. Trans. Roy. Soc. London, (A), in the press.