THE NUCLEAR COLLECTIVE MODEL AND THE SYMPLECTIC GROUP

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1. Introduction

The collective models are among the most successful of nuclear models. Nevertheless, for the most part, they remain black-box models and no-one seems to have any very clear idea of what goes on inside the box. For example, in the rotational model, one supposes that the nucleus has some deformed intrinsic shape that rotates adiabatically, giving rise to rotational bands and strong electric quadrupole transitions. One frequently has a good idea, from experiment, what the shape is, in terms of intrinsic quadrupole moments, etc., and one knows the moment of inertia. But one has virtually no experimental knowledge of the dynamical motions that go on inside the deformed nucleus. Some of the possibilities are illustrated in Fig. 1. It would be most informative, for example, if one could measure transverse electromagnetic currents for rotational nuclei. In principle this is possible by inelastic electron scattering. But in practice, the cross sections are dominated by the longitudinal currents which again tell one only about the intrinsic

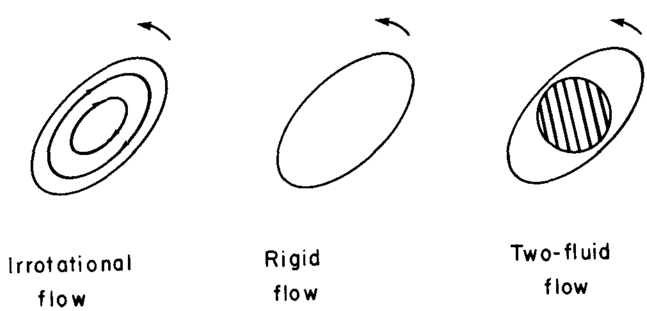

Fig. 1. How does a deformed nucleus rotate?

density distribution.

Theoretical predictions of the intrinsic dynamics have likewise been conspicuously absent, although a number of speculative models have been proposed. The cranking model, for example, has had considerable success in predicting moments of inertia for a wide range of rotational nuclei. But its significance and its dynamical content remain unclear.

Our objective is the formulation of a practical microscopic theory of collective states which, if it accords with experiment, will enable one to understand the nature of the collective states it describes. It appears that the attainment of this objective is at hand in terms of the algebraic theory of collective motion, which we review here.

The algebraic theory of collective motion has evolved over a period of many years. However we mention in this review only the major contributions. The program was initiated in 1955 by Tomonaga[1], who used algebraic means to derive canonical collective coordinates. It was followed in 1958 by the SU(3) model of Elliott[2]. Elliott's work was of fundamental importance because it introduced the *spectrum generating algebra* into the theory and because it provided the essential link between the phenomenological collective models and the shell model. However the precise nature of the link remained obscure until recently. The importance of the symplectic group, in relating the collective and independent-particle models, was observed by Goshen and Lipkin[3,4], who studied the problem in 1 dimension in 1959 and in 2 dimensions in 1968. As we shall show in this paper, the extension to 3 dimensions and the formulation of the algebraic $sp(3,\mathbb{R})$ model finally does indeed provide the means to realize collective states microscopically and thereby enables one to investigate the nature of collective dynamics.

The algebraic approach is first of all to formulate a phenomenological collective model in terms of an algebra of observables, such that the observables can be realized in a standard way as operators on many-particle state space. Thus one is enabled to determine microscopic collective states in a basis of many-particle states which carry an irreducible unitary representation of the algebra.

For the adiabatic rotational model[5], the algebra of observables is $[\mathbb{R}^5]so(3)$, which was introduced by Ui (1970)[6] and also investigated by Weaver, Biedenharn and Cusson (1973)[7]. For the Bohr collective model of rotations and quadrupole vibrations[5] the algebra is $cm(3)\sim[\mathbb{R}^6]sl(3,\mathbb{R})$. As shown in Fig. 2 these algebras are both subalgebras of $sp(3,\mathbb{R})$. However, $sp(3,\mathbb{R})$ also contains $su(3)$, the symmetry algebra of the harmonic oscillator, and it is for this reason that $sp(3,\mathbb{R})$ is instrumental in enabling the collective models to be expressed in harmonic oscillator shell model terms.

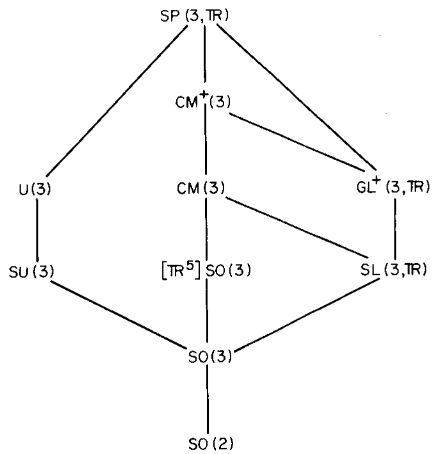

Fig. 2. Subgroups (subalgebras) of SP(3,ℝ) important in the microscopic theory of nuclear collective motion.

In the first part of this review we discuss the irreducible unitary representations of sp(3,ℝ) classified according to the subalgebra chain

$$sp(3,\mathbb{R}) \supset u(3) \supset su(3) \supset so(3).$$

In the second part we discuss the kinematical groups

$$GL^{+}(3,\mathbb{R}) \supset SL(3,\mathbb{R}) \supset SO(3)$$

and their actions as groups of deformations and rotations on many-particle states. In the third part we discuss the $[\mathbb{R}^5]so(3)$ and cm(3) collective models. Finally we discuss the sp(3,ℝ) model and the practical procedure for deriving collective states in the framework of the shell model.

2. The group SP(n,ℝ) and its algebra sp(n,ℝ).

SP(n,ℝ) is defined as the group of invertible (2n×2n) real matrices which, acting as a group of transformations on a real 2n-dimensional vector space V_n, leaves invariant the skew-symmetric bilinear form $v^t Jw$ where v and w are vectors in V_n and

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \quad (2.1)$$

Thus $SP(n,\mathbb{R})$ is the set of matrices

$$SP(n,\mathbb{R}) = \{M \in GL(2n,\mathbb{R}) \,|\, M^t J M = J\}. \quad (2.2)$$

The group $SP(n,\mathbb{R})$ and its algebra $sp(n,\mathbb{R})$ are of considerable physical interest for two very important reasons. The first reason is that $SP(n,\mathbb{R})$ is the group of linear canonical transformations in 2n-dimensional phase space[8]. The second reason, and the one we will focus on in this review, derives from their subgroup (subalgebra) structures. In particular, $SP(n,\mathbb{R})$ contains both the unitary group $U(n)$, which is the degeneracy group of the n-dimensional harmonic oscillator, and the general linear group $GL^+(n,\mathbb{R})$, which is the kinematical group of deformations and rotations. The major subgroup chains of $SP(3,\mathbb{R})$ are shown, for example, in Fig. 2. The algebras of the subgroups including $CM^+(3)$ play a fundamental role in the theory of nuclear collective motion; indeed they provide S.G.A.'s (spectrum generating algebras) for the calculation of collective states. On the other hand, $u(3)$ and its subalgebra $su(3)$ provide the microscopic link with the shell model[2]. Thus $sp(3,\mathbb{R})$ and its subalgebras provide not only a precise and succinct algebraic formulation of collective models but also the means to realize these models microscopically in the framework of the shell model.

The significance of the symplectic groups for the microscopic theory of collective motion was observed by Goshen and Lipkin[3] and emphasized by Ui[6]. Algebraic microscopic models, based on $sp(n,\mathbb{R})$, were proposed for n=1 and 2 by Goshen and Lipkin[3,4], who also derived the irreducible unitary representations for $sp(1,\mathbb{R})$[3] and $sp(2,\mathbb{R})$[4]. However, their approach, which was to exploit the isomorphism of $C_1=sp(1,\mathbb{R})_{\mathbb{C}}$ with A_1 and of $C_2=sp(2,\mathbb{R})_{\mathbb{C}}$ with B_2 does not extend to $n\geq 3$. The appropriate representations of $SP(n,\mathbb{R})$ were derived by Godement[9] and expressed in a form suitable for application to the microscopic theory of collective motion by Rosensteel.[10]

3. Unitary representations of $sp(n,\mathbb{R})$

The principal and discrete series of unitary representations have been discussed in detail in two recent papers[10,11]. In physical applications one is primarily interested in the latter[10].

The discrete series representations are readily constructed on many-fermion state space by the realization of the algebra $sp(n,\mathbb{R})$ as the vector space of all skew-adjoint one-body bilinear products in the position $x_{j\alpha}$ and momentum $p_{j\alpha}$ observables ($j=1,\ldots,N$, $\alpha=1,\ldots,n$) of an N-particle system in an n-dimensional space. A basis for $sp(n,\mathbb{R})$ is given by

$$i\,L_{\alpha\beta} = i\sum_{j=1}^{N} (x_{j\alpha}p_{j\beta}-x_{j\beta}p_{j\alpha}) \tag{3.1}$$

(angular momentum)

$$i\,S_{\alpha\beta} = i\sum_{j=1}^{N} (x_{j\alpha}p_{j\beta}+p_{j\alpha}x_{j\beta}) \tag{3.2}$$

(shear momentum)

$$i\,Q_{\alpha\beta} = i\sum_{j=1}^{N} x_{j\alpha}\,x_{j\beta} \tag{3.3}$$

(quadrupole tensor)

$$i\,K_{\alpha\beta} = i\sum_{j=1}^{N} p_{j\alpha}\,p_{j\beta} \tag{3.4}$$

(quadrupole momentum tensor)

It is clear that the transformation

$$\left.\begin{array}{l} x_{i\alpha} \to X_{i\alpha} = e^{S}\,x_{i\alpha}\,e^{-S} \\ p_{i\alpha} \to P_{i\alpha} = e^{S}\,p_{i\alpha}\,e^{-S} \end{array}\right\}, \quad S \in sp(n,\mathbb{R}) \tag{3.5}$$

is a linear canonical (symplectic) transformation, since it conserves the commutation relations

$$[X_{i\alpha},\,P_{j\beta}] = e^{S}\,[x_{i\alpha},p_{j\beta}]e^{-S} = i\hbar\,\delta_{ij}\,\delta_{\alpha\beta}. \tag{3.6}$$

Furthermore it can readily be ascertained by construction that any linear canonical transformation of the N-particle coordinates can be generated in this way.

Thus each such realization of $sp(n,\mathbb{R})$ defines a unitary representation on many-particle state space. It turns out that many of these representations are representations of the two-fold covering group of $SP(3,\mathbb{R})$; viz. the metaplectic group. They are of course, in general, reducible. For example, for a single-particle in one dimension, Itzykson[12] has shown that the above realization of $sp(1,\mathbb{R})$ defines a unitary representation of the two-fold covering group of $SP(1,\mathbb{R})$ and that this representation is the direct sum of two irreducible discrete representations.

The physically relevant irreducible representations are precisely those that occur in the decomposition of the above representation on many particle state space. The procedure for constructing these irreducible unitary representations becomes transparent when the sp(n,$\mathbb{R}$) basis is expressed in terms of the boson annihilation and creation operators

$$a_{j\alpha} = \frac{1}{\sqrt{2}} (x_{j\alpha} + i\, p_{j\alpha})$$
$$a^{\dagger}_{j\alpha} = \frac{1}{\sqrt{2}} (x_{j\alpha} - i\, p_{j\alpha}) \qquad (3.7)$$

One finds

$$i\, L_{\alpha\beta} = \sum_{j=1}^{N} (a^{\dagger}_{j\alpha}\, a_{j\beta} - a^{\dagger}_{j\beta}\, a_{j\alpha}) \qquad (3.8)$$

$$i\, S_{\alpha\beta} = \sum_{j=1}^{N} (a_{j\alpha}\, a_{j\beta} - a^{\dagger}_{j\alpha}\, a^{\dagger}_{j\beta}) \qquad (3.9)$$

$$i\, Q_{\alpha\beta} = \frac{i}{2} \sum_{j=1}^{N} (a^{\dagger}_{j\alpha}\, a_{j\beta} + a_{j\alpha}\, a^{\dagger}_{j\beta} + a_{j\alpha}\, a_{j\beta} + a^{\dagger}_{j\alpha}\, a^{\dagger}_{j\beta}) \qquad (3.10)$$

$$i\, K_{\alpha\beta} = \frac{i}{2} \sum_{j=1}^{N} (a^{\dagger}_{j\alpha}\, a_{j\beta} + a_{j\alpha}\, a^{\dagger}_{j\beta} - a_{j\alpha}\, a_{j\beta} - a^{\dagger}_{j\alpha}\, a^{\dagger}_{j\beta}) \qquad (3.11)$$

Next one observes that sp(n,$\mathbb{R}$) is the vector space direct sum of a unitary subalgebra u(n), of boson number conserving operators, and a complementary space of boson pair annihilation and creation operators, where u(n) is spanned by the operators $i\, L_{\alpha\beta}$ and $i\, \mathcal{Q}_{\alpha\beta} = (i/2)(Q_{\alpha\beta}+K_{\alpha\beta})$. We recall that u(n) is the degeneracy algebra of the n-dimensional harmonic oscillator Hamiltonian[2]. Suppose then that (λ,μ) is an irreducible unitary representation of u(n) carried by a basis of degenerate lowest-energy oscillator states $|(\lambda,\mu)\nu\rangle$; i.e. states for which

$$\sum_{j=1}^{N} (a_{j\alpha}\, a_{j\beta}) |(\lambda,\mu)\nu\rangle = 0 \qquad \text{all } \alpha,\beta,\nu. \qquad (3.12)$$

It then follows that one can generate the basis for an irreducible unitary representation of sp(n,$\mathbb{R}$) by the repeated application of the sp(n,$\mathbb{R}$) two-boson creation operators to these lowest energy u(n) basis states. Thus these unirreps are standard cyclic irreducible representations determined by the lowest energy u(n) representation.

This construction has been derived rigorously in ref. 10. It is precisely the generating procedure that is needed for a microscopic shell-model theory of nuclear collective motion. However, before discussing how it may be deployed in the microscopic theory of nuclear collective motion, we must first demonstrate that, for collective motion in 3-dimensional space, sp(3,$\mathbb{R}$) indeed contains the appropriate spectrum generating algebras.

4. The linear subgroups of SP(3,$\mathbb{R}$)

The linear subgroups of SP(3,$\mathbb{R}$)

$$SL(3,\mathbb{R}) \subset GL^{+}(3,\mathbb{R}) \subset SP(3,\mathbb{R})$$

play a fundamental role in the algebraic formulation of collective models because they are the kinematical groups of collective vibrations and rotations.

Consider, for example, the subgroup $GL^{+}(1,\mathbb{R})$ of $GL^{+}(3,\mathbb{R})$ corresponding to collective motions in 1-dimension. Acting on N-particle states, $GL^{+}(1,\mathbb{R})$ has the realization

$$GL^{+}(1,\mathbb{R}) = \{e^{-i\alpha S} \mid S=\sum_{j=1}^{N}(x_j p_j + p_j x_j)\}. \tag{4.1}$$

The action on particle coordinates, defined by eq.(3.5),

$$\left.\begin{array}{l} e^{-i\alpha S}\, x_j\, e^{i\alpha S} = e^{-\frac{1}{2}\hbar\alpha}\, x_j \\ e^{-i\alpha S}\, p_j\, e^{i\alpha S} = e^{+\frac{1}{2}\hbar\alpha}\, p_j \end{array}\right\} \quad j=1,\ldots,N \tag{4.2}$$

is seen to be a simple scale transformation, cf. Fig. 3. The action on N-particle states is correspondingly given simply by a compression or dilation of the coordinate scale.

In 3-dimensional space, the subgroup $GL^{+}(1,\mathbb{R}) \otimes GL^{+}(1,\mathbb{R}) \otimes GL^{+}(1,\mathbb{R}) \subset GL^{+}(31,\mathbb{R})$ can likewise effect scale transformations. In the special case of equal scale transformations of each of the three coordinate axes, one has monopole (breathing mode) deformations. But, in general, one has also quadrupole deformations. The generators for this subgroup are the diagonal shear momenta ($i\,S_{\alpha\alpha}$, $\alpha=1,2,3$) of eq. (3.2). However, $GL^{+}(3,\mathbb{R})$ possesses a much richer structure than this subgroup, for its generators include also the off-diagonal shear momenta ($i\,S_{\alpha\beta}$) and the angular momenta ($i\,L_{\alpha\beta}$). One can easily show that the shear momenta are the infinitesimal generators of linear curl-free displacements,

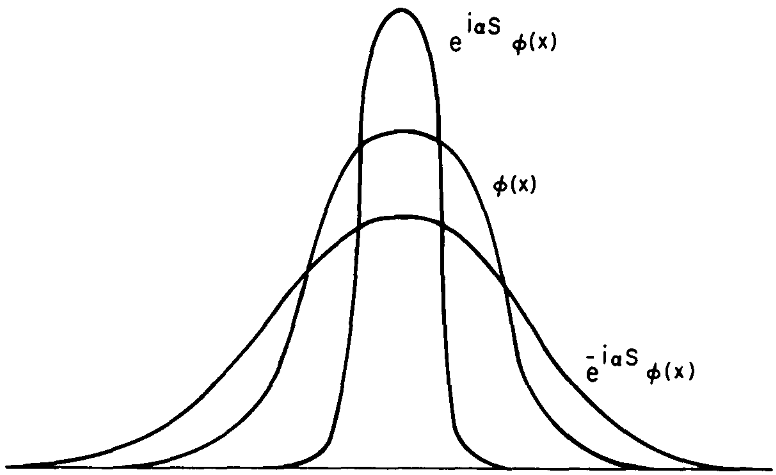

Fig.3. The action of the $GL^+(1,\mathbb{R})$ operator $e^{-i\alpha S}$ on a single-particle wave function $\phi(X)$.

$$\nabla_j \times \delta\vec{x}_j = 0,$$

and that, in addition to deformations, they generate irrotational surface wave rotations. On the other hand, the angular momentum operators, as usual, are the generators of rigid flow rotations.

To summarize then, $GL^+(3,\mathbb{R})$ effects both rigid and irrotational flow rotations plus irrotational monopole and quadrupole deformations. The subgroup $SL(3,\mathbb{R})$ simply excludes the monopole deformations.

5. The SL(3,ℝ) model

The principal series[13] of irreducible unitary representations of $SL(3,\mathbb{R})$ were employed by Weaver and Biedenharn[14], who, following a suggestion of Gell'Mann, proposed an algebraic $sl(3,\mathbb{R})$ model of nuclear collective motion by identifying the basis states of their representation with collective states of the nucleus.

A problem with the $sl(3,\mathbb{R})$ model is that $sl(3,\mathbb{R})$ is not obviously an S.G.A. for any known Hamiltonian. Another problem is that the physical significance of the chosen basis states is unclear. Nevertheless Weaver and Biedenharn do obtain impressive agreement with experimental data for a number of deformed nuclei and it therefore seems to us desirable that the physical foundations of their model should be explored in greater depth.

The reason $sl(3,\mathbb{R})$ is not obviously an S.G.A. for a collective Hamiltonian is that, while it contains the collective momentum observables, it fails to include the corresponding observables of orientation and shape. The same is true of $gl(3,\mathbb{R})$ and its other subalgebras. The following examples illustrate how algebraic collective models can be constructed by augmenting $gl(3,\mathbb{R})$ and its subalgebras by the addition of suitable shape observables.

6. The $[\mathbb{R}^5]so(3)$ rotational model

The kinematical group for rotational motion is

$$SO(3) \subset SL(3,\mathbb{R})$$

To obtain an algebraic rotational model, it is necessary to augment the so(3) angular momentum algebra with some space of observables to characterize the orientation of the nucleus. In the adiabatic rotational model[5], the extra observables are the quadrupole moments. Thus one obtains an S.G.A. for the adiabatic rotational model Hamiltonian by augmenting so(3) to form the semidirect product $[\mathbb{R}^5]so(3)$, where $\mathbb{R}^5$ is the abelian subalgebra spanned by the 5 traceless quadrupole moments

$$i\, Q_{\alpha\beta} = \sum_{j=1}^{N} (x_{j\alpha}x_{j\beta}-\tfrac{1}{3}\delta_{\alpha\beta} \sum_{\gamma} x_{j\gamma}x_{j\gamma}) \tag{6.1}$$

The group $[\mathbb{R}^5]SO(3)$ was employed in the study of nuclear rotations by Ui[6], who derived the irreducible unitary representations algebraically, and by Weaver, Biedenharn and Cusson who used the method of induced group representations.

The carrier space for these unitary representations is the set of square integrable functions on SO(3), spanned by the Wigner rotation matrices $D^L_{KM}(g)$, $g\in SO(3)$. Thus the basis states are labelled by the $[\mathbb{R}^5]SO(3)$ invariants

$$a_2 = -\tfrac{1}{2}\, Tr\, (Q^2),\quad a_3 = -\tfrac{1}{3}\, Tr(Q^3) \tag{6.2}$$

and by the angular momentum invariants LM of the subgroup chain

$$[\mathbb{R}^5]SO(3) \supset SO(3) \supset SO(2)$$

plus an additional quantum number K. Note that this representation is reducible. The irreducible subspaces are certain invariant subspaces which transform appropriately under the action of the discrete little group D_2 (See Mackey[15]). The basis states have an obvious physical interpretation as rotational states of a deformed nucleus with well-defined intrinsic quadrupole moments. Furthermore the imposition of D_2 symmetry corresponds directly to the usual symmetry constraints imposed in the rotational model. The intrinsic moments are in fact the principal moments (I_k, k=1,2,3) of the quadrupole tensor Q and are known functions of a_2 and a_3[16]. Furthermore the K quantum number is the component of the angular momentum along the intrinsic 3 axis.

Thus the $[\mathbb{R}^5]SO(3)$ basis states have a direct one-to-one correspondence with those of the adiabatic rotational model. The algebra $[\mathbb{R}^5]SO(3)$ is seen to be an S.G.A. for any rotational Hamiltonian of the form

$$H_{rot} = \sum_{k=1}^{3} \frac{\hbar^2}{2\mathcal{I}_k(I)} L_k^2 \tag{6.3}$$

where $\mathcal{I}_k$, k=1, 2, 3, are the principal moments of an inertia tensor which must however be rational functions of I_k. For example, in the rigid-flow rotational model[17]

$$\mathcal{I}_j(I) = I_k + I_\ell \qquad (j,\ k,\ \ell \text{ cyclic}) \tag{6.4}$$

and in the irrotational flow model[18,19]

$$\mathcal{I}_j(I) = \frac{(I_k - I_\ell)^2}{I_k + I_\ell} \qquad (j,\ k,\ \ell \text{ cyclic}) \tag{6.5}$$

7. The cm(3) collective model

The algebra cm(3) is obtained by adjoining the 6 quadrupole moments $Q_{\alpha\beta}$ of eq.(3.3) to the generators $L_{\alpha\beta}$ and $S_{\alpha\beta}$ of $sl(3,\mathbb{R})$. Note that we now relax the temporary trace-zero restriction on Q of section 6. Thus

$$cm(3) \equiv [\mathbb{R}^6]sl(3,\mathbb{R}) \tag{7.1}$$

The CM(3) group was proposed as the relevant group for collective motion in 3 dimensions by Weaver, Biedenharn and Cusson[7], based on some observations of Tomonaga[1]. The cm(3) model was formulated by Rosensteel and Rowe[20] who also determined the irreducible unitary representations of the group.

We shall consider here the slightly larger, but very closely related, group

$$CM^+(3) \equiv [\mathbb{R}^6]GL^+(3,\mathbb{R}) \tag{7.2}$$

where $GL^+(3,\mathbb{R})$ is the general linear group of 3×3 matrices with positive determinant.

In parallel with $[\mathbb{R}^5]SO(3)$, the carrier space for the unitary representations of $[\mathbb{R}^6]GL^+(3,\mathbb{R})$ is the set of square integrable functions on $GL^+(3,\mathbb{R})$. The functions have a natural interpretation through the decomposition

$$\begin{aligned} GL^+(3,\mathbb{R}) &= SO(3).D.SO(3) \\ g &= r_1 .\ d.\ r_2 \end{aligned} \tag{7.3}$$

where

$$D = \left\{ \begin{pmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{pmatrix} \middle| \ d_1 > d_2 > d_3 > 0 \right\} \tag{7.4}$$

Thus a basis for $CM^+(3)$ is given by the set of functions

$$\{\mathcal{D}^L_{KM}(r_1)\ \ \Phi_\nu(d)\ \ \mathcal{D}^L_{KM}(r_2)\}$$

where

$$\{\Phi_\nu(d) = \Phi_{\nu_1}(d_1)\Phi_{\nu_2}(d_2)\Phi_{\nu_3}(d_3)\}$$

is, for example, a complete set of vibrational states indexed by positive integers (ν_1, ν_2, ν_3). The irreducible subspaces correspond to fixed L.

The physical interpretation of this basis is immediately apparent from the $GL^+(3,\mathbb{R})$ decomposition. $\mathcal{D}^L_{KM}(r_2)$ describes rotations of the system in the laboratory coordinate frame with angular momentum L. $\Phi_\nu(d)$ describes intrinsic vibrations, corresponding to scale transformations along each of the three principal axes of the quadrupole mass tensor. These vibrations include β and γ quadrupole vibrations, in the terminology of the collective model[5], plus monopole vibrations. With the restriction of $CM^+(3)$ to $CM(3)$, the monopole degrees of freedom are simply omitted. Finally $\mathcal{D}^L_{KM}(r_1)$ describes an internal rotational motion of the system called 'vorticity'. The vorticity interpretation of L was made by Cusson[21], in his classical $SL(3,\mathbb{R})$ discussion of nuclear hydrodynamic flows. It was later confirmed by Gulshani and Rowe[19] and again by

Weaver, Biedenharn and Cusson[22] in quantal treatments and explicit expressions were given for the vorticity so(3) algebra.

The algebras cm(3) and $cm^+(3)$ are S.G.A.'s for collective Hamiltonians which are expressible in terms of the cm(3) algebras. For example, they are S.G.A.'s for Bohr's hydrodynamic irrotational flow model[5]. The interpretation of L, mentioned above, reveals that irrotational (vortex free) states are in fact realized in the $L=0$ irreducible subspace of cm(3) states. Thus the cm(3) model is more general than the Bohr model, admitting a variety of vorticities as emphasized in refs.19, 22.

8. The sp(3,$\mathbb{R}$) model[23]

A common feature of all of the above algebras is that their generators have simple realizations as one-body operators; cf. eqs.(3.1)-(3.4). Thus the possibility exists of realizing the corresponding collective models microscopically. For the rotational model, for example, this would be achieved by finding simultaneous eigenstates of the $[\mathbb{R}^5]$so(3) invariants a_2 and a_3 and of the angular momentum operators L^2 and L_z on many-particle state space[16]. In practice one would of course need to truncate to a finite shell-model space. Even so, to achieve any degree of accuracy one would need a very large basis of harmonic oscillator shell-model states. It is therefore clearly of enormous benefit if one can find subspaces of the many-particle space which, even if not irreducible with respect to the unitary actions of $[\mathbb{R}^5]$so(3) or cm(3), are nevertheless invariant. This is not only possible, it is eminently practicable. Since sp(3,$\mathbb{R}$) $\supset$ cm(3) $\supset$ $[\mathbb{R}^5]$so(3), it follows that the desired objective is achieved if one can decompose the shell-model space into irreducible invariant subspaces with respect to sp(3,$\mathbb{R}$); i.e. find the irreducible unitary representations of sp(3,$\mathbb{R}$) on shell-model state space. The solution of this problem[10] was given in section 3, and was made possible as a result of the subalgebra chain

$$\mathrm{sp}(3,\mathbb{R}) \supset \mathrm{u}(3) \supset \mathrm{su}(3) \supset \mathrm{so}(3)$$

and the observation that su(3) is the degeneracy algebra of the 3-dimensional harmonic oscillator[2]. In fact, since sp(3,$\mathbb{R}$) is the smallest algebra that contains both cm(3) and su(3), it is the smallest algebra that serves this purpose.

As shown in section 3, the sp(3,$\mathbb{R}$) representations can be generated by starting with the basis states of an su(3) representation in the shell-model (0ħω) valence space and by acting repeatedly on them with the sp(3,ℝ) two-boson operators. The results are shown, starting from the $(\lambda,\mu) = (0,0)$ and (8,0) su(3) representations in Figs. 4 and 5. On the basis of the su(3) model, these particular representations are the ones appropriate for the observation of collective motion in ^{16}O and ^{20}Ne, for example.

(λ,μ)	E_o	L
	8ħω	
(6,0),(2,2),(0,0)	6ħω	$0^3,2^3,3,4^2,6$
(4,0),(0,2)	4ħω	$0^2,2^2,4$
(2,0)	2ħω	0,2
(0,0)	0ħω	0

Fig. 4. SU(3) irreducible representations (λ,μ) spanning the sp(3,ℝ) irreducible representation whose 0ħω subspace transforms as (0,0).

(λ,μ)	E_o	L
	8ħω	
(14,0),(12,1),$(10,2)^2$,(9,1), $(8,3)^2$,$(8,0)^2$,(7,2),$(6,4)^2$, (6,1),(5,3),(4,5),(4,2),(2,6)	6ħω	$0^9,1^8,2^{23},3^{19},4^{29},5^{22},6^{28},7^{19},8^{21},9^{12},10^{11},11^5,12^4,13,14$
(12,0),(10,1),(6,3),(4,4), $(8,2)^2$,(7,1),(6,0)	4ħω	$0^5,1^3,2^{11},3^7,4^{13},5^8,6^{12},7^7,8^9,9^4,10^4,11,12$
(10,0),(8,1),(6,2)	2ħω	$0^2,1,2^4,3^2,4^4,5^2,6^4,7^2,8^3,9,10$
(8,0)	0ħω	0,2,4,6,8

Fig. 5 SU(3) irreducible representations (λ,μ) spanning the sp(3,ℝ) irreducible representation whose 0ħω subspace transforms as (8,0).

The important observation to be made from these figures is that, even up to the $6\hbar\omega$ harmonic oscillator shell, the number of states of any given angular momentum L is still a very small number in terms of the dimensions of matrices that can be diagonalized with present day computers. Thus the diagonalization of collective Hamiltonians within a shell-model space, including all shells up to say $10\hbar\omega$ is quite feasible. By way of contrast, the corresponding calculations without the $sp(3,\mathbb{R})$ decomposition would already be prohibitively large at the $2\hbar\omega$ level.

To date only a few pilot calculations have been performed to assess the practicality of the above programme. The results are extremely encouraging. In the first application, we attempted to generate pure rotational states by diagonalizing the $[\mathbb{R}^5]so(3)$ invariants, as indicated above. As expected, 'rotational' bands of states emerged of which the lowest members, truncated at the $0\hbar\omega$, $2\hbar\omega$ and $4\hbar\omega$ levels, are compared in Table 1 with the full-space (rotational model) result, derived algebraically. In particular, one observes the extent to which the members of the band have common intrinsic quadrupole moments, as measured by the constancy of a_2 and a_3, and the extent to which the matrix elements of the quadrupole operator between various states, as measured by the corresponding B(E2) values (reduced transition rates), satisfy the rotational model relationships. Although the rate of convergence is not quite as rapid as one might have hoped, one can nevertheless anticipate that if these calculations were extended up to say the $10\hbar\omega$ level one would realize pure rotational states, for quite a large number of levels, to an accuracy of within a few percent. To our knowledge this is the first time that such a result has been achieved.

For applications to real nuclei, one can be even more optimistic. For in real nuclei, one does not expect to find rotors with sharp intrinsic quadrupole moments, due to vibrational shape fluctuations and the usual shell-model suppression of contributions from higher shells. Thus a reasonable collective Hamiltonian might be

$$H = H_o + V(\beta,\gamma) \tag{8.1}$$

where H_o is the harmonic oscillator shell-model Hamiltonian and $V(\beta,\gamma)$ is a potential in the Bohr-Mottelson quadrupole deformation parameters, β, γ[5]. $V(\beta,\gamma)$ can be designed to have a minimum at some suitably chosen intrinsic shape. Furthermore it can be expressed in terms of the $[\mathbb{R}^5]so(3)$ invariants by means of the identities

	L	0	2	4	6	8
$0\hbar\omega$	$a_2(L)/a_2(0)$	1.0	0.97	0.93	0.85	0.75
	$a_3(L)/a_3(0)$	1.0	0.95	0.83	0.65	0.39
	$\frac{B(E2;L\to L-2)}{B(E2;2\to 0)}$			1.27	1.07	0.64
$2\hbar\omega$	$a_2(L)/a_2(0)$	1.0	0.98	0.94	0.87	0.78
	$a_3(L)/a_3(0)$	1.0	0.96	0.88	0.75	0.57
	$\frac{B(E2;L\to L-2)}{B(E2;2\to 0)}$			1.34	1.29	1.07
$4\hbar\omega$	$a_2(L)/a_2(0)$	1.0	0.99	0.95	0.90	0.82
	$\frac{B(E2;L\ L-2)}{B(E2;2\ 0)}$			1.37	1.39	1.28
R.M.	$a_2(L)/a_3(0)$	1.0	1.0	1.0	1.0	1.0
	$a_3(L)/a_3(0)$	1.0	1.0	1.0	1.0	1.0
	$\frac{B(E2;L\to L-2)}{B(E2;2\to 0)}$			1.43	1.57	1.65

Table 1. The ratios $a_2(L)/a_2(0)$ for the eigenvalues of a_2, $a_3(L)/a_3(0)$ for the expectation values of a_3, and $B(E2;L\to L-2)/B(E2;2\to 0)$ for reduced E2 transition probabilities for the $sp(3,\mathbb{R})$ irreducible representation whose $0\hbar\omega$ subspace transforms as (8,0) under su(3) for mass number N=20. The results are given for the most deformed band in each of the subspaces up to $0\hbar\omega$, $2\hbar\omega$ and $4\hbar\omega$, respectively. The corresponding full space (rotational model) results are given for comparison.

$$a_2 = -\frac{3}{20\pi} N^2 R_o^4 \beta^2$$

$$a_3 = -\frac{1}{20\sqrt{5\pi}} N^3 R_o^6 \beta^3 \cos 3\gamma \qquad (8.2)$$

where R_o is the nuclear radius. In the following application, we have used the simple expression for V

$$V(\beta,\gamma) = b[a_2+b_3 a_3+\tfrac{1}{2}b_4(a_2)^2] \qquad (8.3)$$

which is manifestly in the sp(3,$\mathbb{R}$) enveloping algebra. Fig. 6 shows the potential $V(\beta,\gamma)$ used here for a range of β,γ in the neighbourhood of its minimum. The spectrum for H, eq.(8.1), is compared with experiment for ^{20}Ne in Fig.7, for a reasonable set of parameters b, b_3 and b_4 and for truncation of the (8,0) space of states at the $4\hbar\omega$ level.

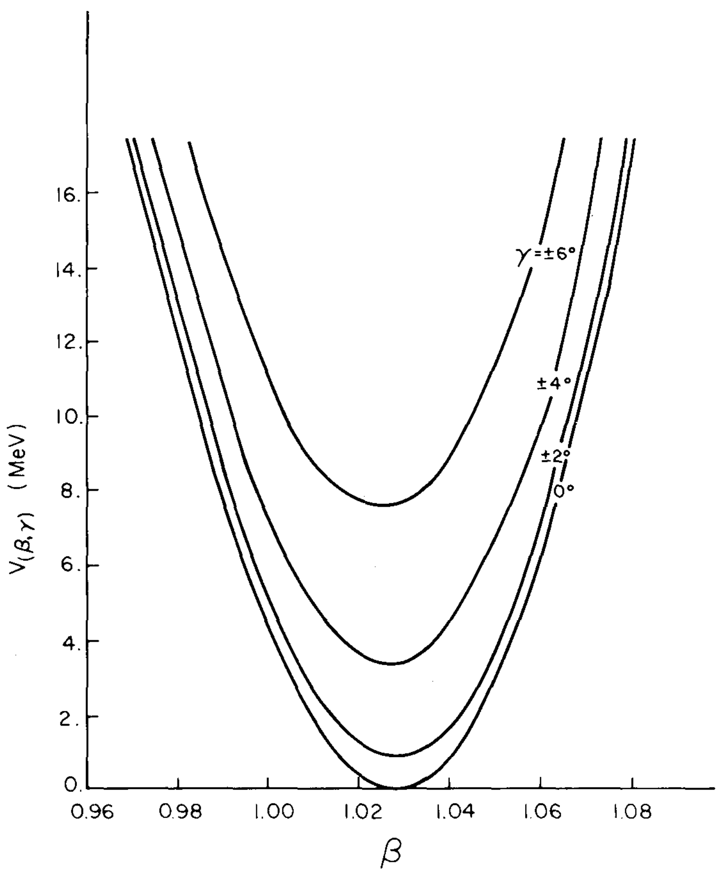

Fig. 6. The potential $V(\beta,\gamma)$ of eq. (8.3) in the neighbourhood of its minimum.

For comparison, the su(3) results, corresponding to truncation at the $0\hbar\omega$ level, are also shown. The important observation is not so much the energy spectrum, which can be fitted more or less exactly by a suitable choice of potential, but the dramatic improvement in the quadrupole transition rates, which the su(3) model underestimates by a factor of ~3. One can be optimistic therefore that, with the inclusion of a few more shells, such calculations will be able to explain fully the collectivity of the ^{20}Ne spectrum and those of other light nuclei.

9. Summary

We have shown that the sp(3,$\mathbb{R}$) algebra and its subalgebras are S.G.A.'s for various collective models and that the decomposition of the shell-model space into irreducible sp(3,$\mathbb{R}$) subspaces provides a practical means to realize collective states microscopically. The programme is not yet complete, however. In particular, one would like

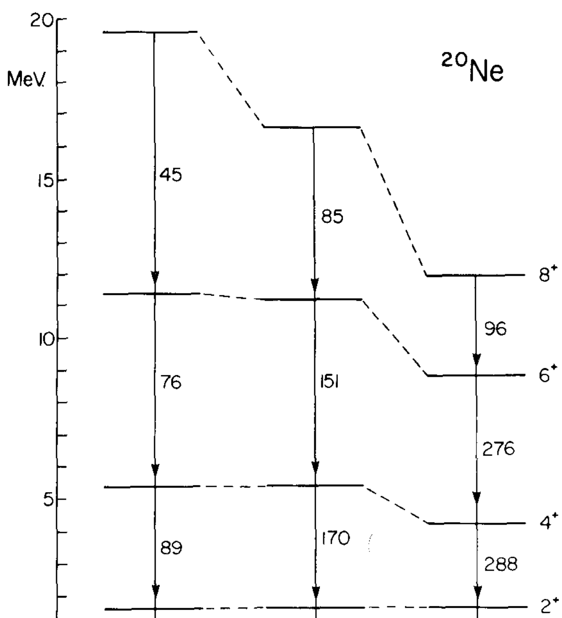

Fig. 7. Comparison of the su(3), sp(3,$\mathbb{R}$) and experimental spectra and reduced E2 transition rates (indicated in single-particle units) for ^{20}Ne.

to extend the calculations to higher levels of truncation, perform calculations with realistic two-nucleon interactions rather than phenomenological potentials, and investigate the effects of the sp(3,ℝ) breaking spin-orbit and tensor forces. All of these things can be done. What is important, from a general shell-model point of view, is that one now has a well-defined procedure for augmenting any shell-model space, to admit the possible development of collective dynamics, by the addition of those states which one has identified as being necessary for that purpose. Finally, by realizing collective states on many-particle (shell-model) state space in this way one is enabled to exploit the much larger algebra of all many-particle observables to probe the currents and other dynamical properties of collective states which the collective models, in themselves, are powerless to predict.

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