GEOMETRY OF REAL AND COMPLEX CANONICAL TRANSFORMATIONS IN QUANTUM MECHANICS

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1. INTRODUCTION

Quantum mechanics of finitely many particles necessarily involves the group of linear (and affine) canonical transformations. A well-defined ray representation of this group acts in the space of states of any quantum-mechanical system with finitely many degrees of freedom and plays a central role in many different contexts. [15, 19, 20, 21, 22, 23, 33, 35, 36, 37, 38, 44, 49, 50, 53, 54, 55, 56 , 43].

This representation appears quite naturally in quantum mechanics over phase space (Weyl-Wigner correspondence) [2, 9, 16, 17, 18, 25, 26, 27, 28, 29, 30, 31, 32, 34, 39, 40, 42, 45, 46, 47, 48, 57, 58]. We shall see that it becomes, when suitably written, just a matter of looking at one object from different symplectic reference frames. This is particularly interesting for complex canonical transformations which are represented, in general, by unbounded operators [1, 8, 10, 11, 12, 35, 37, 52].

The list of references is far from being systematic or complete. It is meant to give an idea cf the variety of motivations and points of view in the subject.

2. REAL SYMPLECTIC SPACE (PHASE SPACE) [6, 34, 42]

E will denote an even-dimensional vector space over real numbers

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$$\dim_R E = 2\nu < \infty$$

and σ will be a real-valued, bilinear, nondegenerate antisymmetric form on E.

We shall use the notation

$$e^{a}(b) = e^{2i\pi\sigma(a,b)} \qquad (a, b \in E).$$

There are two features to remember about the symplectic space $E(\sigma)$:

(i) $e^{a}(b)$ is a nontrivial multiplier for the additive group of E (i.e. there exists a nontrivial projective (ray) representation of E with multiplier $e^{a}(b)$).

(ii) If one defines Fourier transform on E by

$$(Ff)(a) = \int e^{a}(b)\, f(b)\, db$$

then the volume element db can be normalized so that

$$F^2 = 1.$$

Both statements become false if σ is replaced by an orthogonal form.

Translations and symmetries with respect to points [28, 29, 31] :

Given an $a \in E$, consider the translation (in phase space)

$$\tau_a \quad : \quad b \rightarrow b + a \quad (b \in E)$$

and the symmetry around the point $\frac{1}{2}a$ (notice the factor $\frac{1}{2}$!)

$$\pi_a \quad : \quad b \rightarrow -b + a \quad .$$

The group $E^{(2)}$ consisting of the translations and symmetries is non-abelian. Its multiplication table can be conveniently written as follows : Denote by Z_2 the multiplicative group having as elements the numbers $+1$ and -1 . Write τ_a as the pair $\{1, a\}$, and π_a as the pair $\{-1, a\}$. Then the group operation in $E^{(2)}$ is given by

$$\{\varepsilon, a\}\{\varepsilon', a'\} = \{\varepsilon\varepsilon', a + \varepsilon a'\} \qquad (\varepsilon, \varepsilon' \in Z_2)$$

We have

$$E^{(2)} = E^{(+)} \cup E^{(-)}$$

where $E^{(+)}$ consists of all translations and $E^{(-)}$ of all symmetries. Furthermore $E^{(+)}$ is a normal subgroup of $E^{(2)}$, and $E^{(2)}/E^{(+)} \simeq Z_2$. The connected component of the identity $\{1,0\}$ in $E^{(2)}$ is $E^{(+)}$.

3. WEYL-WIGNER SYSTEMS

Definition : Let $E(\sigma)$ be a finite-dimensional real symplectic space, and let $E^{(2)}$ be the group of translations and symmetries around points of E. A Weyl-Wigner system is a strongly continuous map $\{\varepsilon, a\} \longrightarrow W^{(\varepsilon)}(a)$ from $E^{(2)}$ to unitary operators in a separable Hilbert space $\mathcal{H}$, such that

$$W^{(\varepsilon)}(a)\, W^{(\varepsilon')}(a') = e^{a}(\varepsilon a')\, W^{(\varepsilon\varepsilon')}(a+\varepsilon a')$$

$$(\{\varepsilon, a\} \in E^{(2)}\ ,\ \{\varepsilon', a'\} \in E^{(2)})$$

i.e. a projective representation of $E^{(2)}$ with multiplier $e^{a}(\varepsilon a')$. The restriction of a Weyl-Wigner system to $E^{(+)}$ is called a Weyl system.

Weyl systems have been long in use in the study of canonical commutation relations. The Wigner operators $W^{(-)}(a)$ are implicit in Wigner's definition of quasi-probability [58]. Namely : The Wigner quasiprobability associated to a state ρ of a quantum-mechanical system is essentially $\mathrm{tr}(\rho\, W^{(-)}(a))$.

Weyl-Wigner systems are given by certain representations of the Weyl-Wigner-Heisenberg group, which can be defined as follows :

Denote by T the multiplicative group of complex number of modulus one. Consider the set $G = \{Z_2 \times E \times T\} = \{E^{(2)} \times T\}$, and introduce the group operation

$$\{\varepsilon, a, \tau\}\{\varepsilon', a', \tau'\} = \{\varepsilon\varepsilon',\ a+\varepsilon a',\ \tau\tau' e^{a}(\varepsilon a')\}$$

Remarks on Weyl-Wigner systems :

(i) The uniqueness theorem of Stone and von Neumann, together with an easy additional argument shows that there exists (up to unitary equivalence) exactly two irreducible Weyl-Wigner systems for a given $E(\sigma)$. An irreducible Weyl system can be extended to an irreducible Weyl-Wigner system acting in the same space.

(ii) The representation space of a Weyl-Wigner system is always infinite-dimensional

Every Wigner operator $W^{(-)}(\alpha)$ is both unitary and self-adjoint (Weyl operators $W^{(+)}(\alpha)$ are always unitary).

In the litterature [27] on Wigner quasi-probability distributions one can find the equation

$$\mathrm{tr}(W^{(-)}(\alpha)) = 1$$

for every α. We shall see that this is essentially a correct statement, even through $W^{(-)}(\alpha)$, being a unitary operator in an infinite-dimensional Hilbert space, cannot be of trace class.

4. WEYL-WIGNER CORRESPONDENCE

From now on, $W^{(\varepsilon)}(\alpha)$ will denote an irreducible Weyl-Wigner system, acting in a Hilbert space $\mathcal{H}$.

Remember that for suitable pairs of operators in $\mathcal{H}$ one can define the Hilbert-Schmidt scalar product

$$((A,B))_{HS} = \mathrm{tr}(A^{*}B).$$

Now consider the map $A \rightarrow f_A^{(\cdot)}(.)$ from operators in $\mathcal{H}$ to functions on $E^{(2)}$, defined by

$$f_A^{(\varepsilon)}(\alpha) = ((A^{*}, W^{(\varepsilon)}(\alpha)))_{HS} \cdot \mathrm{const} \tag{1}$$

The correspondence $A \rightarrow f_A^{(\cdot)}(.)$ can be established first, say, for trace class operators A and then extended to other classes of operators and (generalized) functions.

Properties of this the correspondence are summarized by the

Theorem :

i) For given A, the functions $f_A^{(+)}$ and $f_A^{(-)}$ are symplectic Fourier transforms of each other :

$$f_A^{(+)} = F\, f_A^{(-)} \qquad\qquad f_A^{(-)} = F\, f_A^{(+)}.$$

ii) The correspondence $A \rightarrow f_A^{(\cdot)}(.)$ is inverted by

$$A = \int_E f_A^{(-)}(\tfrac{a}{2}) W^{(-)}(a)\,da = \int_E f_A^{(+)}(a) W^{(+)}(\tfrac{a}{2})\,da \qquad (2)$$

(with a suitable definition of operator-valued integrals) [28].

iii) The correspondence $A \rightarrow f_A^{(\cdot)}(.)$ is "unitary" in the sense that

$$((A,B))_{HS} = (f_A^{(+)}, f_B^{(+)})_{L^2(E)} = (f_A^{(-)}, f_B^{(-)})_{L^2(E)}.$$

Examples : Take a fixed $a_0 \in E$, and consider $A = W^{(-)}(a_0)$. Then

$$f_A^{(-)}(.) = \delta_{a_0}(\cdot) \qquad (\ \delta\text{-function at } a_0\ )$$

$$f_A^{(+)}(.) = e^{a_0}(\cdot)$$

If $B = W^{(+)}(b_0)$, then

$$f_B^{(-)}(.) = e^{b_0}(.) \quad \text{and} \quad f^{(+)}(.) = \delta_{b_0}(\cdot)$$

Now it is easier to understand the sense in which the trace of $W^{(-)}(a_0)$ is unity :

$$tr(W^{(-)}(a_0)) = ((W^{(-)}(a_0),1))_{HS} = (\delta_{a_0},1)_{L^2(E)} = 1.$$

The domain of the "inner product" on the r.h.s. has to be extended beyond $L^2(E)$ [3].

5. COHERENT STATE REPRESENTATIONS AND HOLOMORPHIC REPRESENTATIONS

[7, 13, 14, 28, 35, 36, 37, 49, 51, 59, 60].

We shall now look at a family of concrete realizations for the irreducible representation space .

An allowed complex structure [34] for $E(\sigma)$, is a linear map J such that

i) $J \in Sp(E)$ (i.e. $\sigma(Ja, Ja') = \sigma(a, a')$)

ii) $J^2 = -1$

iii) $\sigma(a, Ja) > 0$ if $a \neq 0$.

<u>Remark</u> : Any symplectic basis in E gives rise to an allowed complex structure: let $\alpha_1, \beta_1, \ldots, \alpha_\nu, \beta_\nu$ be such that $\sigma(\alpha_i, \alpha_j) = \sigma(\beta_i, \beta_j) = 0$, $\sigma(\alpha_i, \beta_j) = \delta_{ij}$. Define J by $J\alpha_i = \beta_i$, $J\beta_i = -\alpha_i$. Then J satisfies i), ii) and iii).

Given J, define

a) A "vacuum"

$$\Omega_J(a) = e^{-\pi\sigma(a, Ja)} \tag{3}$$

b) A space $\mathcal{H}_J$, consisting of complex-valued functions Φ on E, satisfying the two conditions below :

b1) Φ is of the form

$$\Phi(a) = \varphi(a)\Omega_J(a)$$

where φ is entire holomorphic with respect to J, i.e. satisfies the Cauchy-Riemann equations

$$\nabla^{Ja}\varphi = i\nabla^{a}\varphi \qquad (a \in E)$$

(Here $(\nabla^a\varphi)(b) = \left(\frac{d}{d\lambda}\varphi(b+\lambda a)\right)_{\lambda=0}$

b2) Φ belongs to $L^2(E)$, i.e. $\int |\Phi(a)|^2 da < \infty$

It can be shown that $\mathcal{H}_J$ is a <u>closed</u> subspace of $L^2(E)$, i.e. that $\mathcal{H}_J$ is a Hilbert space.

It is important to notice that the inner product in $\mathcal{H}_J$, inherited from $L^2(E)$, <u>is independent of the choice of J</u>.

In the Hilbert space $\mathcal{H}_J$ we define a family $W^{(\varepsilon)}(a)$ of operators by

$$(W^{(\varepsilon)}(a)\Phi)(b) = e^{a}(b)\Phi(\varepsilon b - \varepsilon a) . \tag{4}$$

<u>Theorem</u> [28]:

i) $\mathcal{H}_J$ is transformed onto itself by the operators $W^{(\varepsilon)}(a)$ just defined.

ii) The family $W^{(\varepsilon)}(a)$ is an irreducible Weyl-Wigner system in $\mathcal{H}_J$.

iii) On every $\Phi \in \mathcal{H}_J$, the operator $W^{(-)}(0)$ (i.e. parity) coincides with the operator F (symplectic Fourier transform)

$$W^{(-)}(0)\,\Phi = F\Phi \,.$$

Remark : Again, it is important to notice that our definition of Weyl-Wigner operators does not involve J . The same operator $W^{(\varepsilon)}(\alpha)$ can be restricted to any one of the spaces $\mathcal{H}_J$. This is a reason for working in $\mathcal{H}_J$ rather than in the holomorphic (Bargmann) representation space of functions φ , which are trivially related to Φ :

$$\varphi(v) = \Omega_J(v)^{-1}\Phi(v).$$

Analytic kernels [5, 13, 14, 15] :

Consider in $\mathcal{H}_J$ the family of coherent states, defined by

$$\Omega_J^{a} = W^{(+)}(\alpha)\Omega_J \quad (= W^{(-)}(\alpha)\Omega_J)$$

where Ω_J is the "vacuum" defined in (3).

Theorem :

If $\Phi \in \mathcal{H}_J$, then $(\Omega_J^{a}, \Phi) = 2^{-\nu}\Phi(a)$

If A is a bounded operator from $\mathcal{H}_J$ to $\mathcal{H}_{J'}$, (when J' is again allowed for σ and if $A(\alpha,b)$ is defined by

$$A(\alpha,b) = (\Omega_{J'}^{a}, A\Omega_J^{b})_{\mathcal{H}_{J'}}$$

then the action of A is given by

$$(A\Phi)(\alpha) = 4^{\nu}\int A(\alpha,b)\Phi(b)\, db \tag{5}$$

The function $A(\alpha,b)$ will be called the analytic kernel corresponding to A.

With the help of analytic kernels one can represent not only bounded operators, but much more singular objects -essentially unbounded quadratic forms defined on the linear span of the family of coherent states. For instance, a multiplication operator can be represented by an analytic kernel if the corresponding function increases at infinity not faster than an inverse gaussian. For purpose of quantum mechanics, we can safely disregard operators that cannot be represented by analytic kernels.

The analytic kernel corresponding to $W^{(\varepsilon)}(v)$ is

$$W(a, b; v) = e^{v}(a)\, e^{v}(\varepsilon b)\, e^{\varepsilon b}(a)\, \Omega_J(v + \varepsilon b - a)$$

6. CANONICAL TRANSFORMATIONS, REAL

Let us summarize, in slightly different terms, what we have done untill now ; we have considered, in coordinateless phase space, a regular representation of canonical commutation relations, which is something very elementary. It acts on the Hilbert space $L^2(E)$ of all functions square integrable on phase space. The Weyl operator $W^{(+)}(\alpha)$ is a displacement times a multiplication by $e(.)$. The regular representation is reducible ; in our case, it is an infinite sum of mutually equivalent irreducibles. In such a situation, there is a great deal of arbitrariness in a choice of an irreducible subspace, and we need irreducibility in order to play the games of everyday quantum mechanics. We have decided to look at closed invariant subspaces $\mathcal{H}_J \subset L^2(E)$ given by allowed complex structures J ; (remember that any symplectic reference frame gives rise to an allowed complex structure). Any $\mathcal{H}_J$ carries the well-known coherent state representation (i.e. the Bargmann representation in which the Gaussian sits in the wave function instead of sitting in the measure). The Weyl-Wigner operators of the regular representation restrict to any $\mathcal{H}_J$; so we can define, with the help of (1), (2), the Weyl-Wigner correspondence on the linear span of all the spaces $\mathcal{H}_J$, using the Weyl-Wigner operators of the regular representation.

Any restriction of a quantum-mechanical operator A to a space $\mathcal{H}_J$ can be viewed as a description of A in a symplectic reference frame.

What is the relationship between two restrictions of ane and the same A ?

Let J and J' be allowed, with $J' = SJS^{-1}$, where S is symplectic ; $S \in Sp(E)$. (Any two allowed J are conjugated in $Sp(E)$). We want to find a map $T = T(S)$ from $\mathcal{H}_J$ onto $\mathcal{H}_{J'}$, which intertwines the restriction $W^{(\cdot)}(.)_J$ of $W^{(\cdot)}(.)$ to $\mathcal{H}_J$ with the restriction to $\mathcal{H}_{J'}$:

$$TW^{(\varepsilon)}(a)_J = W^{(\varepsilon)}(\alpha)_{J'}T .$$

This is now a trivial task : Quite generally, an intertwining operator between representation spaces can also be described by the corresponding invariant sesquilinear functional. In our case, finding T is equivalent to finding a sesquilinear functional $\mathcal{B}(\Phi';\Psi)$ on $\mathcal{H}_{J'}\times\mathcal{H}_{J}$, such that

$$B(W^{(\varepsilon)}(\alpha)_{J'}\,\Phi'\,;\,W^{(\varepsilon)}(a)_J\Psi) = B(\Phi';\Psi)$$

for all $\Phi'\in\mathcal{H}_{J'}$, $\Psi\in\mathcal{H}_J$, $\varepsilon\in Z_2$, $\alpha\in E$. The intertwining operator T is then given by

$$(\Phi',T\Psi)_{J'} = \mathcal{B}(\Phi';\Psi)$$

It is clear how to find a B with the required properties. All the Hilbert spaces $\mathcal{H}_J$ have inherited their scalar product from $L^2(E)$, and all the Weyl-Wigner operators $W^{(\varepsilon)}(\alpha)$ are unitary in $L^2(E)$. Consequently we may take

$$\mathcal{B}(\Phi';\Psi) = (\Phi',\Psi)_{L^2(E)} = \int \overline{\Phi}'(a)\,\Psi(a)\,da$$

Remarks :

1) In order to have an entirely explicit expression for T, we may compute the corresponding analytic kernel :

$$T(\alpha,b) = (\Omega^{\alpha}_{J'},\Omega^{b}_{J})_{L^2(E)}$$

Then $(T\Phi)(\alpha)$ is given by (5).

2) If one prefers to work within one irreducible representation space $\mathcal{H}_J$, this can be done as follows : consider the unitary map $U(S)$ from $\mathcal{H}_J$ to $\mathcal{H}_{J'}=\mathcal{H}_{SJS^{-1}}$, defined by $(U(S)\Phi)(\alpha) = \Phi(S^{-1}\alpha)$. Then $R(S) = U(S)T(S)$ is unitary in $\mathcal{H}_J$ and satisfies

$$R(S)W^{(\varepsilon)}(\alpha)_J R(S)^{-1} = W^{(\varepsilon)}(S\alpha)_J.$$

3) The correspondence $S\rightarrow R(S)$ is a projective representation of $Sp(E)$. The corresponding true representation of a (two-sheeted) covering group of $Sp(E)$ (the metaplectic representation) has given rise to an extensive literature : see S. Sternberg's lecture at this conference [53], and [15], [20], [21], [41], [55], [56].

4) Starting with an affine symplectic space, one extends easily the above discussion to affine canonical transformations (inhomogeneous symplectic group).

7. CANONICAL TRANSFORMATIONS, COMPLEX [22, 25]

In order to study complex canonical transformations, it is best to start afresh, and consider an even-dimensional vector space V over complex numbers, carrying a complex -valued symplectic form σ. The group of complex-linear transformations of V that preserve σ is denoted by $Sp(V)$. This is the complex symplectic group discussed by P. Kramer at this conference.

The space V carries its natural complex structure $i : v \to iv$. As a motivation for the definitions that follow, notice that a symplectic basis (reference frame) in V defines an additional complex structure J that commutes with i. Namely if $\alpha_1, \beta_1, \dots, \alpha_\nu, \beta_\nu$ is a basis for the vector space V, such that $\sigma(\alpha_j, \beta_k) = \delta_{jk}$ and the other inner products are zero, define J :

$$J\alpha_j = \beta_j \qquad \text{and} \qquad J\beta_j = -\alpha_j \tag{6}$$

The reference frame defines also a real subspace of V, consisting of the real linear combinations of basis elements. It is convenient to treat separately additional complex structures and real subspaces of V.

<u>Definition</u> : <u>An allowed additional complex structure</u> on V is any $J \in Sp(V)$ such that $J^2 = -1$.

<u>Remarks</u> :

1) $+i$ and $-i$ are not allowed additional complex structures, since they do not belong to $Sp(V)$.

2) The form $s_J(u,v) = \sigma(u,Jv)$ is complex-valued and symmetric. We are not yet imposing on it any conditions of positive definiteness, in contrast to the real case.

Given an allowed additional complex structure J on V, consider the space $Z_J(V)$ of all complex-valued functions of the form

$$\Phi(v) = \varphi(v)\Omega_J(v)$$

where $\Omega_J(v) = e^{-\pi\sigma(v,Jv)}$ and where φ is <u>entire holomorphic with respect to both</u> i <u>and to</u> J, i.e. satisfies

$$\nabla^{iu}\varphi = i\nabla^{u}\varphi$$

and

$$\nabla^{Ju}\varphi = i\nabla^{u}\varphi$$

for every $u \in V$.

The space $Z_J(V)$ can also be characterized as follows : in the vector space V , consider the complementary projections $P_J^{(\pm)} = \frac{1}{2}(1 \pm iJ)$. Call $P_J^{(+)}V$ <u>the creation subspace</u> and $P_J^{(-)}V$ <u>the annihilation subspace</u> (with respect to J) ; the names will be justified presently. A function Φ belongs to $Z_J(V)$ if and only if it is entire holomorphic with respect to i , and satisfies the following covariance condition along the creation subspace :

$$\Phi(u - v_+) = e^{2i\pi\sigma(v_+,t)}\Phi(u) \qquad (7)$$

$$(u \in V , \quad v_+ \in P_J^{(+)}V).$$

The subspaces $P_J^{(+)}V$ and $P_J^{(-)}V$ are maximal isotropic for σ . In the language of geometric quantization theory, [19] , [21] , [62], the decomposition $V = P_J^{(+)}V + P_J^{(-)}V$ is a <u>complex polarization</u>.

On $P_J^{(-)}V$, the complex structures i and J coincide. The covariance condition (7) allows us to identify $Z_J(V)$ with the space of all i (or J) -holomorphic functions, of arbitrary growth, on the complex subspace $P_J^{(-)}V \subset V$.

If Φ is any i-holomorphic function on V , if $\varepsilon \in Z_2$ and if $v \in V$, define $W^{(\varepsilon)}(v)$ by

$$(W^{(\varepsilon)}(v)\Phi)(u) = e^{2i\pi\sigma(v,u)} \quad (\varepsilon u - \varepsilon v)$$

<u>Proposition</u> :
If $\Phi \in Z_J(V)$, then, for every $\varepsilon \in Z_2$ and every $v \in V$, the function $W^{(\varepsilon)}(v)\Phi$ also belongs to $Z_J(V)$.

We have so defined <u>complex</u> Weyl-Wigner operators $W^{(\varepsilon)}(v)$ in the (topological) vector spaces $Z_J(V)$ and will soon define them in a suitable partial inner product space. Intuitively, the $W^{(\varepsilon)}(v)$ are displacements and symmetries by complex momenta and coordinates. Infinitesimal generators of $W^{(+)}(v)$ along the creation (resp. annihilation) subspaces are easily shown to correspond to the usual creation (resp. annihilation) operators ; hence the names.

For every $v \in V$, define the coherent state Ω_J^v as $\Omega_J^v = W^{(+)}(v)\ {}_J$.

As in the real case, the Weyl-Wigner operators are defined independently of J ; they restrict to every $Z_J(V)$.

In contrast to the real case, we have as yet no way to speak about adjoints, or to compute matrix elements. In order to do so, and in order to understand the relationships between the real and complex case, we need some additional definitions.

A complex conjugation for the complex symplectic space $V(\sigma)$ is a real-linear map τ such that

i) $$\tau^2 = 1$$

ii) $$\tau(iv) = -i\tau v$$

and

iii) $$\sigma(\tau u, \tau v) = \overline{\sigma}(u,v)$$

$(u \in V, \ v \in V)$ (complex conjugate in C).

A complex conjugation in V defines a "real subspace" $E_\tau = \{v | \tau v = v\}$.

Now let J be an allowed additional complex structure in V. We shall say that J and τ are matched if

a) $$J\tau = \tau J$$

and

b) $$\sigma(\tau v, Jv) > 0$$

whenever $v \neq 0$.

Let $\alpha_1, \cdots \beta_\nu$ be a reference frame in V, and J the additional complex structure defined by (6). Define in V a complex conjugation by $\tau(\lambda\alpha_j) = \overline{\lambda}\alpha_j$, $\tau(\lambda\beta_j) = \overline{\lambda}\beta_j$ and real linearity. Then J and τ are matched.

Remark : With the help of a matched pair one can make V into a finite-dimensional (right) Hilbert space over quaternions ; the form σ becomes the "complex symplectic part" of the quaternion-valued inner product. This is analogous to the well-known construction in which a real symplectic form appears as the imaginary part of a complex Hilbert scalar product on real phase space.

Let J, τ be a matched pair. Then E_τ (with the restriction of σ to E_τ) is a real symplectic space. The restriction of J to E is an allowed complex structure in the sense of Sec.5. We define consequently a Hilbert space $\mathcal{H}_{J\tau}$ as follows :

The elements of $\mathcal{H}_{J\tau}$ are functions $\Phi \in Z_J(V)$ that are square integrable when restricted to E_τ :

$$\int |\Phi(u)|^2 d_\tau \xi < \infty$$

where $d_\tau \xi$ is the invariant measure on E_τ , normalized according to Sec 2.

For all $\alpha \in E_\tau$, the Weyl-Wigner operators $W^{(\varepsilon)}(\alpha)$ restrict to unitary operators $W^{(\varepsilon)}(\alpha)_{J\tau}$ in $\mathcal{H}_{J\tau}$. So each $\mathcal{H}_{J\tau}$ carries a perfectly conventional quantum mechanics, based on this Weyl-Wigner system.

If $u \in V$ does not belong to E , then $W^{(\varepsilon)}(u)$ is unbounded in $\mathcal{H}_{J\tau}$. The adjoint of this restriction is

$$(W^{(\varepsilon)}(u)_{J\tau})^* = W^{(\varepsilon)}(-\varepsilon\tau u)_{J\tau}.$$

The unbounded operator $W^{(\varepsilon)}(u)_{J\tau}$ in $\mathcal{H}_{J\tau}$ can be represented by an analytic kernel.

Remark : It is convenient to consider the space of all $\Phi \in Z_J(V)$ such that $\int |\Omega^a \Phi| d\xi_\tau < \infty$ for all $\alpha \in E_\tau$, and to extend the inner product in $\mathcal{H}_{J\tau}$ to suitable pairs of vectors in this space. One obtains so a partial inner product space ([3]), which is the natural domain for complex Weyl-Wigner operators -from the point of view of $\{\tau J\}$.

As in the real case, we shall now look at one and the same quantum-mechanical operator from various reference frames. For the sake of definiteness, consider a Weyl-operator $W^{(+)}(\alpha)$ ($\alpha \in V$). Let $\{J, \tau\}$ be a matched pair (the relevant part of a reference frame) ; let $S \in Sp(V)$, and let $\tau' = S\tau S^{-1}$, $J' = SJS^{-1}$. (The point α need not belong to either of the subspaces E_τ, $E_{\tau'}$). We want to compare $W^{(+)}(\alpha)_{J\tau}$ and $W^{(+)}(\alpha)_{J'\tau'}$. The results can be described as follows :

Consider in $\overline{\mathcal{H}_{J'\tau'} \times \mathcal{H}_{J\tau}}$ the set of pairs $\{\Phi', \Psi\}$ ($\Phi' \in \mathcal{H}_{J'\tau'}$, $\Psi \in \mathcal{H}_{J\tau}$ such that the function $\overline{\Phi}'(\tau' v)\Psi(v)$ is absolutely integrable over every affine subspace $\alpha + E_\tau$, parallel to E_τ . On this set of pairs, define the sesquilinear functional

$$B(\Phi', \Psi) = \int \overline{\Phi}'(\tau' v)\ \Psi(v)\ d_\tau \xi$$

If there exists an operator T from (a domain in) $\mathcal{H}_{J\tau}$ to $\mathcal{H}_{J'\tau'}$ such that

$$B(\Phi', \Psi) = (\Phi', T\Psi)_{\tau'}$$

then T intertwines $W^{(+)}(\alpha)_{J\tau}$ and $W^{(+)}(\alpha)_{J'\tau'}$:

$$T\, W^{(+)}(\alpha)_{J\tau} = W^{(+)}(\alpha)_{J'\tau'} T$$

for every $\alpha \in V$.

The functional B and the corresponding operator T are in general unbounded. The reason is, essentially, that the integration domain E_{τ} is "slanted" with respect to $E_{\tau'}$; this brings about additional growth in $\Phi(\tau' v)$.

It is not difficult, however, to find conditions under which $T(S)$ is represented by an analytic kernel, and to compute that kernel.

Complex canonical transformations and spectral properties of quantum-mechanical observables.

Any quantum-mechanical observable is a superposition of Weyl (or Wigner) operators ; consequently, it restricts to any one of the spaces
The restrictions are intertwined by the unbounded (a fortiori non-unitary) operators $T(S)$.

If we take this seriously, then the natural object of study is not one operator in a fixed Hilbert space, but rather the "analytic" family of restrictions $A_{J\tau}$ of an observable A. Such families have already been studied for particular subgroups of (inhomogenous) Sp(V) [1] [12] [24] [52] [8] [11] and special observables. The resulting spectral theory is in several ways more "physical" than the orthodox one. A related point of view can be found in [61].

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