Coherent Structures in Turbulent Flow

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It is customary to say that the problem of turbulence remains the outstanding challenge of fluid dynamics, but in fact the concerted efforts of many scientists and engineers over the years have led to a considerable body of useful knowledge. The three processes of cataloguing, curve fitting and 'postdicting' are well under control. If there is a problem, it is partly because we now ask harder questions and are more concerned with prediction and control; or we hanker for the El Dorado of a general theory and the chimaera of simple understanding.

The purpose of this meeting is to consider if coherent structures are the road to the promised land, to decide if we are right to retreat from the statistical approaches of energy spectra, correlations, the Reynolds equation, etc., and to examine critically the prospects for a deterministic approach to turbulent flow. Put briefly, is the understanding and description of vortex dynamics the key to turbulence?

The concept of coherent structure or organized eddy is of course not exactly new. In a sense, the mixing length theories of Taylor and Prandtl, in which momentum or vorticity is convected by a blob of fluid, use a vague kind of physical entity as the basis for transport or mixing. More formally, Synge & Lin [1] proposed 37 years ago a model of isotropic turbulence as a random superposition of Hill Spherical Vortices (H.S.V's). The H.S.V. is an exact solution of the Euler equations (in fact it would be an exact solution of the Navier-Stokes equations except that the viscous stresses are not continuous across the boundary) in which the vorticity is confined to the interior of a sphere, the only non-zero component being the azimuthal with magnitude $(15U/2R^2)y$, where $U = |\underset{\sim}{U}|$ is the speed and R the radius of the sphere. Associated with the H.S.V. is a kinetic energy $(10/7)\rho\pi U^2R^3$, and a hydrodynamic impulse $\underset{\sim}{I} = 2\rho\pi R^3\underset{\sim}{U}$. The velocity field at a point $\underset{\sim}{r}$ relative to the vortex center can be written in the form

$$\underset{\sim}{u}_{HSV}(\underset{\sim}{r},\underset{\sim}{U},R) = U\,A(r/R)\underset{\sim}{r} + B(r/R)\underset{\sim}{U}.$$

Then Synge & Lin expressed the velocity field of the fluid as

$$\underset{\sim}{u}(\underset{\sim}{x}) = \iiint W(R,\underset{\sim}{\xi},\underset{\sim}{U})\, \underset{\sim}{u}_{HSV}(\underset{\sim}{x}-\underset{\sim}{\xi},\underset{\sim}{U},R)\, dR\, d\underset{\sim}{\xi}\, d\underset{\sim}{U}$$

where W is a random function giving the probability density of finding at the point $\underset{\sim}{\xi}$ a H.S.V. of radius R moving with speed $\underset{\sim}{U}$. From this expression, can be synthesized all the statistical properties of $\underset{\sim}{u}$. This appears to be the first formal kinematic expression of a turbulent velocity field as the random superposition of laminar structures or coherent vortices.

It is ironic that the behavior $f(r) \sim 1/r^3$ for the asymptotic behavior of the longitudinal velocity correlation was predicted in this way in 1943 but ignored, while the statistical theory which assumed exponential decay considered the Loitsianskii invariant $\int_0^\infty r^4 f(r)dr$, which of course does not exist if $f(r) \sim r^{-3}$. Later work confirmed the correctness of the r^{-3} prediction and provided a belated recognition of the possible advantages of physical models over purely statistical considerations.

A later development was Townsend's [2] attempt in 1951 to construct a physical model for the dissipation eddies in isotropic turbulence as a random superposition of vortex sheets and tubes, stimulated in part by the success of Kolmogorov k,ε,ν scaling (k = wave number, not kinetic energy) for the velocity covariance and energy spectrum at Reynolds numbers too small for the assumptions of the theory to be valid, and in part by the strong implications from the Batchelor & Townsend [3] observations, confirmed later by numerous other measurements (see Van Atta & Antonia [4] for a recent survey) that flatness factors of high order are large and strongly Reynolds number dependent This suggests that the energy dissipating regions are spatially intermittent, as predicted on theoretical grounds by Landau [5]. The basic structures employed by Townsend were the equilibrium viscous vortex sheet and tube (apparently due to Bergers and rediscovered several times) in an irrotational straining field

$$\omega_x = \omega_o\, e^{-\alpha y^2/2\nu},\quad u = \alpha x,\quad v = -\alpha y,\quad w = -\int \omega_x\, dy$$

or

$$\omega_x = \omega_o\, e^{-\alpha r^2/2\nu},\quad u = 2\alpha x,\quad v_r = -\alpha r,\quad v_\theta = \frac{\nu\omega_o}{\alpha}\,(1-e^{-\alpha r^2/2\nu})$$

where $r = (x^2+y^2)^{\frac{1}{2}}$. Random superposition of sheets and tubes (by a formula like that for H.S.V's) gives reasonable agreement with fine scale spectra observations. More sophisticated developments of Townsend's idea have been tried by several authors, e.g. Saffman [6] and Tennekes [7]. Saffman's appears to be the most detailed, and was an attempt to give physical significance to the Taylor micro-scale $\lambda = (5q^2)^{\frac{1}{2}}/\varepsilon$ and Kolmogorov length $\ell = (\nu^3/\varepsilon)^{\frac{1}{4}}$. The former was envisaged as an internal boundary layer between large eddies (e.g. H.S.V's) and the latter a scale produced by Taylor-Gortler instability of the internal boundary layers.

However, none of these physical or deterministic models can be regarded as being satisfactory or more than suggestive speculation. There is no real agreement on the kinematics of the structures, and they contain no dynamics. Also their predictions (e.g. for the dependence of flatness factors on Reynolds number) are inferior to those obtained by non-physical statistical dimensional analysis. Thus it cannot be said that the concept of coherent structures has been particularly useful or productive to date in the study of homogeneous turbulence. Its main value has been perhaps to suggest another point of view, in which the dynamics of vorticity is emphasized, spatial structure is important, and the possibility that Fourier components have strongly correlated phases is recognized. The concept found its real support in the studies of shear flow turbulence, most spectacularly in the turbulent mixing layer, and there is little doubt that the now classic work of Brown and Roshko was the impetus that changed a marginal activity into an active industry.

The remaining candidate to date for the role of building block or fundamental particle for a coherent structure is the vortex ring or more generally the vortex filament. The classical vortex ring is the thin torus filled with uniform vorticity, which propagates with speed

$$U_V = \frac{\Gamma}{4\pi R}\left(\log\frac{8R}{\delta} - \frac{1}{4} + O\left(\frac{a^2}{R^2}\right)\right),$$

where R is the radius of the ring, δ the radius of the core, and Γ the circulation about the axis. The vortex ring and H.S.V. are actually the opposite ends of a continuous family of finite cored vortex rings, which were computed by Norbury [8]. The vortex ring can also be regarded as the special case of a vortex filament, or thin tube of vorticity, in the shape of a circle. The velocity of a filament is given to leading order by the Biot-Savart law (or equivalent)

$$\underset{\sim}{u} = \frac{\Gamma}{4\pi}\oint \frac{d\underset{\sim}{s} \times \underset{\sim}{r}}{r^3}$$

where $\not{f}$ means that a cut-off, of length proportional to δ, has to be employed in the evaluation of the integral for a point on the filament. (A careful and thorough study of the equations of motion of a vortex filament, going to higher order than the Biot-Savart law and discussing in addition the equations for the internal structure, which can have an $O(1)$ effect on the speed, was carried out by Moore & Saffman [9]).

The use of such vortex structures as models for the quasi-permanent coherent structures of a turbulent flow makes the study of their interactions and stability a matter of pressing interest. There has been some fascinating work on the interaction of vortex rings and filaments, showing remarkably strange behavior (e.g. Oshima & Asaka [10], Oshima [11]), which indicates that the dynamics of interacting vortex rings is a far from simple problem. In particular, the breaking and reforming of vortex lines (clearly noticeable for instance in condensation trails [12]) seems to play a vital role in the interactions of vortex rings, and we have no idea as yet of the basic processes involved. There has, however, been much work in the related problem of magnetic field line reconnection in magnetohydrodynamics which may be relevant, e.g. [13].

The three dimensional stability of pairs or arrays of line vortices needs further investigation in order to clarify the interaction properties of coherent structures. The long axial wavelength disturbances with $k\delta << 1$ (k = axial wavenumber, δ = core radius) can be studied using the Biot-Savart law (with a suitable cut-off for the self induction) to calculate the speeds of the deformed vortices. This was done by Schlayer [14] and Rosenhead [15] for the Karman vortex street, but these papers are obscure and it is not clear exactly what their results are. Simpler treatments of a pair of counter-rotating and co-rotating vortices were carried out by Crow [16] and Jimenez [17], respectively. These results are rather interesting. Crow found weak instability for sufficiently small k while Jimenez found that the configuration was always stable under the long wave approximation. These results are consistent with the apparent slow instability to three-dimensional disturbances of the Karman vortex street, which contains vortices of both signs, while the mixing layer, which contains coherent vorticity of just one sign, appears to be stable to long wave instabilities (other than the pairing instability) and is accordingly a more permanent structure. The observations of trailing vortices show that counter-rotating vortices break and reform into loops if the instability grows to finite amplitude, and a similar phenomenon can perhaps be expected in the Karman vortex street. The relevance or importance of the short wave parametric instability of vortex filaments with $k\delta = O(1)$, see Moore & Saffman [18], is not completely certain but may be related to the spanwise structures seen in the mixing layer.

Pierrehumbert [19] has recently calculated the two and three dimensional linear stability of the inviscid mixing layer described by the stream function

$$\psi = \frac{1}{2} \log(\cosh 2y - \rho \cos 2x) \quad .$$

As ρ goes from 0 to 1, this flow field changes from a hyperbolic tangent to an array of point vortices distance π apart. A good overall fit with experimental profiles is claimed for $\rho = 0.25$. This flowfield exhibits the subharmonic (wavelength 2π) pairing instability, with growth rate a slowly increasing function of ρ. Thus, it appears that finite core size may decrease slightly the pairing instability growth rate of a row of point vortices, but not by any significant amount. It was also found that the instability persists when the disturbance is three-dimensional, but is most unstable when two-dimensional and stabilizes when the spanwise wavelength is comparable to the vortex separation. The parametric instability of a straight filament corresponds to disturbance of wavelength π, in which all vortices are disturbed in the same way. The disturbance is neutrally stable when two-dimensional, and has a maximum growth rate when its spanwise wavelength is comparable to vortex separation. However, it is uncertain how typical these results are. Saffman & Szeto's [20] investigation of the two-dimensional stability of an array of uniform finite cored vortices showed stability to disturbances of wavelength π until the vortices reached a certain size when they became unstable, but their arguments were qualitative and they did not calculate actual eigenvalues.

The extent to which the different and various aspects of turbulent flow can be modelled or understood as a collection of H.S.V's, or vortex tubes and sheets, or vortex rings and filaments, or alternative large coherent flow structures, or large number of two-dimensional point vortices is one of the problems which this meeting will address. There is one significant point which should perhaps be made. Apart from the difficulty in dealing adequately with the dynamical interactions of the vortex structures, which raises some challenging mathematical problems, and incorporating viscosity, instabilities and 'fine scale turbulence', it may not be a too good approximation to assume that the vortex structures are surrounded by irrotational fluid. We perhaps need a theory for the structure of vortices in a background of weak vorticity. Some first steps have been taken by Kiya & Arie [21], [22].

An interesting comment has also been made by Taganov & Dudoladov [23] who have modelled the turbulent mixing layer by finite and point vortices. They suggest that two classes of vortex formation should be considered. The first class comprises structures which are essentially inviscid, and are relevant to jets, wakes, mixing layers, etc. While the second class contains those structures in which viscosity plays a large role in the momentum transfer and are modelled by low Reynolds number solutions. Those of the second class would satisfy no-slip boundary conditions at solid walls and be relevant to the viscous sublayer, etc.

The experimental study of the coherent structure is most firmly established in the mixing layer, boundary layer, jet and wake. Theoretical models are also developed most extensively for these flows, as we shall be hearing in the following talks, and considerable progress has been made in describing and understanding these flows in terms of vortex interactions. These are, however, free flows, apart from the boundary layer. There is also evidence of coherent or organized structure in bounded flows, which in fact predates the discovery of free flow phenomena, and it is worth suggesting that theoreticians should perhaps also direct their efforts to these. Coles [24] in 1961 described the intermittency and periodic turbulent flow structures in pipe flow and circular Couette flow. There has lately been renewed interest in the Taylor vortices at large Taylor number, e.g. Koschmieder [25]. To what extent the intermittent and periodic nature of these bounded flows can be explained in terms of coherent structures or vortex dynamics remains to be elucidated.

One aspect of vortex dynamics that will be mentioned explicitly here is the information that can be gathered from steady solutions of the Euler equations. The philosophy behind coherent structures is that at least the gross features of the flow are determined by the interaction of vorticity, and thus the possible equilibrium configurations are of interest. These have been calculated recently by Saffman & Szeto [20] and independently by Pierrehumbert & Widnall [26] for a linear array of uniform vortices which models the turbulent mixing layer. Equilibrium shapes were computed and it was found that solutions only exist if the vortices are not too large. For vortices with uniform constant vorticity, the area A has to be less than $.238L^2$, where L is the distance between centers. This provides support for a deterministic model of the evolution of the coherent structures in a shear layer, in which the vortices grow by the continuous ingestion of irrotational fluid until they reach such a size that they can no longer exist, at which stage tearing occurs and the array is reformed with a larger L and A but smaller A/L^2, and the process repeats. Of course, a lot of questions are left unanswered, the most important being the rate or frequency of the process. An alternative model is to recognise the instability of the array to the pairing instability in which neighboring vortices rotate around each other and coalesce. The objection here that there is no coalescence mechanism is answered by further work of Saffman & Szeto [27] on the steady states of two rotating vortices. They showed that for this flow there are no steady solutions when $L < 1.8A^{\frac{1}{2}}$, and that since the process of pairing will reduce the separation to less than this, it is expected that coalescence can take place. Both models make the same kind of quantitative prediction, giving values of a/b around 2 and δ_ω/L around 0.3. While there is clearly some uncertainty in deciding what is the basic physical process, it does appear that the processes of coalescence and mixing layer growth are consistent with the inviscid dynamics of vorticity.

The deterministic approach is certainly an exciting one, holding out the possibility of a fundamental advance in understanding. The search for quasi-permanent coherent structures and the study of their interactions promise to be challenging tasks. But there are clearly serious obstacles. It is not yet easy to make quantitative predictions. The most glaring omission is the inability to predict in any reasonable way the logarithmic law of the wall of the turbulent boundary layer. The steady Reynolds stress equation (and the associated 'invariant modelling') would certainly seem to be irrelevant, having as much physical content as the statement that the average vehicle has 3.4 wheels or the average family 2.7 children, but perhaps a conditionally sampled Reynolds equation or an integro-differential equation for the probability distribution function based on the pioneering work of Lundgren and Monin would be a useful subject for study. A recent step in this latter direction has been made by Onufriyev [28]. Such an approach would certainly not be inconsistent with intermittency if a bimodal probability distribution were employed.

One issue that needs to be carefully considered is the role of the computer in theoretical turbulence research. One day the computer may be able to answer all questions, or more precisely predict any number that is required (whether this constitutes 'understanding' is a subjective question), but this is still in the distant future. At the moment, the computer can churn out millions of random numbers, as in sub-grid modelling or following clouds of point vortices, or it can be used to solve specific equations for well posed models, such as those that give the structure of vortex rings or steady vortex arrays. It is hoped that this meeting will clarify its role in the coherent structure approach to turbulence.

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