SELF-CONSISTENT COLLECTIVE COORDINATE METHOD FOR THE MAXIMALLY DECOUPLED COLLECTIVE MOTION

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1. INTRODUCTION

In recent years, the concept of a collective path or, more generally, of a collective subspace in the many-particle Hilbert space has been increasingly arousing interest in attempts of microscopic description of the large amplitude collective motion like vibrations of soft nuclei, fission, heavy-ion reactions, etc., and the problem of how to determine the collective subspace (rather than assuming it a priori) has been discussed by many authors[1)∿8)].

The main purpose of this talk is to develop a new quantum theory which is capable by itself of determining the maximally decoupled collective subspace as well as a maximally decoupled collective Hamiltonian, on the basis of a fundamental principle called the invariance principle of the Schrödinger equation[4),6),7),8)].

First, the basic ideas of the theory are explained by formulating the invariance principle within the framework of the time-dependent Hartree-Fock (TDHF) theory[7)]. In order to overcome the essential drawback of the TDHF theory that the theory itself does not give any prescription how to quantize the "classical" collective motion derived by the theory, we then formulate the invariance principle within the framework of the unitary-transformation method with auxiliary variables[9),10),11)]. It is demonstrated that the requirement of the invariance principle of the Schrödinger equation is completely equivalent to requiring existence of an invariant collective subspace of the Hamiltonian, and the unitary-transformation method with auxiliary variables associated with the invariance principle enables us to develop a full quantum theory of the maximally decoupled collective motion.

2. BASIC IDEAS
- SEMI-CLASSICAL THEORY WITHIN THE FRAMEWORK OF THE TDHF[7) -

An essential element in the study of the microscopic dynamics underlying the collective mode of motion is to find the "particle-collective coupling" which is just what organizes self-consistently the collective modes under consideration out of the particle excitations. A clue for such an investigation is provided by the cranking model[12) for the collective rotation. In the cranking model, one considers the particle motion in such a mean field with a deformed equilibrium that is uniformly rotating with frequency ω_{rot}. In the rotating (time-dependent) coordinate frame, the "particle-collective coupling", which organizes the collective rotation out of the particle excitations, manifests itself as the Coriolis coupling $-\vec{\omega}_{rot}\cdot\vec{J}$. ($\vec{J}$ is the total angular-momentum operator of particles, and we use $\hbar=1$ here and hereafter.) This coupling in the rotating frame gives rise to an increase in the energy of the particle motion, which is identified with the collective rotational energy,

$$\mathcal{H}_{rot}(\omega_{rot}) \equiv \langle\phi(\omega_{rot})|\hat{H}|\phi(\omega_{rot})\rangle_\beta - \langle\phi_0|\hat{H}|\phi_0\rangle_\beta \quad . \tag{2.1}$$

Here $|\phi_0\rangle_\beta$ is the Hartree-Fock ground state with deformation β and $|\phi(\omega_{rot})\rangle_\beta$ is the Hartree-Fock state in the rotating frame satisfying

$$\delta\langle\phi(\omega_{rot})|\hat{H} - \vec{\omega}_{rot}\cdot\vec{J}|\phi(\omega_{rot})\rangle_\beta = 0 \quad , \tag{2.2}$$

where $\hat{H}' \equiv \hat{H} - \vec{\omega}_{rot}\cdot\vec{J}$ is the time-displacement operator describing the time-evolution of the system in the rotating frame.

Now let us generalize this idea in order to describe the large-amplitude collective motion around the Hartree-Fock ground state $|\phi_0\rangle$ with a "spherical" equilibrium. An essential difference of this case from the above cranking model is in the fact that we do not know, in advance, the "particle-collective coupling" organizing the large amplitude collective vibration under consideration, out of the particle excitations. In order to find the "particle-collective coupling" in this case, we therefore introduce a generalized moving frame by a time-dependent unitary transformation with a finite number of pairs of parameters $\{\eta_r^*(t), \eta_r(t)\}$,

$$|\phi(\eta_r{}^*(t),\eta_r(t))\rangle = \hat{U}^{-1}(\eta_r{}^*(t),\eta_r(t))|\phi_0\rangle,$$
$$\hat{U}^{-1} = \exp\{i\hat{G}_0(\eta_r{}^*(t),\eta_r(t))\}, \tag{2.3}$$

where $|\phi(\eta_r^*(t),\eta_r(t))\rangle$ is the Hartree-Fock state in the moving frame. The set of parameters $\{\eta_r^*(t),\eta_r(t)\}$, which corresponds to a c-number version of collective bosons in the Heisenberg representation

$$\eta_r{}^*(t) \leftrightarrow b_r^+(t) \equiv \exp\{i\mathbb{H}_B t\} b_r^+ \exp\{-i\mathbb{H}_B t\} , \tag{2.4}$$

with a collective boson Hamiltonian $\mathbb{H}_B(b_r^+,b_r)$, specifies the time-dependent variations of the mean field associated with the collective motion described by a set of collective coordinates $\alpha_r(t)$ and their conjugates $\pi_r(t)$,

$$\alpha_r(t) = \frac{1}{\sqrt{2}}(\eta_r^*(t)+\eta_r(t)), \quad \pi_r(t) = \frac{i}{\sqrt{2}}(\eta_r^*(t)-\eta_r(t)). \tag{2.5}$$

In order to simplify the presentation of the theory, hereafter we restrict ourselves to a single pair of parameters $(\eta^*(t),\eta(t))$.

Since the time-dependence of the parameters $(\eta^*(t), \eta(t))$ completely specifies the time-dependence of $|\phi(\eta^*,\eta)\rangle$ in Eq. (2.3), we have

$$i\frac{\partial}{\partial t}|\phi(\eta^*,\eta)\rangle = \{i\dot{\eta}\cdot\hat{O}_0^+(\eta^*,\eta)-i\dot{\eta}^*\hat{O}_0(\eta^*,\eta)\}|\phi(\eta^*,\eta)\rangle, \tag{2.6}$$

where the operator $\hat{O}_0{}^+(\eta^*, \eta)$ is the local infinitesimal generator with respect to η, and is defined by

$$\hat{O}_0{}^+(\eta^*,\eta) \equiv \{\frac{\partial}{\partial\eta} \hat{U}^{-1}(\eta^*,\eta)\}\hat{U}(\eta^*,\eta). \tag{2.7}$$

In order to determine the motion of the moving frame specified by the time-dependence of $(\eta^*(t), \eta(t))$ as well as the structure of the operators $(\hat{O}_0{}^+(\eta^*,\eta), \hat{O}_0(\eta^*,\eta))$, we employ the <u>invariance principle of the Schrödinger equation</u>[4),6),7),8)] which specifies the concept of the <u>maximally decoupled</u> collective path. The principle can be simply stated as follows: <u>The time-dependence of the parameters $(\eta^*(t),\eta(t))$ must be introduced so as to keep the Schrödinger equation invariant.</u> Within the framework of the Hartree-Fock theory, the principle is expressed as

$$\delta_0\langle\phi(\eta^*,\eta)|\{(i\frac{\partial}{\partial t} - \hat{H})|\phi(\eta^*,\eta)\rangle\} = 0, \quad \text{and} \quad \text{h.c.}, \tag{2.8}$$

with the boundary conditions $\hat{U}^{-1}(\eta^*=0,\ \eta=0) = 1$ and $\delta\langle\phi_0|\hat{H}|\phi_0\rangle = 0$ at $\eta^* = \eta = 0$, where the variation $|\delta_0\phi(\eta^*,\eta)\rangle$ is defined by

$$|\delta_0\phi(\eta^*,\eta)\rangle \equiv \hat{U}^{-1}(\eta^*,\eta)|\delta\phi_0\rangle .$$

With the use of Eq. (2.6), Eq. (2.8) can be written as

$$\delta_0\langle\phi(\eta^*,\eta)|\hat{H}-i\dot{\eta}\hat{O}_0^+(\eta^*,\eta)+i\dot{\eta}^*\hat{O}_0(\eta^*,\eta)|\phi(\eta^*,\eta)\rangle = 0,\ \text{i.e.,}$$

$$\delta\langle\phi_0|e^{-i\hat{G}_0}\hat{H}e^{i\hat{G}_0}-i\dot{\eta}e^{-i\hat{G}_0}\frac{\partial}{\partial\eta}e^{i\hat{G}_0}-i\dot{\eta}^*e^{-i\hat{G}_0}\frac{\partial}{\partial\eta^*}e^{i\hat{G}_0}|\phi_0\rangle = 0 \ . \qquad (2.9)$$

Corresponding to Eq. (2.2), thus, the "particle-collective" coupling in the moving frame, which is highly nonlinear in general, manifests itself as the coupling $\hat{H}'-\hat{H} \equiv -i\dot{\eta}\hat{O}_0^+(\eta^*,\eta)+i\dot{\eta}^*\hat{O}_0(\eta^*,\eta)$. This coupling gives rise to an increase in the energy of the particle motion, which is identified with the energy of the collective motion

$$\mathcal{H}_0(\eta^*,\eta) \equiv \langle\phi(\eta^*,\eta)|\hat{H}|\phi(\eta^*,\eta)\rangle-\langle\phi_0|\hat{H}|\phi_0\rangle . \qquad (2.10)$$

This corresponds to the c-number version of the collective boson Hamiltonian $\mathbb{H}_B(b^+,b)$ in Eq. (2.4); $\mathcal{H}_0(\eta^*,\eta) \leftrightarrow \mathbb{H}_B(b^+,b)$.

At this stage, we require the following <u>self-consistency condition</u> on the collective motion: The collective Hamiltonian identified by Eq. (2.10) must self-consistently determine the time-dependence of the moving frame (i.e., the time-dependence of $(\eta^*(t),\ \eta(t))$) by the canonical equation of motion

$$i\dot{\eta} = \frac{\partial}{\partial\eta^*}\mathcal{H}_0(\eta^*,\eta), \qquad -i\dot{\eta}^* = \frac{\partial}{\partial\eta}\mathcal{H}_0(\eta^*,\eta) \ . \qquad (2.11)$$

It is easily shown[7)] that this self-consistency condition is fulfilled when we choose the parameters (η^*,η) so as to satisfy

$$\langle\phi(\eta^*,\eta)|[\hat{O}_0(\eta^*,\eta),\hat{O}_0^+(\eta^*,\eta)]|\phi(\eta^*,\eta)\rangle$$

$$= \langle\phi_0|[e^{-i\hat{G}_0}\frac{\partial}{\partial\eta}e^{i\hat{G}_0},e^{-i\hat{G}_0}\frac{\partial}{\partial\eta^*}e^{i\hat{G}_0}]|\phi_0\rangle = 1. \qquad (2.12)$$

It can be also verified that such a choice of the parameters (η^*,η) is generally possible. (See Appendix C of Ref. (7)). It is now clear that the problem to solve a set of the basic equations (2.9), (2.10), (2.11) and (2.12) self-consistently can be reduced to finding the hermitian operator $\hat{G}_0(\eta^*,\eta)$ satisfying these equations.

One of the simplest ways to determine $\hat{G}_0(\eta^*,\eta)$ is a method called an <u>(η^*,η)-expansion</u>[7]: Since $\hat{G}_0(\eta^*,\eta)$ is restricted to be a one-body operator, we can make the following expansion of $\hat{G}_0(\eta^*,\eta)$ with respect to (η^*,η);

$$\hat{G}_0(\eta^*,\eta) = \Sigma_\lambda\{g_\lambda(\eta^*,\eta)\hat{X}_\lambda^+ + g_\lambda^*(\eta^*,\eta)\hat{X}_\lambda\},$$

$$g_\lambda(\eta^*,\eta) = \sum_{n\geqq 1} g_\lambda(n), \quad g_\lambda(n) \equiv \sum_{(r+s\geqq n)} {}_{r,s}g_\lambda(r,s)(\eta^*)^r(\eta)^s, \tag{2.13}$$

where we have used the complete set of the RPA (or the Tamm-Dancoff) eigenmodes $\{\hat{X}_\lambda^+,\hat{X}_\lambda\}$ instead of the particle-hole pairs $\{c_\mu^+c_i,\ c_i^+c_\mu\}$,

$$\langle\phi_0|[\hat{X}_\lambda,\hat{X}_{\lambda'}^+]|\phi_0\rangle = \delta_{\lambda\lambda'}, \quad \langle\phi_0|[\hat{X}_\lambda,\hat{X}_{\lambda'}]|\phi_0\rangle = 0,$$
$$\langle\phi_0|[\hat{X}_\lambda,[\hat{H},\hat{X}_{\lambda'}^+]]|\phi_0\rangle = \omega_\lambda\delta_{\lambda\lambda'}, \quad (\omega_\lambda > 0). \tag{2.14}$$

(We adhere to the convention of denoting occupied single-particle orbits by the indices $i,j,\cdots$, and unoccupied single-particle orbits by the indices $\mu,\nu,\cdots$.) Since the basic equations (2.9) with (2.11) and (2.12) with the (η^*,η)-expansion (2.13) are supposed to be valid for continuous ranges of η^* and η, we can equate the coefficients of each power of (η^*,η) in these equations to zero. Thus, by starting with the coefficients with the lowest power of (η^*,η) and by proceeding to the higher (η^*,η)-coefficients step by step, we can determine the unknown quantities $g_\lambda(r,s)$ of $\hat{G}_0(\eta^*,\eta)$ in Eq. (2.13) as well as the collective Hamiltonian $\mathcal{H}_0(\eta^*,\eta)$ self-consistently. The important task in this expansion method is to set up some specific conditions which characterize the collective motion under consideration in "small-amplitude" (i.e., small η) limit. In the case of the collective band whose lowest excited state in the small-amplitude limit is described by the lowest-energy eigenmode $\hat{X}_{\lambda_0}^+$ in Eq. (2.14), we may set up the condition so as to satisfy $g_\lambda(1,0) = g_\lambda(0,1) = 0$ for $\lambda \neq \lambda_0$.

3. UNITARY-TRANSFORMATION METHOD WITH AUXILIARY BOSONS ASSOCIATED WITH THE INVARIANCE PRINCIPLE OF THE SCHROEDINGER EQUATION

We are now in a stage to formulate the above basic ideas in a form of full quantum theory. For this purpose, it is convenient to employ the unitary-transformation method with auxiliary variables[9),10),11)]. Let us suppose that the nuclear many-body system under consideration exhibits a class of collective excitation spectra and all the collective states can be expanded in terms of a set of orthonormal state vectors

$$\{|n\rangle, \quad n = 0,1,2,\cdots\}, \tag{3.1}$$

which defines a collective subspace. The orthogonal complement of the collective subspace is then specified to be spanned by a set of orthonormal state vectors

$$\{|i\rangle\};\langle n|i\rangle = 0 \quad \text{for} \quad |n\rangle\in\{|n\rangle\}. \tag{3.2}$$

The collective subspace is characterized by idealized creation and annihilation operators $\hat{K}^+$ and $\hat{K}$, which may be complicated functions of fermion operators and are defined by

$$\hat{K}^+ \equiv \sum_n \sqrt{n+1}|n+1\rangle\langle n|, \quad \hat{K} \equiv \sum_n \sqrt{n+1}|n\rangle\langle n+1| \tag{3.3a}$$

$$[\hat{K}, \hat{K}^+] = 1\cdot\hat{P} . \tag{3.3b}$$

Here $\hat{P}$ is the projection operator into the collective subspace; $P = \Sigma_n|n\rangle\langle n|$. We also denote, hereafter, the projection operator into the orthogonal complement of the collective subspace as $\hat{Q} = 1 - \hat{P} = \Sigma_i|i\rangle\langle i|$.

In the unitary-transformation method with auxiliary bosons, we introduce auxiliary bosons (b^+,b) satisfying

$$[b,b^+] = 1, \tag{3.4}$$

as a new degree of freedom independent of the fermion degrees of freedom, and the state vectors are extended into a product space $\{|\Psi\rangle\rangle\} \equiv \{|\Psi\rangle\}\otimes\{|\Phi)_B\}$ of the fermion-state space $\{|\Psi\rangle\}$ and the boson-state space $\{|\Phi)_B\}$. We then employ a unitary transformation satisfying the following properties;

$$\hat{\mathbb{V}}^{-1}\hat{K}^{+}\hat{\mathbb{V}} = b^{+}\cdot\hat{P}, \quad \hat{\mathbb{V}}^{-1}b^{+}\hat{P}\hat{\mathbb{V}} = \hat{K}^{+}, \tag{3.5a}$$

$$\hat{\mathbb{V}}\cdot\hat{Q} = \hat{Q}\cdot\hat{\mathbb{V}} = \hat{Q}, \tag{3.5b}$$

$$\hat{\mathbb{V}}^{-1}|n_1>\otimes|n_2)_B = |n_2>\otimes|n_1)_B, \tag{3.5c}$$

where $\{|n)_B\}$ is a set of orthonormal boson states spanning the boson-state space $\{|\Phi)_B\}$;

$$|n)_B = \frac{1}{\sqrt{n!}}(b^{+})^{n}|0)_B, \quad b|0)_B = 0. \tag{3.6}$$

A formal solution of $\hat{\mathbb{V}}$ is given by[8)]

$$\hat{\mathbb{V}} = \exp(i\pi b^{+}b)\cdot\exp\{\frac{\pi}{2}(\hat{K}^{+}b-\hat{K}b^{+})\}\cdot\hat{P} + \hat{Q}. \tag{3.7}$$

Equation (3.5) implies that the unitary transformation $\hat{\mathbb{V}}$, which we call the transformation into "collective representation", interchanges the collective subspace $\{|n>\}$ and the boson space $\{|n)_B\}$ while it keeps the orthogonal complement $\{|i>\}$ invariant.

Since we have not yet made any specification of the collective subspace so far, the Hamiltonian of the system is generally decomposed into three parts;

$$\begin{aligned}
\hat{H} &= \hat{H}_{coll} + \hat{H}_{intr} + \hat{H}_{coupl}, \\
\hat{H}_{coll} &\equiv \hat{P}\hat{H}\hat{P} - \varepsilon_{intr}\hat{P}, \quad \hat{H}_{intr} \equiv \hat{Q}\hat{H}\hat{Q} + \varepsilon_{intr}\hat{P}, \\
\hat{H}_{coupl} &\equiv \hat{Q}\hat{H}\hat{P} + \hat{P}\hat{H}\hat{Q}.
\end{aligned} \tag{3.8}$$

Thus, the Hamiltonian in the collective representation is given by

$$\hat{\mathbb{V}}^{-1}\hat{H}\hat{\mathbb{V}} = \mathbb{H}_B(b^{+}b)\cdot\hat{P} + \hat{H}_{intr} + \hat{\mathbb{V}}^{-1}\hat{H}_{coupl}\hat{\mathbb{V}}, \tag{3.9}$$

where the boson Hamiltonian $\mathbb{H}_B(b^{+},b)\hat{P}\equiv\hat{\mathbb{V}}^{-1}\hat{H}_{coll}\hat{\mathbb{V}}$ manifests itself as a collective Hamiltonian. However, it must be emphasized that, only if the coupling $\hat{\mathbb{V}}^{-1}\hat{H}_{coupl}\hat{\mathbb{V}}$ is "weak", the collective motion under consideration is to have physical reality.

We are now in a position to discuss the problem of how to specify the maximally decoupled collective subspace as well as the maximally decoupled collective (boson) Hamiltonian. The fundamental principle for this specification is the invariance principle of the Schrödinger equation. To formulate the principle in the present full quantum case, we employ the boson operators in the Heisenberg representation

$b^+(t) = \exp(i\mathbb{H}_B t)b^+\exp(-i\mathbb{H}_B t)$, satisfying the equation of motion, $idb^+(t)/dt = [b^+(t),\mathbb{H}_B(b^+,b)]$. This corresponds to Eq. (2.4). The collective boson Hamiltonian $\mathbb{H}_B(b^+,b)$ is not yet specified and will be self-consistently determined later. We then introduce a time-dependent unitary transformation

$$\hat{\mathbb{V}}(t) = e^{i\mathbb{H}_B t}\hat{\mathbb{V}}e^{-i\mathbb{H}_B t} \equiv \hat{\mathbb{V}}(b^+(t),b(t)), \tag{3.10}$$

where $\hat{\mathbb{V}}$ is the unitary transformation into the collective representation given by Eq. (3.7). The invariance principle is now formulated as follows: The boson Hamiltonian $\mathbb{H}_B(b^+,b)$, which specifies the time-dependence of the boson operators $(b^+(t),b(t))$, must be determined in such a way that the Schrödinger equation after the time-dependent unitary transformation $\hat{\mathbb{V}}(t)$ remains invariant, i.e.,

$$i\frac{\partial}{\partial t}|\tilde{\Psi}(t)\gg_c = \hat{H}|\tilde{\Psi}(t)\gg_c, \qquad |\tilde{\Psi}(t)\gg_c = \hat{\mathbb{V}}^{-1}(t)|\Psi(t)\gg, \tag{3.11}$$

where $|\Psi(t)\gg$ satisfies the original Schrödinger equation

$$i\frac{\partial}{\partial t}|\Psi(t)\gg = \hat{H}|\Psi(t)\gg, \quad |\Psi(t)\gg = |\Psi(t)>\otimes|\Phi)_B. \tag{3.12}$$

Equation (3.11) can be expressed as

$$\begin{bmatrix} \hat{P}\hat{\mathbb{L}}\hat{P} & \hat{P}\hat{\mathbb{L}}\hat{Q} \\ \hat{Q}\hat{\mathbb{L}}\hat{P} & \hat{Q}\hat{\mathbb{L}}\hat{Q} \end{bmatrix} \begin{bmatrix} \hat{P}\cdot\exp\{-i\cdot\hat{H}_{coll}t\}|\Psi(t)\gg_c \\ \hat{Q}\cdot\exp\{-i\cdot\mathbb{H}_B t\}|\Psi(t)\gg_c \end{bmatrix} = 0, \quad |\Psi(t)\gg_c \equiv \hat{\mathbb{V}}^{-1}|\Psi(t)\gg, \tag{3.13}$$

where

$$\hat{\mathbb{L}} \equiv \hat{\mathbb{V}}^{-1}\{\hat{H}+\mathbb{H}_B(b^+,b)\}\hat{\mathbb{V}} - \{\hat{H}+\mathbb{H}_B(b^+,b)\}. \tag{3.14}$$

With the use of Eq. (3.5b), we have $\hat{Q}\hat{\mathbb{L}}\hat{Q}=0$ so that Eq. (3.13) can be decomposed into the maximal-decoupling condition

$$\hat{P}\hat{\mathbb{L}}\hat{Q} = \hat{Q}\hat{\mathbb{L}}\hat{P} = 0, \qquad \text{i.e.,} \tag{3.15a}$$

$$\hat{P}\hat{H}\hat{Q} = \hat{Q}\hat{H}\hat{P} = 0, \tag{3.15b}$$

and the definition of the maximal decoupled collective boson Hamiltonian

$$\hat{P}\hat{\mathbb{L}}\hat{P}=0, \qquad \text{i.e.,} \tag{3.16a}$$

$$\mathbb{H}_B(b^+,b)\hat{P}=\hat{P}\{\hat{\mathbb{V}}^{-1}\hat{H}\hat{\mathbb{V}}-\hat{H}+\hat{\mathbb{V}}^{-1}\mathbb{H}_B(b^+,b)\hat{\mathbb{V}}\}\hat{P}=\hat{P}\{\hat{\mathbb{V}}^{-1}\hat{H}\hat{\mathbb{V}}-\varepsilon_{intr}\}\hat{P}, \tag{3.16b}$$

where we have used the relation $\hat{\mathbb{V}}^{-1}\mathbb{H}_B(b^+,b)\hat{\mathbb{V}}\hat{P} = \hat{H}_{coll}$ together with the expression (3.8) of the Hamiltonian. Equation (3.15) clearly demonstrates that the collective subspace $\{|n>\}$ associated with the invariance principle must be an invariant subspace of the Hamiltonian.

The physical meaning of the invariance principle is now clear: The invariance principle of the Schrödinger equation is completely equivalent to requiring existence of the unitary transformation $\hat{\mathbf{V}}$ into the collective representation which leads us to the invariant collective subspace as well as the maximally decoupled collective boson Hamiltonian $\mathbb{H}_B(b^+,b)$. Thus, in the collective representation associated with the invariance principle, the transformed Hamiltonian $\hat{\mathbf{V}}^{-1}\hat{H}\hat{\mathbf{V}}$ is given by

$$\hat{\mathbf{V}}^{-1}\hat{H}\hat{\mathbf{V}}=\mathbb{H}_B(b^+,b)\hat{P}+\hat{H}_{intr}, \quad \hat{H}_{intr}=\hat{Q}\hat{H}\hat{Q}+\varepsilon_{intr}\hat{P}. \tag{3.17}$$

4. BASIC EQUATIONS IN THE THEORY

In order to compensate for the redundancy in the degrees of freedom due to the introduction of the auxiliary bosons, it is necessary to impose a subsidiary condition which physical state vectors $\{|\Psi_{phys}\gg\}$ in the enlarged fermion-boson product space have to fulfill. The simplest one is

$$b|\Psi_{phys}\gg = 0, \tag{4.1}$$

which simply implies that the physical state vectors are given with the boson vacuum by

$$\{|\Psi_{phys}\gg\} = \{|\Psi>\}\otimes|0)_B, \quad b|0)_B = 0. \tag{4.2}$$

In the collective representation, we thus obtain

$$\{\hat{P}|\Psi_{phys}\gg_c\} = \{|\Phi)_B\}\otimes|0>, \quad \hat{K}|0> = 0. \tag{4.3}$$

The invariance principle (3.15a) and (3.16a) connected with the state $|0>$ is now written as

$$\hat{Q}\cdot[\hat{\mathbf{V}}^{-1}\{\hat{H}+\mathbb{H}_B(b^+,b)\}\hat{\mathbf{V}} - \{\hat{H}+\mathbb{H}_B(b^+,b)\}]|0> = 0, \tag{4.4}$$

$$\hat{P}\cdot[\hat{\mathbf{V}}^{-1}\{\hat{H}+\mathbb{H}_B(b^+,b)\}\hat{V} - \{\hat{H}+\mathbb{H}_B(b^+,b)\}]|0> = 0, \tag{4.5}$$

$$\mathbb{H}_B(b^+,b) = \langle 0|\hat{\mathbb{V}}^{-1}\hat{H}\hat{\mathbb{V}}|0\rangle - \langle 0|\hat{H}|0\rangle, \tag{4.6}$$

where we have supposed $(0|\mathbb{H}_B(b^+,b)|0)_B=0$, i.e., $\langle 0|\hat{\mathbb{V}}^{-1}\mathbb{H}_B(b^+,b)\hat{\mathbb{V}}|0\rangle \equiv \langle 0|\hat{H}_{coll}(\hat{K}^+,\hat{K})|0\rangle = 0$. (This supposition is always fulfilled when the boson Hamiltonian has a form $\mathbb{H}_B = \sum_{r,s} h_{rs}(b^+)^r(b)^s$ with $h_{00} = 0$.)

In order to find the unitary transformation $\hat{\mathbb{V}}$ associated with the invariance principle without having to know explicit functional forms $(\hat{K}^+, \hat{K})$, it is important to express $(\hat{K}^+, \hat{K})$ in terms of "local infinitesimal generators" $(\hat{\mathbb{O}}^+, \hat{\mathbb{O}})$ of $\hat{\mathbb{V}}$ with respect to (b^+,b), which are defined by

$$\hat{\mathbb{O}}^+ = \hat{\mathbb{V}}^{-1}[\hat{\mathbb{V}},b^+], \quad \hat{\mathbb{O}} = \hat{\mathbb{V}}^{-1}[\hat{\mathbb{V}},b] \quad . \tag{4.7}$$

With the use of the fundamental property (3.5a) of $\hat{\mathbb{V}}$, we then can express $(\hat{K}^+, \hat{K})$ as

$$\hat{K}^+ = (b^+ - \hat{\mathbb{O}}^+)\cdot\hat{P}, \quad \hat{K} = (b - \hat{\mathbb{O}})\cdot\hat{P}. \tag{4.8}$$

Summarizing the above discussion, we have a set of the basic equations to be solved for the determination of $\hat{\mathbb{V}}$ and $\mathbb{H}_B(b^+,b)$; (I) <u>the self-consistency condition for the maximally decoupled collective boson Hamiltonian (4.6)</u>, (II) <u>the invariance principle of the Schrödinger equation</u> (4.4) and (4.5) and (III) <u>the subsidiary condition for the physical states</u> (4.3) <u>associated with</u> (3.3), i.e.,

$$\hat{K}|0\rangle \equiv (b-\hat{\mathbb{O}})|0\rangle = 0, \qquad \langle 0|[\hat{K},\hat{K}^+]|0\rangle = 1. \tag{4.9}$$

Here, it is rather convenient to reformulate the set of these basic equations by employing the following theorem concerning the coherent state:

<u>[theorem]</u> The normal-ordered form of any function $\mathbb{F}(b^+,b)$ is given by

$$\mathbb{F}(b^+_{\cdot} b) = \mathbb{N}\cdot\{F(\eta^*,\eta)\}, \quad F(\eta^*,\eta) \equiv (\eta|\mathbb{F}(b^+,b)|\eta)_B, \tag{4.10}$$

where $|\eta)_B$ is the coherent state with a complex c-number η ,

$$|\eta)_B \equiv \exp(b^+\eta - b\eta^*)|0)_B, \quad b|\eta)_B = \eta|\eta)_B, \tag{4.11}$$

and $\mathbb{N}$ is an operator which arranges all c-number η in $F(\eta^*,\eta)$ to the right and all η^* to the left and then replaces η^* and η by the boson operators b^+ and b respectively.

By making use of this theorem, the basic equations can be written as

$$\mathbb{H}_B(b^+:b)=\mathbb{N}\cdot\{\ \mathcal{H}_B(\eta^*,\eta)\}$$

$$\mathcal{H}_B(\eta^*,\eta)=\langle 0|\{(\eta|\hat{\mathbb{V}}^{-1}\hat{H}\hat{\mathbb{V}}|\eta)_B\}\ |0\rangle\ -\langle 0|\hat{H}|0\rangle, \qquad \text{(I)}$$

$$[(\eta|\hat{\mathbb{V}}^{-1}\{\hat{H}+\mathbb{H}_B(b^+,b)\}\hat{\mathbb{V}}|\eta)_B - \{\hat{H}+\ \mathcal{H}_B(\eta^*,\eta)\}]|0\rangle =0, \qquad \text{(II)}$$

$$\hat{K}|0\rangle\equiv\{\eta-\hat{O}(\eta^*,\eta)\}|0\rangle=0\ ,$$

$$\langle 0|[\ \hat{O}(\eta^*,\eta),\ \hat{O}^+(\eta^*,\eta)]|0\rangle =1, \qquad \hat{O}(\eta^*,\eta)\ \equiv(\eta|\hat{\mathbb{O}}|\eta)_B\ . \qquad \text{(III)}$$

From Eqs. (II) and (III), of course we obtain

$$[\hat{K},\{(\eta|\hat{\mathbb{V}}^{-1}(\hat{H}+\mathbb{H}_B)\hat{\mathbb{V}}|\eta)_B-(\hat{H}+\mathcal{H}_B)\}]|0\rangle = 0. \qquad (4.12)$$

So far we have not yet made any essential approximation in deriving the basic equations (I), (II) and (III). At this stage, we make the following fundamental approximation: The state $|0\rangle$ is approximated by a single slater determinant $|\phi\rangle$. Generally this single slater-determinant state need not satisfy the condition $\delta\langle\phi|\hat{H}|\phi\rangle=0$, because the state $|0\rangle$ generally need not an eigenstate of $\hat{H}$ and is merely a state satisfying

$$\hat{H}_{intr}|0\rangle = \varepsilon_{intr}|0\rangle\ . \qquad (4.13)$$

(As is easily seen from Eq. (3.17), any state belonging to the collective subspace $\{|n\rangle\}$ can satisfy Eq. (4.13).) In this approximation, the basic equations (I), (II) and (III) are reduced respectively to

$$\mathbb{H}_B(b^+:b)\ =\mathbb{N}\cdot\{\mathcal{H}_B(\eta^*,\eta)\},$$

$$\mathcal{H}_B(\eta^*,\eta)\ =\ \langle\phi|\{(\eta|\hat{\mathbb{V}}^{-1}\hat{H}\hat{\mathbb{V}}|\eta)_B\}|\phi\rangle-\langle\phi|\hat{H}|\phi\rangle, \qquad \text{(I')}$$

$$\delta\langle\phi|\{(\eta|\hat{\mathbb{V}}^{-1}\hat{H}\hat{\mathbb{V}}|\eta)_B-\hat{H}+(\eta|\hat{\mathbb{V}}^{-1}\mathbb{H}_B\hat{\mathbb{V}}|\eta)_B\}|\phi\rangle = 0, \qquad \text{(II')}$$

$$\{\eta-\hat{O}(\eta^*,\eta)\}|\phi\rangle = 0,\quad \hat{O}(\eta^*,\eta)\ \equiv\ (\eta|\hat{\mathbb{V}}^{-1}[\hat{\mathbb{V}},b]|\eta)_B,$$

$$\langle\phi|[\hat{O}(\eta^*,\eta),\hat{O}^+(\eta^*,\eta)]|\phi\rangle = 1. \qquad \text{(III')}$$

and Eq. (4.12) is written as

$$\langle\phi|[\hat{O}(\eta^*,\eta),\{(\eta|\hat{\mathbb{V}}^{-1}(\hat{H}+\mathbb{H}_B)\hat{\mathbb{V}}|\eta)_B-\hat{H}\}]|\phi\rangle = 0. \tag{4.14}$$

5. DETERMINATION OF THE MAXIMALLY DECOUPLED COLLECTIVE BOSON HAMILTONIAN

We are now in a position to discuss the problem of determining the maximally decoupled collective boson Hamiltonian $\mathbb{H}_B(b^+,b)$ by solving the set of the approximate basic equations (I'), (II') and (III'). Although the set of the basic equations (I), (II) and (III) has been derived by employing the fundamental property (3.5) of the unitary transformation $\hat{\mathbb{V}}$, where the idealized operators $(\hat{K}^+,\hat{K})$ have been imaged as if they were known, we must not employ the property in solving the set of the basic equations (I'), (II') and (III') to keep a consistency of the method.

Thus, the problem of solving the set of the basic equations can be reduced to finding a hermitian operator $\hat{\mathbb{G}}(b^+:b)$, which is defined through

$$\hat{\mathbb{V}} = \exp\{i\hat{\mathbb{G}}(b^+:b)\} \tag{5.1}$$

and is a normal-ordered operator with respect to (b^+,b). With the use of a transformation

$$\begin{aligned} b^+ &= \eta^*+a^+, \quad b = \eta+a \ , \\ [a,a^+] &= 1, \quad a|\eta)_B = 0 \ , \end{aligned} \tag{5.2}$$

let us make the Taylor expansion of $\hat{\mathbb{G}}(b^+:b)$ around (η^*,η)

$$\begin{aligned} \hat{\mathbb{G}}(b^+:b) &= \hat{G}(\eta^*,\eta)+\sum_{m\geq 1}\frac{1}{m!} : \left(a^+\frac{\partial}{\partial\eta^*} + a\frac{\partial}{\partial\eta}\right)^m : \hat{G}(\eta^*,\eta), \\ \hat{G}(\eta^*,\eta) &\equiv (\eta|\hat{\mathbb{G}}(b^+:b)|\eta)_B \quad , \end{aligned} \tag{5.3}$$

where the symbol : : means the normal product with respect to (a^+,a). With the use of Eq. (5.3), we then can express

$$(\eta|\hat{\mathbb{V}}^{-1}\hat{H}\hat{\mathbb{V}}|\eta)_B,\quad (\eta|\hat{\mathbb{V}}^{-1}\mathbb{H}_B\hat{\mathbb{V}}|\eta)_B \quad \text{and} \quad \hat{O}(\eta^*,\eta)\equiv(\eta|\hat{\mathbb{V}}^{-1}[\hat{\mathbb{V}},b]|\eta)_B$$

in the basic equations in terms of $\hat{G}(\eta^*,\eta)$ and its derivatives with respect to (η^*,η).

For example, in the case of the "semi-classical" approximation[8], <u>in which only the first order terms of the Taylor expansion of $\hat{\mathbb{G}}(b^+:b)$ is retained</u>:

$$\hat{\mathbb{G}}(b^+:b) \approx \hat{G}(\eta^*,\eta)+\left(a^+\frac{\partial}{\partial\eta^*} + a\frac{\partial}{\partial\eta}\right)\hat{G}(\eta^*,\eta), \tag{5.4}$$

we obtain

$$\begin{aligned}
(\eta|\hat{O}|\eta)_B &\approx -e^{-i\hat{G}}\frac{\partial}{\partial\eta^*}\, e^{i\hat{G}} \equiv \hat{O}^{(0)}(\eta^*,\eta),\\
(\eta|\hat{V}^{-1}\hat{H}\hat{V}|\eta)_B &\approx e^{-i\hat{G}}\hat{H}e^{i\hat{G}}, \qquad\qquad (5.5)\\
(\eta|\hat{V}^{-1}\mathbb{H}_B\hat{V}|\eta)_B &\approx \mathcal{H}_B(\eta^*,\eta) - \frac{\partial}{\partial\eta^*}\mathcal{H}_B(\eta^*,\eta)\cdot\hat{O}^{(0)+}(\eta^*,\eta)\\
&\quad - \frac{\partial}{\partial\eta}\mathcal{H}_B(\eta^*,\eta)\cdot\hat{O}^{(0)}(\eta^*,\eta)\ .
\end{aligned}$$

where $\hat{G}$ stands for $\hat{G}(\eta^*,\eta)$ and we have used the property of the coherent state; $(\eta|[\mathbb{H}_B,a^+]|\eta)_B = \partial\,\mathcal{H}_B(\eta^*,\eta)/\partial\eta$.

Thus, in the semi-classical approximation, the set of the basic equations (I'), (II') and (III') can be written in the forms;

$$\mathcal{H}_B(\eta^*,\eta) = \langle\phi|(e^{-i\hat{G}}\hat{H}e^{i\hat{G}}-\hat{H})|\phi\rangle \equiv \mathcal{H}_0(\eta^*,\eta), \tag{5.6}$$

$$\begin{aligned}
\delta\langle\phi|\{e^{-i\hat{G}}\hat{H}e^{i\hat{G}}-\hat{H}\} &- \frac{\partial}{\partial\eta^*}\mathcal{H}_B(\eta^*,\eta)\cdot\hat{O}^{(0)+}(\eta^*,\eta)\\
&- \frac{\partial}{\partial\eta}\mathcal{H}_B(\eta^*,\eta)\cdot\hat{O}^{(0)}(\eta^*,\eta)|\phi\rangle = 0,
\end{aligned} \tag{5.7}$$

$$\langle\phi|[\hat{O}^{(0)}(\eta^*,\eta),\ \hat{O}^{(0)+}(\eta^*,\eta)]|\phi\rangle = 1, \tag{5.8}$$

and Eq. (4.14) is reduced to

$$\langle\phi|[\hat{O}^{(0)}(\eta^*,\eta),\{e^{-i\hat{G}}\hat{H}e^{i\hat{G}}-\hat{H}\}]|\phi\rangle = \frac{\partial}{\partial\eta^*}\mathcal{H}_B(\eta^*,\eta) \tag{5.9}$$

The forms of these equations (5.6), (5.7) and (5.8) are completely the same as Eq. (2.10), Eq. (2.9) with (2.11) and Eq. (2.12), except that the single slater determinant $|\phi\rangle$ need not be the Hartree-Fock ground state $|\phi_0\rangle$ and $\hat{G}(\eta^*,\eta)$ has not yet specified to be a one-body operator.

In Section 2, we have discussed the (η^*,η)-expansion method to determine $\hat{G}_0(\eta^*,\eta)$ as well as $H_0(\eta^*,\eta)$. Provided that $\hat{G}(\eta^*,\eta)$ is assumed to be a one-body operator, we can also employ the (η^*,η)-expansion method in solving the set of basic equations (I') and (II') and (III'). The "quantum" effects are then obtained by taking account of the higher order terms of the Taylor expansion of $\hat{\mathbb{G}}(b^+:b)$ in Eq. (5.3).

6. CONCLUDING REMARKS

Finally, we would like to make two remarks.

(i) In this talk, we have discussed the set of basic equations (I'), (II') and (III'), which is obtained from the set of (I), (II) and (III) by assuming the state $|0\rangle$ to be a single slater determinant. When we make a further assumption specifying the collective subspace so as to satisfy

$$|\phi\rangle = |0\rangle, \quad |n\rangle = \frac{1}{\sqrt{n!}}(\hat{K}^+)^n|\phi\rangle \in \{|np\text{-}nh\rangle\}, \tag{6.1}$$

where $\{|np\text{-}nh\rangle\}$ is a subspace consisting of n-particle-n-hole excited states with respect to the single slater determinant $|\phi\rangle$, the set of basic equations can be greatly simplified. In this case, we can show[13] that the set of basic equations leads us to a Marumori-Yamamura-Tokunaga-type boson-mapping theory[14] <u>of the maximally decoupled collective subspace</u>.

(ii) In this talk, we have used the simplest subsidiary condition (4.1), i.e., $b|0)_B=0$. In the collective representation, the correspondent of the boson vacuum $|0)_B$ is the state $|0\rangle$ which satisfies Eq. (4.13). As has been discussed in Section 4, however, an "intrinsic" state $|\Psi_{intr}\rangle$ which fulfills

$$\hat{H}_{intr}|\Psi_{intr}\rangle = \varepsilon_{intr}|\Psi_{intr}\rangle \tag{6.2}$$

<u>need not</u> to be the state $|0\rangle$, because any state in the collective subspace $\{|n\rangle\}$ can fulfill Eq. (6.2). If the system under consideration undergoes a phase transition to a "deformed" phase, therefore, it is rather convenient to start with a subsidiary condition

$$(b-\beta)|\Psi_{phys}(\beta)\rangle\rangle = 0 \tag{6.3}$$

which implies that the physical state vectors are given with a coherent state with "deformation" β by

$$\{|\Psi_{phys}(\beta)>>\} = \{|\Psi>\}\otimes|\beta)_B \ , \qquad (6.4)$$
$$|\beta)_B = \exp(b^+\beta - b\beta^*)\ |0)_B \ .$$

This case will be discussed elsewhere.

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