A MICRO-deSITTER SPACETIME WITH CONSTANT TORSION: A NEW VACUUM SOLUTION OF THE POINCARÉ GAUGE FIELD THEORY

Peter Baekler and Friedrich W. Hehl
Institute for Theoretical Physics
University of Cologne
D-5000 Cologne 41, West Germany

ABSTRACT

We study the Poincaré gauge field theory with Lagrangians of the type (curvature scalar + torsion^2)/l^2 + curvature^2/$\varkappa$. Here l is the Planck length and $\varkappa$ the coupling constant of the hypothetical Lorentz gauge bosons. We find a new vacuum solution with a deSitter metric and with constant microscopic torsion $\sim \sqrt{\varkappa}/\ell$ and curvature $\sim \varkappa/\ell^2$. Its curvature displays double duality properties.

CONTENTS

1. INTRODUCTION AND SUMMARY

Soon after Yang and Mills /1/ developed the SU(2)-gauge concept in the context of the conserved isotopic spin current, Utiyama /2/ extended this idea to any semi-simple Lie-symmetry group and demonstrated furthermore that general relativity (GR) can be understood as a gauge theory related in some sense to local Lorentz invariance. A most lucid presentation of Utiyama's gauge theoretical ideas can be found in his more recent paper /3/. This approach to GR can be traced back to an earlier attempt of Weyl /4/.

Subsequently Sciama /5/ and Kibble /6/ have shown that, in a gauge theoretical set-up, the ad-hoc assumption of a symmetric connection of spacetime, used by Utiyama, should be dropped. Thus one arrives at a gravitational theory formulated in the framework of a Riemann-Cartan spacetime U_4. It is most natural to interpret the resulting theory as a gauge theory with local Poincaré invariance, cf. Cartan /7/. We will call it a Poincaré gauge field theory[+] (PG). The present article is based on our four Erice lectures on this subject /32/.

Einstein "deduced" GR and its Riemannian spacetime V_4 by studying the motion of forcefree point-particles in non-inertial frames of reference and by applying the equivalence principle. In a gauge theoretical approach one studies the Lagrangian of a fermionic field, typically of a Dirac field, in non-inertial frames of reference and applies again a principle of equivalence[++]: A Riemann-Cartan spacetime U_4 is the outcome.

It is the heuristic power of the Einsteinian type of procedure as applied to a fermionic field which lends support to the PG. For fermions a U_4 is a much more natural "habitat" than a V_4. The latter is naturally adapted to point-particles. -

In Sect. 2 we will formulate the two field equations of the PG. In Sect. 3 the most general polynomial Lagrangian in torsion and curvature will be displayed yielding quasi-linear 2nd-order field equations. For a detailed motivation and derivation in the context of both of these sections see /32/.

Subsequently we are are going to look for exact vacuum solutions of the field equations with non-vanishing torsion. Such solutions with spherical symmetry and reflection invariance were found earlier for specific Lagrangians of the PG by Neville /36/, Baekler /37/, Baekler, Hehl and Mielke /38/, by Benn, Dereli and Tucker /39,40/ and by Baek-

+) Cf. also Hayashi and Shirafuji /8/ and Hayashi and Nakano /9/. There is a vast literature on this field. References can be found by starting, say, from the articles of Ne'eman /10/, Trautman /11/ and Hehl, Nitsch and Von der Heyde /12/ in the GRG-Einstein Centennial Volume (Held /13/). More recent work includes the articles of Hennig and Nitsch /14/, Mielke /15/, Ne'eman /16/, Nieh and Yau /17/, see also /18,19/, Schweizer /20/, Szczyrba /21,22/, Thirring /23/, Tseytlin /24/, Wallner /25,26/, Yasskin /27/, and, on the equations of motion, Audretsch /28/ and Rumpf /29,30/; cf. also Drechsler /31/.

++) Cf. Von der Heyde /34/, Mack /35/, and Utiyama /3/. In the gravitational part of Utiyama's paper on p. 2218 $\Gamma_\varsigma = 0$ (locally) is postulated in order "to derive the Riemannian Γ from a gauge theoretical approach". If one put $A_\varsigma = 0$ (locally) instead, one would arrive at a Riemann-Cartan spacetime. Utiyama writes in a letter to one of the authors furthermore: "I know that the requirement $A_\varsigma = 0$, locally (D) is more general than the requirement $\Gamma_\varsigma = 0$, locally (E). Einstein's equivalence principle should be replaced with the requirement (D) which is applicable even to microscopic cases. The condition (D) fits the orthodox view of the general theory of gauge fields.... this requirement is a generalized principle of equivalence." We completely agree.

ler /42/. J.D. McCrea /43/ found corresponding cylindrically symmetric solutions and Adamowicz /44/ plane wave solutions, see also Chen and Chern /45/. Mielke /46/ developed a general method for the generation of exact solutions.

In Sect. 4 we turn to spherical symmetry with reflection invariance. Earlier work on this subject, besides the articles cited in the last paragraph, has been done by Ramaswamy and Yasskin /47/, Baekler and Yasskin /48/, Nieh and Rauch /49,50/, Rauch, Shaw and Nieh /51/ and Rauch /52/, see also Tsamparlis /53,54/. In the papers /49,50,51/ the assumption of reflection invariance has been relaxed.

In Sect. 5 we specialize the tetrad such as to find a spatially homogeneous time-dependent solution. First, in Sect. 6, we execute this program for the purely quadratic model Lagrangian (6.1). For this purpose we use the two LISP-based algebraic manipulation schemes REDUCE /55/ and ORTOCARTAN /56,57/, cf. d'Inverno /58/. Our REDUCE-programs rely heavily on similar programs written by J.D. McCrea, cf. /43/. We present the new constant curvature and constant torsion solution in eqs. (6.13) and (6.14) and show that it is weakly double self dual.

In Sect. 7 it is shown that our new micro-deSitter solution with constant torsion is, by a suitable adaption of the constants, see. eq. (6.14)', also a solution of the field equations of the general polynomial Lagrangian (3.1), provided the fairly weak constraints (7.1) and (7.2) on the corresponding coupling constants are fulfilled.

2. THE TWO FIELD EQUATIONS OF THE POINCARÉ GAUGE FIELD THEORY

The independent gravitational potentials of the Riemann-Cartan spacetime of the Poincaré gauge field theory (PG) are the tetrad coefficients $e_i{}^\alpha$ and the connection coefficients $\Gamma_i{}^{\alpha\beta} = -\Gamma_i{}^{\beta\alpha}$. Here $i,j \ldots = 0 \ldots 3$ are holonomic (world) indices and $\alpha, \beta \ldots = 0 \ldots 3$ anholonomic (Lorentz) indices. The tetrads are chosen orthonormal, i.e. the local metric $\eta_{\alpha\beta}$ coincides with the Minkowski metric diag. (-+++). The tetrad coefficients $e_i{}^\alpha$ can be interpreted as translational and the connection coefficients $\Gamma_i{}^{\alpha\beta}$ as Lorentz (or rotational) gauge potentials. The corresponding field strength tensors are torsion

$$F_{ij}{}^{\alpha} := 2 D_{[i} e_{j]}{}^{\alpha} = 2\left(\partial_{[i} e_{j]}{}^{\alpha} + \Gamma_{[i|\beta|}{}^{\alpha} e_{|j]}{}^{\beta}\right) \tag{2.1}$$

and curvature

$$F_{ij\alpha}{}^{\beta} = 2\left(\partial_{[i}\Gamma_{j]\alpha}{}^{\beta} + \Gamma_{[i|\varepsilon}{}^{\beta}\Gamma_{|j]\alpha}{}^{\varepsilon}\right), \tag{2.2}$$

respectively. The operator D_i represents the covariant exterior derivative. Let be given a matter field represented by a Poincaré spinor-tensor $\Psi(x^i)$ and referred to a local tetrad; its indices are suppressed. The action function W of a matter field Ψ that is interacting with the Poincaré gauge fields, can be put into the form

$$W = \int d^4x\, e\left[L(\eta_{\alpha\beta}, \gamma^{\alpha} \ldots \Psi, D_{\alpha}\Psi) + V(\varkappa_1, \varkappa_2, \ldots, \eta_{\alpha\beta}, F_{\alpha\beta}{}^{\gamma}, F_{\alpha\beta\gamma}{}^{\delta})\right]. \tag{2.3}$$

Here L is the special relativistic matter Lagrangian of, say, a Dirac field minimally coupled to the geometry, γ^{α} = Dirac matrices, e: = det $e_i{}^{\alpha}$, and $\mathcal{L}$: = eL, whereas $\mathcal{V}$: = eV is the gauge field Lagrangian depending on some coupling constants $\varkappa_1, \varkappa_2$ ..., on the local metric, and on the anholonomic components of torsion and curvature, respectively

Let be given the field momenta by

$$\mathcal{H}_{\alpha}{}^{ij} = 2\frac{\partial \mathcal{V}}{\partial F_{ji}{}^{\alpha}}, \qquad \mathcal{H}_{\alpha\beta}{}^{ij} = 2\frac{\partial \mathcal{V}}{\partial F_{ji}{}^{\alpha\beta}}, \tag{2.4}$$

the momentum current of the gauge fields by

$$\mathcal{E}_{\alpha}{}^{i} := e^{i}{}_{\alpha}\mathcal{V} - F_{\alpha j}{}^{\gamma}\mathcal{H}_{\gamma}{}^{ji} - F_{\alpha j}{}^{\gamma\delta}\mathcal{H}_{\gamma\delta}{}^{ji}, \tag{2.5}$$

and the spin current of the gauge fields by

$$\mathcal{E}_{\alpha\beta}{}^{i} := \mathcal{H}_{[\beta\alpha]}{}^{i}, \tag{2.6}$$

then the field equations of the PG read

$$D_j\mathcal{H}_{\alpha}{}^{ij} - \mathcal{E}_{\alpha}{}^{i} = e\,\Sigma_{\alpha}{}^{i}, \tag{2.7}$$

$$D_j\mathcal{H}_{\alpha\beta}{}^{ij} - \mathcal{E}_{\alpha\beta}{}^{i} = e\,\tau_{\alpha\beta}{}^{i}. \tag{2.8}$$

The sources on the right hand sides are the canonical momentum current and the canonical spin current of the matter field. For simplicity we will concentrate in this article on <u>vacuum</u> solutions of the field equations, i.e. we will put the material sources in (2.7,2.8) equal to zero later on.

The field equations are supplemented by the two Bianchi identities for torsion and curvature, respectively:

$$D_{[i}F_{jk]}{}^{\beta} \equiv F_{[ijk]}{}^{\beta}, \tag{2.9}$$

$$D_{[i}F_{jk]\alpha}{}^{\beta} \equiv 0. \tag{2.10}$$

3. THE MOST GENERAL LAGRANGIAN YIELDING QUASI-LINEAR 2ND ORDER FIELD EQUATIONS

The guiding principle for the construction of the gauge field Lagrangian V is that we allow at most second derivatives of the gauge potentials $e_i{}^{\alpha}$ and $\Gamma_i{}^{\alpha\beta}$, and these second derivatives should appear in the field equations only linearly (hypothesis of quasi-linearity). If we further assume V to be polynomial in the field strengths, we find

$$
\begin{aligned}
(3.1)\qquad V = {} & \frac{1}{\mu l^4} + \frac{1}{l^2}\Big[\frac{1}{2\chi}F + \frac{1}{4}F_{\alpha\beta}{}^{\gamma}\big(d_1 F^{\alpha\beta}{}_{\gamma} + d_2 F_{\gamma}{}^{\beta\alpha} + d_3\delta_{\gamma}^{\beta} F^{\alpha\mu}{}_{\mu}\big)\Big] \\
& - \frac{1}{4\varkappa}F_{\alpha\beta\gamma\delta}\Big[F^{\alpha\beta\gamma\delta} + f_1 F^{\alpha\gamma\beta\delta} + f_2 F^{\gamma\delta\alpha\beta} + f_3\eta^{\alpha\delta}F^{\beta\gamma} \\
& + f_4\eta^{\alpha\delta}F^{\gamma\beta} + f_5\eta^{\alpha\delta}\eta^{\beta\gamma}F\Big] \;.
\end{aligned}
$$

The U_4-Ricci tensor is defined by $F_{\alpha\beta} := F_{\gamma\alpha\beta}{}^{\gamma}$, its contraction, the curvature scalar, by $F := F_{\gamma\mu}{}^{\mu\gamma}$. The Planck length is denoted by l, $1/\mu l^4$ represents the cosmological constant, and $\chi, \varkappa$, f_A and d_A are dimensionless coupling constants. In (3.1) we hypothesized that, in addition to the usual gravitational potential $e_i{}^{\alpha}$ ("gravitons", weak Einstein gravity with coupling constant l^2), a propagating Lorentz gauge potential ("tordions", strong Yang-Mills gravity with coupling constant $\varkappa$) does exist in nature.[+]

By using (2.4) we derive the corresponding gauge field momenta, which are linear in torsion and curvature, respectively:

$$
(3.2)\qquad \mathcal{H}_{\alpha}{}^{ij} = \frac{e}{l^2}\big(-d_1 F^{ij}{}_{\alpha} + d_2 F_{\alpha}{}^{[ij]} + d_3 e^{[i}{}_{\alpha} F^{j]\gamma}{}_{\gamma}\big) \;,
$$

$$
\begin{aligned}
(3.3)\qquad \mathcal{H}_{\alpha\beta}{}^{ij} = {} & \frac{e}{\chi l^2} e^{i}{}_{[\alpha} e^{j}{}_{\beta]} + \frac{e}{\varkappa}\big(F^{ij}{}_{\alpha\beta} + f_1 F^{i}{}_{[\alpha}{}^{j}{}_{\beta]} + f_2 F_{\alpha\beta}{}^{ij} \\
& + f_3 e^{[i}{}_{[\beta} F^{j]}{}_{\alpha]} + f_4 e^{[i}{}_{[\beta} F_{\alpha]}{}^{j]} + f_5 F e^{i}{}_{[\beta} e^{j}{}_{\alpha]}\big) \,.
\end{aligned}
$$

By substituting (3.1),(3.2) and (3.3) into (2.5)-(2.8),we find the two field equations of the PG in their explicit form:

+) This was proposed by Hehl, Ne'eman, Nitsch and Von der Heyde /59/. For a corresponding ansatz in supergravity see Nishino /60/.

(FIRST)
$$
\begin{aligned}
&\frac{1}{\chi}\left(F_\alpha{}^i - \frac{1}{2}F e^i{}_\alpha + \Lambda e^i{}_\alpha\right) + \frac{1}{e}D_j\left(e d_1 F^{ji}{}_\alpha + e d_2 F_\alpha{}^{[ij]} + e d_3 e^{[i}{}_\alpha F^{j]\mu}{}_\mu\right) \\
&+ F_{\alpha j}{}^\gamma\left(d_1 F^{ij}{}_\gamma + d_2 F_\gamma{}^{[ji]} + d_3 e^{[j}{}_\gamma F^{i]\mu}{}_\mu\right) \\
&- \frac{1}{4}e^i{}_\alpha F_{\mu\nu}{}^\lambda\left(d_1 F^{\mu\nu}{}_\lambda + d_2 F_\lambda{}^{\nu\mu} + d_3 \delta^\nu{}_\lambda F^{\mu\gamma}{}_\gamma\right) \\
&+ \frac{\ell^2}{\varkappa} F_{\alpha j}{}^{\gamma\delta}\Big(F^{ji}{}_{\gamma\delta} + f_1 F^{[j}{}_\gamma{}^{i]}{}_\delta + f_2 F_{\gamma\delta}{}^{ji} + f_3 e^{[j}{}_{[\delta} F^{i]}{}_{\gamma]} \\
&\qquad + f_4 e^{[j}{}_{[\delta} F_{\gamma]}{}^{i]} + f_5 F e^j{}_{[\delta} e^i{}_{\gamma]}\Big) \\
&- \frac{\ell^2}{4\varkappa} e^i{}_\alpha F_{\mu\nu\gamma\delta}\Big(F^{\nu\mu\gamma\delta} + f_1 F^{\nu\gamma\mu\delta} + f_2 F^{\gamma\delta\nu\mu} + \\
&\qquad f_3 F^{\mu\gamma}\eta^{\nu\delta} + f_4 F^{\gamma\mu}\eta^{\nu\delta} + f_5 F \eta^{\nu\delta}\eta^{\gamma\mu}\Big) \\
&= \ell^2 \Sigma_\alpha{}^i \quad ,
\end{aligned}
$$

(SECOND)
$$
\begin{aligned}
&\frac{\varkappa}{2\chi\ell^2}T_{\alpha\beta}{}^i + \frac{1}{e}D_j\Big(e F^{ij}{}_{\alpha\beta} + e f_1 F^{[i}{}_\alpha{}^{j]}{}_\beta + e f_2 F_{\alpha\beta}{}^{ij} \\
&\qquad + e f_3 e^{[i}{}_{[\beta} F^{j]}{}_{\alpha]} + e f_4 e^{[i}{}_{[\beta} F_{\alpha]}{}^{j]} + e f_5 F e^i{}_{[\beta} e^j{}_{\alpha]}\Big) \\
&+ \frac{\varkappa}{\ell^2}\left(d_1 F^i{}_{[\beta\alpha]} - \frac{d_3}{2} e^i{}_{[\alpha} F_{\beta]}{}^\mu{}_\mu + \frac{3}{4} d_2 F_{[\alpha\beta}{}^{i]} + \frac{d_2}{2} F_{\alpha\beta}{}^i\right) = \varkappa \tau_{\alpha\beta}{}^i \; .
\end{aligned}
$$

For brevity we put $\Lambda := -\frac{\chi}{\mu\ell^2}$. With this sign convention, and for $\mu \neq 0$, we formally receive the same Λ as in GR. Observe that one of the constants f_2, f_4 or f_5 can be eliminated by use of the Euler-Gauss-Bonnet theorem. For convenience we choose $f_5 = 0$.

4. SPHERICAL SYMMETRY WITH REFLECTION INVARIANCE

For spherical symmetry and in spherical coordinates, the metric turns out to be

$$
ds^2 = -e^{2\mu(T,R)}dT^2 + e^{2\lambda(T,R)}dR^2 + r^2(T,R)(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) \tag{4.1}
$$

with the unknown functions μ, λ and r. Under the same conditions the torsion tensor has eight independent components, which can be further reduced by imposing spatial reflection invariance: then the only non-zero components of the torsion tensor $F_{\alpha\beta\gamma}$ are

$$
\begin{aligned}
F_{010} &= -f(T,R) \;, \\
F_{011} &= -h(T,R) \;, \\
F_{022} &= F_{033} = -K(T,R) \;, \\
F_{122} &= F_{133} = g(T,R) \;.
\end{aligned} \tag{4.2}
$$

$(\;)^{\cdot}$ will denote differentiation with respect to T and $(\;)'$ with respect to R.

In the following we will calculate the anholonomic components of the various geometrical objects involved. For this purpose we specify a tetrad associated with the metric (4.1). We choose the coframe as follows:

$$
\omega^0 = e^{\mu}\, dT \;, \quad \omega^1 = e^{\lambda}\, dR \;, \quad \omega^2 = r\, d\vartheta \;, \quad \omega^3 = r \sin\vartheta\, d\varphi \;. \tag{4.3}
$$

The corresponding tetrad coefficients $e_i{}^{\alpha}$ can be read off from $\omega^{\alpha} = e_i{}^{\alpha} dx^i$.

The anholonomic connection is determined by

$$
\Gamma_{\alpha\beta\gamma} = -\Omega_{\alpha\beta\gamma} + \Omega_{\beta\gamma\alpha} - \Omega_{\gamma\alpha\beta} + \tfrac{1}{2}\left(F_{\alpha\beta\gamma} - F_{\beta\gamma\alpha} + F_{\gamma\alpha\beta}\right) \tag{4.4}
$$

with the object of anholonomity

$$
\Omega_{\alpha\beta\gamma} = \eta_{\gamma\delta}\, e^{i}{}_{\alpha}\, e^{j}{}_{\beta}\, \partial_{[i} e_{j]}{}^{\delta} \;. \tag{4.5}
$$

For spherical symmetry (4.4) becomes

$$
\begin{aligned}
\Gamma_{010} &= -(\mu' e^{-\lambda} + f) := V(T,R) \;, \\
\Gamma_{101} &= \dot{\lambda} e^{-\mu} + h \quad := X(T,R) \;, \\
\Gamma_{202} &= \Gamma_{303} = \frac{\dot{r}}{r} e^{-\mu} + K \; := Y(T,R) \;, \\
\Gamma_{212} &= \Gamma_{313} = \frac{r'}{r} e^{-\lambda} - g \; := -W(T,R) \;, \\
\Gamma_{323} &= \frac{1}{r} \cot\vartheta \;.
\end{aligned} \tag{4.6}
$$

The curvature tensor

$$
F_{\alpha\beta\gamma}{}^{\delta} = 2\partial_{[\alpha}\Gamma_{\beta]\gamma}{}^{\delta} + 2\Gamma_{[\alpha|\varepsilon}{}^{\delta}\Gamma_{|\beta]\gamma}{}^{\varepsilon} + 2\Omega_{\alpha\beta}{}^{\varepsilon}\Gamma_{\varepsilon\gamma}{}^{\delta} \tag{4.7}
$$

has the following non-vanishing components

$$F_{0101} = -e^{-\mu-\lambda}\left[(Xe^{\lambda})^{\cdot} - (Ve^{\mu})'\right] := A(T,R) \quad , \tag{4.8}$$
$$F_{0220} = F_{0330} = -\frac{e^{-\mu}}{r}(Yr)^{\cdot} - VW := C(T,R) \quad ,$$
$$F_{1220} = F_{1330} = -\frac{e^{-\lambda}}{r}(Yr)' - XW := -D(T,R) \quad ,$$
$$F_{0221} = F_{0331} = \frac{e^{-\mu}}{r}(Wr)^{\cdot} + VY := -G(T,R) \quad ,$$
$$F_{1221} = F_{1331} = \frac{e^{-\lambda}}{r}(Wr)' + XY := H(T,R) \quad ,$$
$$F_{2323} = \frac{1}{r^2} + Y^2 - W^2 := L(T,R) \quad .$$

5. A SPECIFIC TIME-DEPENDENT ANSATZ FOR METRIC AND TORSION

In order to find homogeneous time-dependent and not necessarily isotropic solutions of the field equations, we specialize the ansatz (4.3) to

$$\omega^0 = dT \; , \; \omega^1 = \Phi(T,R)\,dR \; , \; \omega^2 = \Phi(T,R)\,d\vartheta \; , \; \omega^3 = \Phi(T,R) R \sin\vartheta \, d\varphi \tag{5.1}$$

yielding the metric

$$ds^2 = -dT^2 + \Phi^2(T,R)\left(dR^2 + R^2 d\Omega^2\right) \quad . \tag{5.2}$$

For the torsion we assume

$$\begin{aligned} -h(T,R) &= -k(T,R) =: Z(T,R) \; , \\ g(T,R) &= f(T,R) \equiv 0 \; . \end{aligned} \tag{5.3}$$

Then the anholonomic connection reads

$$\Gamma_{101} = \Gamma_{202} = \Gamma_{303} = \frac{\dot\Phi}{\Phi} - Z(T,R) \quad ,$$
$$\Gamma_{212} = \Gamma_{313} = \frac{1}{R} \; , \quad \Gamma_{233} = \frac{\cot\vartheta}{R} \quad , \tag{5.4}$$

and for the anholonomic components of the U_4-curvature $F_{\alpha\beta\gamma\delta}$ we find

$$F_{0101} = F_{0202} = F_{0303} = \ddot\Phi/\Phi - \dot\Phi Z/\Phi - \dot Z \quad ,$$
$$F_{0221} = F_{0331} = -\dot\Phi'/\Phi^2 + \dot\Phi\Phi'/\Phi^3 = -(1/\Phi)(\dot\Phi/\Phi)' \quad ,$$
$$F_{1202} = F_{1303} = \dot\Phi'/\Phi^2 - \dot\Phi\Phi'/\Phi^3 - Z'/\Phi = (1/\Phi)(\dot\Phi/\Phi)' - Z'/\Phi \quad , \tag{5.5}$$
$$F_{1212} = F_{1313} = (1/\Phi^2)(\Phi'/\Phi)' + \Phi'/\Phi^3 - \dot\Phi^2/\Phi^2 + 2\dot\Phi Z/\Phi - Z^2 \quad ,$$
$$F_{2323} = \Phi'^2/\Phi^4 + 2\Phi'/\Phi^3 R - \dot\Phi^2/\Phi^2 + 2\dot\Phi Z/\Phi - Z^2 \quad .$$

A further simplication can be achieved by the separation ansatz (cf. Fennelly and Pavelle /61/)

$$\Phi(T,R) = \frac{f(T)}{R^2} \quad . \tag{5.6}$$

6. THE NEW SOLUTION FOR THE QUADRATIC MODEL LAGRANGIAN

The field equations (FIRST) and (SECOND) of the 10-parameter Lagrangian are much more complicated than the Einstein equations. Therefore, for computational ease, it is advisable to pick a simple model Lagrangian which should have all the characteristic features of the full 10-parameter Lagrangian (3.1). A particular plausible choice is our <u>purely quadratic</u> Lagrangian /59/

$$V_Q = \frac{1}{4\ell^2} F_{\alpha\beta}{}^{\gamma}\left(-F^{\alpha\beta}{}_{\gamma} + 2\delta_{\gamma}^{\beta} F^{\alpha\mu}{}_{\mu}\right) - \frac{1}{4\varkappa} F_{\alpha\beta\gamma\delta} F^{\alpha\beta\gamma\delta} \quad , \tag{6.1}$$

$$\left(\Lambda = 1/\chi = f_\Lambda = 0 \;,\; a_1 = -1 \;,\; a_2 = 0 \;,\; a_3 = 2 \right) \; .$$

On substituting the data of Sect. 5 into (FIRST) and (SECOND) and using the parameter set of (6.1), we find for the vacuum case

$$(6.2) \begin{cases} \text{FIRST}(0,0) = -3\dot{z}^2 + 6\dot{z}\ddot{f}/f - 6\dot{z}z\dot{f}/f - 2z'^2R^4/f^2 - 3\ddot{f}^2/f^2 + 6z\ddot{f}\dot{f}/f^2 + 3\dot{f}^4/f^4 \\ \quad -12z\dot{f}^3/f^3 + 15z^2\dot{f}^2/f^2 - 12z^3\dot{f}/f - 6z(\dot{f}/f)\varkappa/\ell^2 + 3z^4 + 3z^2\varkappa/\ell^2 = 0, \\ \text{FIRST}(0,1) = z'\left(\dot{z} - \ddot{f}/f + z\dot{f}/f\right) = 0, \\ \text{FIRST}(1,0) = z'\left(-2\dot{z} + 2\ddot{f}/f - 2z\dot{f}/f + \varkappa/\ell^2\right) = 0, \\ \text{FIRST}(2,2) = \text{FIRST}(3,3) = \dot{z}^2 - 2\dot{z}\ddot{f}/f + 2\dot{z}z\dot{f}/f - 2\dot{z}\varkappa/\ell^2 + \ddot{f}^2/f^2 - z^4 \\ \quad -2z\ddot{f}\dot{f}/f^2 - \dot{f}^4/f^4 + 4z\dot{f}^3/f^3 - 5z^2\dot{f}^2/f^2 + 4z^3\dot{f}/f - 4z(\dot{f}/f)\varkappa/\ell^2 + z^2\varkappa/\ell^2 = 0, \\ \text{FIRST}(1,1) = \dot{z}^2 - 2\dot{z}\ddot{f}/f + 2\dot{z}z\dot{f}/f - 2\dot{z}\varkappa/\ell^2 + 2z'^2R^4/f^2 + \ddot{f}^2/f^2 - \\ \quad 2\ddot{f}\dot{f}z/f^2 - \dot{f}^4/f^4 + 4z\dot{f}^3/f^3 - 5z^2\dot{f}^2/f^2 + 4z^3\dot{f}/f - z^4 - 4z(\dot{f}/f)\varkappa/\ell^2 + z^2\varkappa/\ell^2 = 0; \end{cases}$$

$$(6.3) \begin{cases} \text{SECOND}(0,0,1) = (1/f)(z'f)^{\cdot} = 0, \\ \text{SECOND}(1,0,1) = -\ddot{z} - 3\dot{z}\dot{f}/f - 2z'R^3/f^2 + \dddot{f}/f + \ddot{f}\dot{f}/f^2 - z\ddot{f}/f \\ \quad - 2\dot{f}^3/f^3 + 5z\dot{f}^2/f^2 - 6z^2\dot{f}/f + 2z^3 + z\varkappa/\ell^2 = 0, \\ \text{SECOND}(2,0,2) = \text{SECOND}(3,0,3) = -\ddot{z} - 3\dot{z}\dot{f}/f + z''R^4/f^2 + z'R^3/f^2 + 2z^3 \\ \quad + \dddot{f}/f + \ddot{f}\dot{f}/f^2 - z\ddot{f}/f - 2\dot{f}^3/f^3 + 5z\dot{f}^2/f^2 - 6z^2\dot{f}/f + z\varkappa/\ell^2 = 0, \\ \text{SECOND}(2,2,1) = \text{SECOND}(3,3,1) = z'\left(\dot{f} - zf\right) = 0. \end{cases}$$

Inspection of the antisymmetric part of (FIRST) reveals that

(6.4) $Z' = 0 \quad \text{or} \quad Z(T,R) = Z(T)$.

The trace of (FIRST) yields

(6.5) $\dot{Z} + 3Z\dot{f}/f - Z^2 = 0$

with the solution

(6.6) $$Z(T) = \frac{1}{f^3(T)\left[C_1 - \int \frac{d\tau}{f^3(\tau)}\right]} \quad ,$$

where C_1 is an integration constant.

Eqs. (6.5) and (6.6) fulfill (SECOND) identically, i.e. the function f(T) should be determined by the tracefree part of (FIRST). Substitution of (6.5) into (FIRST) yields an integro-differential equation for the unknown function f(T)

(6.7) $$2Z(T)\left(2\dot{f}/f - Z(T)\right)\left[\ddot{f}/f + \dot{f}^2/f^2 + \varkappa/(2\ell^2)\right] + \ddot{f}^2/f^2 - \dot{f}^4/f^4 = 0$$

with Z(T) given by (6.6).

Eq. (6.7) admits at least one simple solution for f(T): Choose

(6.8) $Z(T) = \alpha\dot{f}/f \quad ,$

which implies $C_1 = 0$ in (6.6), and

(6.9) $f(T) = C e^{\beta T} \quad ,$

where C, α, and β are constants. Then (SECOND) is fulfilled for arbitrary C and α with β given by

(6.10) $$\beta^2 = -\frac{\varkappa}{2(\alpha-1)(\alpha-2)\ell^2} \quad , \qquad (\alpha \neq 1,\ \alpha \neq 2) \quad .$$

(FIRST), however, restricts the value of α to

(6.11) $\alpha = 3 \quad .$

Accordingly, the metric is found to be

(6.12) $$ds^2 = -dT^2 + \frac{C^2 e^{2\beta T}}{R^4}\left(dR^2 + R^2 d\Omega^2\right)$$

By a coordinate transformation, it can be put into an explicitly conformally flat form. Collecting our results, we finally have the exact vacuum solution

(6.13a) $$ds^2 = \frac{1}{\beta^2 t^2}\, ds^2_{\text{Minkowski}} \quad ,$$

(6.13b) $$F_{011} = F_{022} = F_{033} = \alpha\beta = \text{constant} \quad ,$$

$$(6.13c)\qquad \boxed{F_{\alpha\beta}{}^{\gamma\delta} = (\alpha-1)\beta^2 \cdot \begin{pmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1-\alpha & & \\ & & & & 1-\alpha & \\ 0 & & & & & 1-\alpha \end{pmatrix} = \text{constant}}$$

with

$$(6.14)\qquad \alpha = 3\,, \quad \beta = \sqrt{-\varkappa}/2\ell \;.$$

The antisymmetric index pairs (01,02,03,23,31,12) of the curvature matrix are numbered by (1,2...6) (cf. Misner, Thorne and Wheeler /62/ p. 361).

Consequently, for the purely quadratic Lagrangian solutions exist with constant (anholonomic) torsion and constant (anholonomic) U_4-curvature, but with an $F_{\alpha\beta}{}^{\gamma\delta}$ which does <u>not</u> satisfy $G_{\alpha\beta}(U_4) = \Lambda\eta_{\alpha\beta}$.

The U_4-curvature (6.13c) with (6.14) can be split into a Riemannian V_4-piece, caused by the deSitter metric (6.13a), and a purely torsion dependent piece according to

$$(6.15)\qquad F_{\alpha\beta}{}^{\gamma\delta} = \frac{\varkappa}{4\ell^2}\,\mathbb{1} + \frac{3\varkappa}{4\ell^2}\begin{pmatrix} -1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} .$$

This decomposition is irreducible under the local Lorentz group (cf. Debever /63,64/ and Lenzen /65/), the deSitter piece corresponds to the U_4-curvature scalar and the second piece to the tracefree symmetric U_4-Ricci tensor. The U_4-Weyl curvature tensor $C_{\alpha\beta}{}^{\gamma\delta}$ vanishes.

As can be read off from (6.15), the U_4-curvature of our new solution is weakly double self dual

$$(6.16)\qquad \overset{+}{F}_{\alpha\beta}{}^{\gamma\delta} := \frac{1}{2}\left(F_{\alpha\beta}{}^{\gamma\delta} + {}^*F^*_{\alpha\beta}{}^{\gamma\delta}\right) = \gamma\,\frac{\varkappa}{2\ell^2}\,\mathbb{1}$$

for $\gamma = 1/2$, where

$$(6.17)\qquad {}^*F^*_{\alpha\beta}{}^{\gamma\delta} := -\frac{1}{4}\,\eta_{\alpha\beta\rho\sigma}\,F^{\rho\sigma}{}_{\mu\nu}\,\eta^{\mu\nu\gamma\delta}$$

is the double dual of the curvature tensor ($\eta_{\alpha\beta\rho\sigma}$ = totally antisymmetric unit tensor). A double duality ansatz such as (6.16) can be used in the first place in order to find exact solutions (Baekler, Hehl and Mielke /38/, Benn, Dereli and Tucker /39/).[+]

+) F. Müller-Hoissen has informed us that he rederived our new solution by solving his Friedmann type equations in /66,67/ for the vacuum case, cf. also Minkevich /68/.

7. THE NEW SOLUTION FOR THE GENERAL CASE

In this section we will show that the solution (6.13, 6.14) can be generalized for the 10-parameter Lagrangian (3.1). For this purpose we substitute (5.1), (5.3), (5.6), and (6.8) into (SECOND) and (FIRST). Then (SECOND), together with the double duality ansatz (6.16), yields the two constraints

$$\left\| \begin{array}{ll} d_1 + \frac{d_2}{2} + \frac{d_3}{2} = 0 & (7.1) \\ \text{and} & \\ \frac{1}{\chi} + 2\gamma\rho - \frac{d_3}{2} = 0 \ , & (7.2) \end{array} \right\|$$

cf. Baekler /42/ ($\rho := 1 + \frac{1}{2}f_1 + f_2 + f_3 + f_4$). (FIRST) leads to /42/

$$\frac{d_3}{2}\, G_{\alpha\beta}(V_4) + \left(\frac{\Lambda}{\chi} - 3\gamma^2\rho\,\frac{\kappa}{\ell^2}\right)\eta_{\alpha\beta} = 0 \tag{7.3}$$

or, assuming $G_{\alpha\beta}(V_4) = -3\beta^2\eta_{\alpha\beta}$ to be fulfilled, to

$$-\frac{3}{2}d_3\beta^2 + \frac{\Lambda}{\chi} - 3\gamma^2\rho\,\frac{\kappa}{\ell^2} = 0 \ . \tag{7.4}$$

The constraint (7.1) holds for the "viable set" of torsion-parameters, which, for $F_{\alpha\beta}{}^{\gamma\delta} = 0$, leads to the teleparallelism theory of gravity.[+] Eq. (7.4) classifies the solutions into two different groups: One with $d_3 = 0$ and a second one with $d_3 \neq 0$.

Let us consider at first $d_3 = 0$. Then, as can be seen from (7.3), non-trivial solutions are only possible if we include a non-zero cosmological constant into the Lagrangian (see Benn, Dereli and Tucker /39/). The perhaps more interesting case is, however, that with $d_3 \neq 0$. Then (7.4) yields

$$\beta^2 = -\frac{2}{3d_3}\left(3\gamma^2\rho\,\frac{\kappa}{\ell^2} - \frac{\Lambda}{\chi}\right) \ . \tag{7.5}$$

These solutions can carry, in the case of $\Lambda = 0$, a "cosmological constant" without having a cosmological constant (see Baekler /37/, Baekler, Hehl and Mielke /38/).

For the function f we find, respectively[++],

[+] Recently interest in the teleparallelism theory increased greatly. The articles of Kopczyński /69/ and of Müller-Hoissen and Nitsch /70/ unify the formalism and give a critical evaluation of the viability of the theory. Recent work includes the articles of Cho /71/, Hayashi and Shirafuji /72/, Meyer /73/, Nitsch /74/, Nitsch and Hehl /75/, Schweizer and Straumann /76/, Schweizer, Straumann and Wipf /77/ and Smalley /78/.

[++] For vanishing torsion, the solution $f = C\exp(\beta t)$ was found by

$$(7.6)\qquad f(T) = \begin{cases} C \exp\left[\sqrt{-\frac{2}{3d_3}\left(3\gamma^2\varsigma\frac{\varkappa}{\ell^2} - \frac{\Lambda}{\chi}\right)}\,T\right], & d_3 \neq 0, \\ C \exp(\beta T),\ \beta = \text{arbitrary const.}, & d_3 = 0. \end{cases}$$

Accordingly,

$$(6.14)'\qquad \left\| \begin{array}{lll} \alpha = 3, & \beta^2 = -\frac{2}{3d_3}\left(3\gamma^2\varsigma\varkappa/\ell^2 - \frac{\Lambda}{\chi}\right), & d_3 \neq 0, \\ \alpha = 3, & \beta^2 = \text{arbitrary constant}, & d_3 = 0, \end{array} \right\|$$

together with (6.13), represent solutions for the 10-paramter Lagrangian, provided the two constraints (7.1) and (7.2) are fulfilled.

ACKNOWLEDGEMENTS

One of the authors (F.W.H.) would like to thank Prof. R. Utiyama for the invitation to give a lecture at the Nara Symposium on Gauge Theory and Gravitation upon which this paper is based. Furthermore, we appreciate helpful remarks of B. Mashhoon, H.-J. Lenzen, and J. Nitsch. A. Krasiński brought his ORTOCARTAN to Cologne and installed it, and J. Dermott McCrea (Dublin) showed us how to write effective REDUCE programs for the PG. We are very grateful to them as well as to R. Esser of our Computer Center whose help was indispensible in order to bring our programs to work.

LITERATURE

/1/ Yang, C.N., and R. Mills: Phys. Rev. 96, 191 (1954).
/2/ Utiyama, R.: Phys. Rev. 101, 1597 (1956).
/3/ Utiyama, R.: Progr. Theor. Phys. 64, 2207 (1980).
/4/ Weyl, H.: Z. Phys. 56, 330 (1929).
/5/ Sciama, D.W.: In "Recent Developments in General Relativity". Festschrift for Infeld, p. 415, Pergamon, Oxford (1962).
/6/ Kibble, T.W.B.: J. Math. Phys. 2, 212 (1961).
/7/ Cartan, E.: Ann. Ec. Norm. Sup. 40, 325 (1923); 41, 1 (1924).
/8/ Hayashi, K., and T. Shirafuji: Progr. Theor. Phys. 64, 866, 883, 1435, 2222 (1980); 65, 525, 2079 (E) (1981); 66, 318, 741 (E), 2258 (1981).
/9/ Hayashi, K., and T. Nakano: Progr. Theor. Phys. 38, 491 (1967).
/10/ Ne'eman, Y.: In /13/, Vol. 1, p. 309 (1980).
/11/ Trautman, A.: In /13/, Vol. 1, p.287 (1980).
/12/ Hehl, F.W., J. Nitsch and P. von der Heyde: In /13/, Vol. 1, p. 329 (1980).
/13/ Held, A. (ed.): "General Relativity and Gravitation, One Hundred Years After the Birth of Albert Einstein." Two Volumes. Plenum Press, N.Y. (1980).
/14/ Hennig, J., and J. Nitsch: GRG Journal 13, 947 (1981).
/15/ Mielke, E.W.: Habilitation Thesis, Univ. of Kiel (1982).
/16/ Ne'eman, Y.: In "Some Strangness in the Proportion, A Centennial Symposium to Celebrate the Achievements of Albert Einstein." H. Woolf (Ed.), p. 429, Addison-Wesley, Reading (1980).

Fennelly and Pavelle /61/ in Yang's gauge model of gravity. Moreover, it also solves the first field quation of SKY-gravity (see /38/ and also Goenner and Havas /79/, Mielke /80/, Ni /81/ and Pavelle /82/).

/17/ Nieh, H.T., and M.L. Yau: Ann. Phys. (N.Y.) 138, 237 (1982).
/18/ Nieh, H.T.: J. Math. Phys. 21, 1439 (1980).
/19/ Nieh, H.T., and M.L. Yau: J. Math. Phys. 23, 373 (1982).
/20/ Schweizer, M.A.: Thesis, Univ. of Zürich (1980).
/21/ Szczyrba, W.: In /33/, p. 105 (1980).
/22/ Szczyrba, W.: Phys. Rev. D25, 2548 (1982).
/23/ Thirring, W.: Lecture Notes in Physics (Springer) 116, 272 (1980).
/24/ Tseytlin, A.A.: Phys. Rev. D., to be published (1982).
/25/ Wallner, R.P.: GRG Journal 12, 719 (1980).
/26/ Wallner, R.P.: Thesis, Univ. of Vienna (1982) and references given.
/27/ Yasskin, P.D.: Thesis, Univ. of Maryland (1979).
/28/ Audretsch, J.: Phys. Rev. D24, 1470 (1981); D25, 605 (E) (1982).
/29/ Rumpf, H.: In /33/, p. 93 (1980).
/30/ Rumpf, H.: GRG Journal 14, 773 (1982).
/31/ Drechsler, W.: Phys. Lett. 107B, 415 (1981).
/32/ Hehl, F.W.: "Four Lectures in Poincaré Gauge Field Thoery." In /33/, p. 5 (1980).
/33/ Bergmann, P.G., and V. de Sabbata (Eds.): "Cosmology and Gravitation: Spin, Torsion, Rotation and Supergravity". Proceedings of the Erice-School, May 1979. Plenum, New York (1980).
/34/ Von der Heyde, P.: Lett. Nuovo Cimento 14, 250 (1975).
/35/ Mack, G.: Fortschr. Phys. 29, 135 (1981).
/36/ Neville, D.E.: Phys. Rev. D21, 2770 (1980).
/37/ Baekler, P.: Phys. Lett. 99B, 329 (1981).
/38/ Baekler, P., F.W. Hehl and E.W. Mielke: Preprint IC/80/140, Trieste (1980), In "Proc. 2nd M. Grossmann Meeting on General Relativity", R. Ruffini (Ed.), North-Holland, Amsterdam (1982).
/39/ Benn, I.M., T. Dereli and R.W. Tucker: GRG Journal 13, 581 (1981).
/40/ Benn, I.M., T. Dereli and R.W. Tucker: J. Phys A15, 849 (1982).
/41/ Baekler, P.: Diploma Thesis, Univ. of Cologne (1980).
/42/ Baekler, P.: Preprint, Univ. of Cologne (1982).
/43/ McCrea, J.D.: Preprint, Univ. College, Dublin (1982).
/44/ Adamowicz, W.: GRG Journal 12, 677 (1980).
/45/ Chen, H.-H., and D.-C. Chern: Preprint, National Central Univ., Taiwan (1982).
/46/ Mielke, E.W.: Preprint IC/81/210, Intern. Centre Theor. Physics, Trieste (1981).
/47/ Ramaswamy, S., and P.B. Yasskin: Phys. Rev. D19, 2264 (1979).
/48/ Baekler, P., and P.B. Yasskin: GRG Journal, to be published (1983).
/49/ Nieh, H.T., and R. Rauch: Phys. Lett. 81A, 113 (1981).
/50/ Rauch, R., and H.T. Nieh: Phys. Rev. D24, 2029 (1981).
/51/ Rauch, R., J.C. Shaw and H.-T. Nieh: GRG Journal 14, 331 (1982).
/52/ Rauch, R.T.: Phys. Rev. D25, 577 (1982); D26, 931 (1982).
/53/ Tsamparlis, M.: Phys. Lett. 75A, 27 (1979).
/54/ Tsamparlis, M.: Phys. Rev. D24, 1451 (1981).
/55/ Hearn, A.C.: REDUCE 2 User's Manual, 2nd ed., Univ. of Utah (1973).
/56/ Krasiński, A., and M. Perkowski: The System ORTOCARTAN - User's Manual, 2nd ed., Warsaw (1980).
/57/ Krasiński, A., and M. Perkowski: GRG Journal 13, 67 (1981).
/58/ d'Inverno, R.A.: In /13/, Vol. 1, p. 491 (1980).
/59/ Hehl, F.W., Y. Ne'eman, J. Nitsch and P. Von der Heyde: Phys. Lett. 78B, 102 (1978).
/60/ Nishino, H.: Progr. Theor. Phys. 66, 287 (1981).
/61/ Fennelly, A.J., and R. Pavelle: J. Phys. A12, 227 (1979).
/62/ Misner, Ch. W., K.S. Thorne and J.A. Wheeler: "Gravitation", Freeman, San Francisco (1973).
/63/ Debever, R.: Bull. Acad. Roy. Belgique, Cl.Sc. 42, 313, 608, 1033 (1956).
/64/ Debever, R.: Cahier de Physique 18, 303 (1964).
/65/ Lenzen, H.J.: Diploma Thesis, Univ. of Cologne (1982).
/66/ Müller-Hoissen, F.: Phys. Lett. A, to be published (1982).
/67/ Müller-Hoissen, F.: Preprint, Univ. of Göttingen (1982).

/68/ Minkevich, A.V.: Phys. Lett. 80A, 232 (1980).
/69/ Kopczyński, W.: J. Phys. A15, 493 (1982).
/70/ Müller-Hoissen, F., and J. Nitsch: Preprint, Univ. of Göttingen (1982).
/71/ Cho, Y.M.: Phys. Rev. D14, 2521 (1976).
/72/ Hayashi, K., and T. Shirafuji: Phys. Rev. D24, 3312 (1981) and references given.
/73/ Meyer, H.: GRG Journal 14, 531 (1982).
/74/ Nitsch, J.: In /33/, p. 63 (1980).
/75/ Nitsch, J., and F.W. Hehl: Phys. Lett. 90B, 98 (1980).
/76/ Schweizer, M., and N. Straumann: Phys. Lett. 71A, 493 (1979).
/77/ Schweizer, M., N. Straumann and A. Wipf: GRG Journal 12, 95 (1980)
/78/ Smalley, L.L.: Phys. Rev. D21, 328 (1980).
/79/ Goenner, H., and P. Havas: J. Math. Phys. 21, 1159 (1980).
/80/ Mielke, E.W.: GRG Journal 13, 175 (1981).
/81/ Ni, W.-T.: Phys. Rev. Lett. 35, 319 (1975).
/82/ Pavelle, R.: Phys. Rev. Lett. 34, 1114, 1484 (E) (1975).