STOCHASTIC QUANTIZATION AND LARGE N REDUCTION

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The stochastic quantization method of Parisi and Wu is used in order to understand the quenched momentum prescription for large N theories. The Hermitian matrix field theory model is studied first and then the same method is applied to SU(N) gauge theory.

I. Introduction

After a work of Eguchi and Kawai[1], some very intensive and active studies on the reduction of degrees of freedom at large N were done recently by various groups[2,3] to reach the quenched momentum prescription[3] for large N theories. These authors[3] have based their discussion on a detailed analysis of planar Feynman graphs for all order of perturbation theory. We believe that the large N reduction phenomenon is so general that it is likely to have more transparent explanations.

In this report we present our study on this problem using the stochastic quantization method of Parisi and Wu[4]. Since in this method the average over the random variables is taken at the end of calculation of correlation functions (Green's functions), it is possible to derive the quenched models by viewing a part of the random average as a quenched average.

Since the stochastic quantization method is relatively new and since in this method we believe there are subjects that require further studies especially for gauge theories, we first review the method in the next section. In section III we discuss the reduction of degrees of freedom for large N by taking the Hermitian matrix field model as an example[5]. In IV, we discuss the same problem for SU(N) gauge theory. In this case there exists some confusions whether we really obtained the quenched Eguchi-Kawai model or not. This confusion is mainly due to our lack of knowledge on the stochastic quantization. We suggest our resolution.

II. Stochastic Quantization

In this section, we review the stochastic quantization method of Parisi and Wu[3].

Let us consider an Euclidean field theory of a system of Bose field $\phi_\ell(x)$. x denotes a d-dimensional space time point while ℓ represents a set of internal indices. The correlation functions (Green's functions) of the theory are given by a functional average defined by

$$\langle\phi_{\ell_1}(x_1)\phi_{\ell_2}(x_2)\cdots\phi_{\ell_n}(x_n)\rangle \equiv \frac{\int \mathcal{D}\phi\,\phi_{\ell_1}(x_1)\phi_{\ell_2}(x_2)\cdots\phi_{\ell_n}(x_n)e^{-S[\phi]}}{\int \mathcal{D}\phi\; e^{-S[\phi]}} \tag{2.1}$$

where $S[\phi]$ is the action of the system.

One interprets (2.1) as a statistical average of dynamical variables $\phi_\ell(x)$ with Boltzman statistical weight. The basic idea of stochastic quantization of Parisi and Wu is that one regards the average (2.1) as the large (fictitious) time equilibrium limit of a statistical average in an inequilibrium system. The time evolution of this statistical system can be described by Fokker-Planck equation:

$$-\frac{1}{2}\frac{\partial\psi}{\partial t} = \hat{H}_{HF}\psi \tag{2.2}$$

$$\hat{H}_{HP} = \int dx[-\frac{1}{2}\sum_\ell \frac{\delta^2}{\delta\phi_\ell^2(x)} + \frac{1}{4}\{\frac{1}{2}\sum_\ell (\frac{\delta S}{\delta\phi_\ell(x)})^2 - \sum_\ell \frac{\delta^2 S}{\delta\phi_\ell^2(x)}\}] \,. \tag{2.3}$$

The probability distribution function at time t, $P[\phi,t]$, is related to ψ by

$$\psi[\phi,t] = e^{\frac{1}{2}S[\phi]} P[\phi,t] \,. \tag{2.4}$$

It is easy to see that $\hat{H}_{HF}\psi=0$ when $P=e^{-S[\phi]}$, so that ψ is stationary. One assumes that at large t the system reaches to this stationary state. If this is the case, the average can also be calculated by using Langevin equation

$$\frac{\partial\phi_\ell(x,t)}{\partial t} = -\frac{\delta S}{\delta\phi_\ell(x,t)} + \eta_\ell(x,t) \tag{2.5}$$

where t is a fictitious time. $\eta_\ell(x,t)$ is a random source function with

Gaussian distribution, namely it possesses the following property of averages:

$$<\eta_{\ell_1}(x_1,t_1)\eta_{\ell_2}(x_2,t_2)\cdots\eta_{\ell_n}(x_n,t_n)>$$

$$= \sum_{\substack{\text{possible}\\ \text{combination}}} \prod_{\text{pairs}} <\eta_{\ell_i}(x_i,t_i)\eta_{\ell_j}(x_j,t_j)>_\eta \tag{2.6}$$

$$<\eta_\ell(x,t)\eta_{\ell'}(x',t')>_\eta = 2\delta_{\ell\ell'}\delta(x-x')\delta(t-t') \ . \tag{2.7}$$

In (2.6), when n is odd the average is zero. The connection between Langevin and Fokker-Planck is given by

$$<F[\phi^\eta(\cdot,t)]>_\eta = \int \mathcal{D}\phi F[\phi(\cdot)]P[\phi,t] \tag{2.8}$$

where $\phi^\eta(x,t)$ is a solution of Langevin equation with initial condition $\phi^\eta(x,0)=\phi^0(x)$ while $P[\phi,t]$ is a solution of Fokker-Planck equation with initial condition $P[\phi,0]=\delta(\phi-\phi^0)$. Accordingly, the stochastic quantization prescription of Parisi and Wu is simply expressed by

$$<\phi_{\ell_1}(x_1)\phi_{\ell_2}(x_2)\cdots\phi_{\ell_n}(x_n)>$$

$$= \lim_{t\to\infty} <\phi^\eta_{\ell_1}(x_1,t)\phi^\eta_{\ell_2}(x_2,t)\cdots\phi^\eta_{\ell_n}(x_n,t)>_\eta \ . \tag{2.9}$$

Since $\phi^\eta_\ell(x,t)$ is a solution of Langevin equation (2.5) with a certain initial condition, in general the expression (2.9) depends on the initial field configuration ϕ^0. An implicit assumption made in the stochastic quantization is that the final result is independent of the initial condition. But it seems to us this is a point requires further investigations and a source of confusions. We shall come back to this point in Section IV.

III. Reduction of Degrees of Freedom for Large N

In this section we apply the stochastic quantization method of Parisi and Wu to derive the quenched momentum prescription for large N theories. We illustrate it for the Hermitian matrix model defined by the action

$$S[\phi] = \int dx \ \mathrm{tr}\{\tfrac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!N}\phi^4\}$$

where $\phi(x)$ is an N×N Hermitian matrix field ($\phi(x)=\phi^+(x)$).

We note the action has an SU(N) symmetry and a reflection symmetry

$$\left.\begin{array}{ll} \phi(x) \longrightarrow u\phi(x)u^+ & u \in SU(N) \\ \phi(x) \longrightarrow -\phi(x) & \end{array}\right\} . \qquad (3.2)$$

Therefore, assuming SU(N) symmetry is unbroken we consider only the following Green's functions

$$\langle tr(\phi(x_1)\phi(x_2)\cdots\phi(x_n))\rangle \qquad (3.3)$$

which is invariant by the transformations.

The corresponding Langevin equation to (3.1) is given by

$$\frac{\partial\phi_{ij}(x,t)}{\partial t} = (\Box - m^2)\phi_{ij}(x,t) - \frac{g}{3!N}(\phi^3(x,t))_{ij} + \eta_{ij}(x,t) \qquad (3.4)$$

where the random source matrix function $\eta(x,t)$ is assumed to have the Gaussian distribution:

$$\langle \eta_{ij}(x,t)\eta_{i'j'}(x',t')\rangle_\eta = 2\delta_{ij'}\delta_{ji'}\delta(x-x')\delta(t-t') . \qquad (3.5)$$

One may formally solve Langevin equation (3.4) by iteration, and since each term of the solution is classified by a tree diagram the solution can be expressed as

$$\phi^\eta_{ij}(x,t) = \sum\cdots(\eta\eta\cdots\eta)_{ij} . \qquad (3.6)$$

Inserting (3.6) into (3.3) we can express Green's function (3.3) in terms of the η averages as following:

$$\begin{aligned} &\langle tr(\phi(x_1)\cdots\phi(x_n))\rangle \\ &= \lim_{t\to\infty}\sum_{m=0}^{\infty}\left(\frac{g}{N}\right)^m \int\cdots\int dy_1dt_1dy_2dt_2\cdots dy_{2m+n}dt_{2m+n} \\ &\qquad K_m(x_1\cdots x_n,t\ ;\ y_1t_1,y_2t_2,\cdots,y_{2m+n}t_{2m+n}) \\ &\qquad \langle tr(\eta(y_1,t_1)\eta(y_2,t_2)\cdots\eta(y_{2m+n},t_{2m+n}))\rangle_\eta . \end{aligned} \qquad (3.7)$$

If we insert the Gaussian distribution property of η given by (2.6) and (3.5) into (3.7), we should be able to obtain the standard perturbation expansion as shown by Parisi and Wu. A nice point of (3.7) is that in (3.7) the SU(N) indices appear only through η so that we can discuss the large N limit by examining only the η averages.

Next we show that the reduced form of η defined by

$$\eta_{ij}(x,t) = \left(\frac{\Lambda}{2\pi}\right)^{d/2} e^{i(p_i-p_j)x} \bar{\eta}_{ij}(t) \qquad (3.8)$$

has in a sense the same Gaussian distribution property in the large N limit. In this expression Λ is a momentum cutoff. In a discrete lattice version of the theory, Λ is related to the inverse of lattice distance a; $\Lambda \sim \frac{1}{a}$.

In this report we describe the proof only for $\langle tr(\eta(x,t)\eta(x',t'))\rangle_\eta$, which should have the property

$$\langle tr(\eta(x,t)\eta(x',t'))\rangle_\eta = 2N^2\delta(x-x')\delta(t-t') \ . \qquad (3.9)$$

In the reduced case, using (3.8) we obtain

$$\sum_{ij} \left(\frac{\Lambda}{2\pi}\right)^{d} \langle e^{i(p_i-p_j)(x-x')} \bar{\eta}_{ij}(t)\bar{\eta}_{ji}(t')\rangle_{p,\bar{\eta}} \ . \qquad (3.10)$$

As a reduced average over p_i and $\bar{\eta}$ we choose the integration over p_i in a hypercube $[\frac{\Lambda}{2},-\frac{\Lambda}{2}]^d$ $\left(\int \prod_{\alpha,i=1}^{N} \left(\frac{dp_i^\alpha}{\Lambda}\right)\right)$ and the Gaussian distribution for $\bar{\eta}$:

$$\langle\bar{\eta}_{ij}(t)\bar{\eta}_{i'j'}(t')\rangle = 2\delta_{ij'}\delta_{i'j}\delta(t-t') \ . \qquad (3.11)$$

Then, $i\neq j$ contribution of (3.10) is given by

$$\left(\frac{\Lambda}{2\pi}\right)^{d} 2(N^2-N)\delta(t-t')\int \prod_{k,\alpha} \frac{dp_k^\alpha}{\Lambda} e^{i(p_i-p_j)(x-x')}$$

$$\approx 2N^2\delta(t-t')\delta(x-x') \sim O(N^2)$$

while $i=j$ contribution is given by

$$\left(\frac{\Lambda}{2\pi}\right)^{d} 2N\delta(t-t')\int \sum_{k,\alpha} \frac{dp_k^\alpha}{\Lambda} = \left(\frac{\Lambda}{2\pi}\right)^{d} 2N\delta(t-t') \sim O(N) \ .$$

In the large N limit we neglect $i=j$ contribution against $i\neq j$. Then we obtain the same expression as (3.9).

In order to see the degree of largeness of N we have used we compare these two contributions by integrating over x. We obtain the following criterion:

$$N >> (\frac{\Lambda}{2\pi})^d\ L^d \ . \tag{3.12}$$

Since Λ is the cutoff momentum and it is ~1/a, the criterion (3.12) is equivalent to

N >> number of space-time points .

For a large N which satisfies the condition (3.12) it is possible to prove that the reduced η (3.8) has Wick decomposition property (2.6), provided N>>n.

Next we look for a solution of Langevin equation when η is given by the reduced form (3.8). We first make an ansatz for ϕ,

$$\phi^{\eta}_{ij}(x,t) = e^{i(p_i-p_j)x}\ \bar{\phi}^{\eta}_{ij}(t) \tag{3.13}$$

and insert it into (3.4). Then, we obtain

$$\frac{\partial\bar{\phi}^{\eta}_{ij}(t)}{\partial t} = -[(p_i-p_j)^2+m^2]\bar{\phi}^{\eta}_{ij}(t) - \frac{g}{3!N}\ (\bar{\phi}^{\eta^3})_{ij} + \bar{\eta}_{ij}(t)(\frac{\Lambda}{2\pi})^d \ , \tag{3.14}$$

which can be considered as a reduced Langevin equation for $\bar{\phi}$.

Combining these equations together we obtain finally

$$\begin{aligned}
&\langle tr(\phi(x_1)\phi(x_2)\cdots\phi(x_n))\rangle \\
&= \lim_{t\to\infty} \langle tr(\phi^{\eta}(x_1,t)\cdots\phi^{\eta}(x_n,t))\rangle_{\eta} \qquad \text{(stochastic quantization)} \\
&= \lim_{t\to\infty} \int \Pi\ \frac{dp}{\Lambda} \sum\ e^{i(p_i-p_j)x_1\cdots}\ \ \langle\bar{\phi}^{\eta}_{ij}(t_1)\bar{\phi}^{\eta}_{jk}(t_2)\ \cdots\ \rangle_{\bar{\eta}} \\
&\qquad\qquad \text{(Reduction of } \eta \text{ and } \phi) \\
&= \int \Pi\ \frac{dp}{\Lambda} \sum\ e^{i(p_i-p_j)x_1\cdots}\ \ \langle\bar{\phi}_{ij}\bar{\phi}_{jk}\ \cdots\ \rangle_{\bar{S}} \\
&\qquad\qquad \text{(stochastic quantization)}
\end{aligned} \tag{3.15}$$

where

$$\langle\cdots\rangle_{\bar{S}} \equiv \frac{\int d\bar{\phi}(\cdots)e^{-\bar{S}[\bar{\phi}]}}{\int d\bar{\phi}e^{-\bar{S}[\bar{\phi}]}} \tag{3.16}$$

and

$$\bar{S}[\bar{\phi}] = (\frac{2\pi}{\Lambda})^d [\sum_{ij} \frac{1}{2}((p_i-p_j)^2+m^2)\bar{\phi}_{ij}\bar{\phi}_{ji} + \frac{g}{4!N} \mathrm{tr}(\bar{\phi}^4)] \quad . \qquad (3.17)$$

The expression (3.15) is nothing but the quenched momentum prescription proposed by the authors of reference 2.

IV. SU(N) Gauge Theory

The action of SU(N) gauge theory is given by

$$S = \int dx \, \frac{1}{4e^2} \, \mathrm{tr} \, F_{\mu\nu}F_{\mu\nu} \qquad (4.1)$$

where e is a coupling constant. The field strength $F_{\mu\nu}$ is expressed in terms of N×N Hermitian matrix vector potential A_μ as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \quad . \qquad (4.2)$$

The Langevin equation of this theory is given by

$$\dot{A}_\mu(x,t) = \frac{1}{e^2}\{\partial_\nu F_{\mu\nu}(x,t) - i[A_\nu, F_{\nu\mu}]\} + \eta_\mu(x,t) \quad , \qquad (4.3)$$

where $\eta_\mu(x,t)$ is a random source function with the same Gaussian distribution property as η in III. According to Parisi and Wu, if one solves this Langevin equation by perturbation with the initial condition

$$A_\mu(x,0) = 0 \qquad (4.4)$$

and calculate the gauge invariant correlation functions by the stochastic method one obtains the ordinary perturbation results in Landau gauge. We note that Langevin eq.(4.3) and the η-average property given by (2.6) and (3.5) are invariant by the following (fictitious) time independent gauge transformations:

$$A_\mu(x,t) \longrightarrow u(x)A_\mu(x,t)u^+(x) + iu(x)\partial_\mu u^+(x)$$

$$\eta_\mu(x,t) \longrightarrow u(x)\eta_\mu(x,t)u^+(x) \qquad (4.5)$$

$$u(x) \in SU(N) \quad .$$

Since the SU(N) gauge theory described by the stochastic quantization in this way is a slight generalization of the Hermitian model of the previous section, it is not difficult to see that we arrive at the following large N reduced model:

$$\left.\begin{aligned} \eta_\mu(x,t) &= (\tfrac{\Lambda}{2\pi})^{d/2}\ e^{iP\cdot x}\ \bar{\eta}_\mu(t)\ e^{-iP\cdot x} \\ A_\mu(x,t) &= e^{iP\cdot x}\ \bar{A}_\mu(t)\ e^{-iP\cdot x} \end{aligned}\right\} \tag{4.6}$$

$$\dot{\bar{A}}\ (t) = \frac{1}{e^2}\ [P_\nu-\bar{A}_\nu(t),[P_\nu-\bar{A}_\nu(t),P_\mu-\bar{A}_\mu(t)]] + (\tfrac{\Lambda}{2\pi})^{d/2}\ \bar{\eta}_\mu(t) \tag{4.7}$$

$$\bar{A}_\mu(0) = 0\ , \tag{4.8}$$

where P is a diagonal matrix with momenta p_i's in its diagonal elements.

Now it is straightforward to calculate the gauge invariant correlation functions in the large N limit by using the perturbative stochastic method to obtain the results of the quenched momentum prescription. However, there is an unexpected complication in the formal level. If we change reduced variables from $\bar{A}_\mu$ to $\tilde{A}_\mu$ by

$$\tilde{A}_\mu = \bar{A}_\mu - P_\mu\ , \tag{4.8}$$

the Langevin equation in terms of new variables becomes

$$\dot{\tilde{A}}\ (t) = -\frac{1}{e^2}\ [\tilde{A}_\nu(t)[\tilde{A}_\nu(t),\tilde{A}_\mu(t)]] + (\tfrac{\Lambda}{2\pi})^{d/2}\ \bar{\eta}_\mu(t)\ , \tag{4.9}$$

which is independent of P_μ. On the other hand the initial condition becomes

$$\tilde{A}_\mu(0) = -P_\mu \tag{4.10}$$

which depends on P_μ. Therefore, if the results of the stochastic quantization are independent of the initial condition the momenta are trivially integrated out and we obtain a matrix model (Eguchi-Kawai model) as a large N reduced model rather than the quenched momentum form.

We believe that this is due to the assumption that the stochastic quantization does not depend on the initial condition chosen to solve the Langevin equation. In gauge theories, due to the gauge symmetry there are no drift forces along the direction of gauge orbit. Thus, the system does not reach to the equilibrium distribution along the gauge orbit and the final distribution depends on the initial configuration. This is in a sense a gauge fixing (weighted average). The criterion that this phenomenon occurs is the existence of a continuous energy spectrum above the ground state of Fokker-Planck Hamiltonian because of the flatness of the potential in a certain direction in configuration space. In the reduced Langevin equation (4.9), the situation is more complicated. We note first that the Langevin equation (4.9) is invarinat by the following time independent reduced gauge transformation

$$\left.\begin{aligned} \tilde{A}_\mu(t) &\rightarrow u\tilde{A}_\mu(t)u^+ \\ \eta_\mu(t) &\rightarrow u\eta_\mu(t)u^+ \end{aligned}\quad u \in SU(N) \right\} . \tag{4.11}$$

We also note that $\tilde{A}_\mu(0)=-P_\mu$ cannot be obtained by the gauge transformation from $\tilde{A}_\mu(0)=0$. Thus, if there remains the P_μ dependence it must be due to the continuous spectrum which does not related to the known symmetry. We found an indication that this is the case. We like to devote the rest of this section for our preliminary study[5].

The action and the Fokker-Planck equation corresponding to reduced Langevin equation (4.9) are given by

$$S[\tilde{A}] = -\frac{\alpha^{-1}}{4}\,\mathrm{tr}[\tilde{A}_\mu,\tilde{A}_\nu]^2 \tag{4.12}$$

$$\tilde{H}_R = -\frac{1}{2}\sum_{\mu,ij}\frac{\delta^2}{\delta\tilde{A}_\mu^{ij}\delta\tilde{A}_\mu^{ji}} + \frac{1}{4}V[\tilde{A}] \tag{4.13}$$

$$V[\tilde{A}] = \frac{1}{2}\alpha^{-2}\,\mathrm{tr}[\tilde{A}_\nu,[\tilde{A}_\nu,\tilde{A}_\mu]]^2 - 2\alpha^{-1}(d-1)N\,\mathrm{tr}(\tilde{A}_\mu - \frac{1}{N}\mathrm{tr}\tilde{A}_\mu)^2 \tag{4.14}$$

$$\alpha = e^2\left(\frac{\Lambda}{2\pi}\right)^d . \tag{4.15}$$

We notice that when the configuration of $\tilde{A}_\mu$ is diagonal the first term in Fokker-Planck potential V is zero while the second term negative and bottomless. Thus, at least for the weak coupling (small α) the diagonal components of $\tilde{A}_\mu$ should be treated non-perturbatively. We therefore separate $\tilde{A}_\mu$ into the diagonal component b_μ and the off diagonal component B_μ:

$$\tilde{A}_\mu^{ij} = (b_\mu + B_\mu)^{ij} \equiv b_\mu^i\delta_{ij} + B_\mu^{ij} .$$

We insert it into V and then expand V in the powers of B_μ. To leading order in α we obtain

$$V = \frac{1}{2}\alpha^{-2}\sum_{i\neq j}(b_\sigma^i-b_\sigma^j)^2\{\delta_{\mu\nu}-\frac{(b^i-b^j)_\mu(b^i-b_j)_\nu}{(b_\sigma^i-b_\sigma^j)^2}\}B_\mu^{ij}B_\nu^{ji}$$

$$-2\alpha^{-1}(d-1)N\sum_i(b^i-\frac{1}{N}\sum_k b_\mu^k)^2 . \qquad (4.16)$$

We then calculate the zero point energy due to B_μ fluctuation keeping b_μ fixed (Born-Oppenheimer). We obtain

$$\frac{1}{4}(d-1)\alpha^{-1}\sum_{ij}(b^i-b^j)^2$$

which is precisely cancelled by the second term in V. Therefore at least in the lowest order in α b_μ becomes effectively a cyclic variable near the ground state due to the off diagonal fluctuations, and there is no effective drift force along the direction of diagonal A .

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5. More detailed study on this subject with a possible supersymmetry discussion will be published elsewhere.