RENORMALIZABILITY OF MASSIVE YANG-MILLS THEORY

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In this note, we aim at revealing the renormalizable structure of massive Yang-Mills (YM) theories on the basis of the formalism in our previous work,[1] quoted as [I], though there had so far been found several counter-observations.[2] Our formalism consists of four kinds of auxiliary scalar fields other than a massive gauge field A^a_μ; a massless scalar fields ξ^a, in addition to the usual Lagrange multiplier field η^a and a pair of Faddeev-Popov ghost fields C^a and $\overline{C}^a$. In terms of these fields, the Lagrangian density in [I], takes the form in the Landau gauge

$$\left.\begin{aligned} L &= -\tfrac{1}{4}(G^a_{\mu\nu})^2 - A^a_\mu\partial_\mu\eta^a + i\partial_\mu\overline{C}^a D^{ab}_\mu C^b - \tfrac{1}{2}m^2(K^a_\mu - A^a_\mu)^2, \\ K^a_\mu &= K^{ab}\partial_\mu\xi^b \ , \quad K^{ab} = \frac{\ell}{g}[1 + \sum_{n=1}^{\ell} \frac{1}{(n+1)!}(i\ell\xi)^n]_{ab}, \end{aligned}\right\} \qquad (1)$$

where $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$, $D^{ab}_\mu = \delta^{ab}\partial_\mu + gf^{acb}A^c_\mu$, $\ell = g/m$ and $\xi = T^a\xi^a$. As has been shown in [I], (1) is invariant under a Becchi-Rouet-Stora (BRS) transformation[3] with a nilpotent character. As is elucidated by Kugo and Ojima (KO),[4] such an exact BRS symmetry enables us to impose a KO-type of supplementary conditions on physical states, which guarantees our physical subspace to be of positive semi-definite due to the quartet mechanism confining the members of the quartet (ξ^a, η^a, C^a, $\overline{C}^a$) into zero-norm subspace. Therefore our physical S-matrix elements, which are taken between any states containing spin-triplet physical gauge-bosons (the Proca particles) alone, is manifestly unitary.

In the perturbation approach, the interaction Lagrangian is obtained by subtracting the free part from (1). Here we should note: 1) The Feynman propagator of gauge bosons takes the form

$$D_{\mu\nu}(k) \sim [\delta_{\mu\nu} - k_\mu k_\nu D(k)]\Delta(k;m^2) \qquad (2)$$

instead of the usual Proca type, and has the same asymptotic behavior

($k \to \infty$) as that in massless YM theories; namely, (2) takes a renormalizable form. 2) A nonpolynomial character of interactions emerges with respect to the field ξ^a. Therefore, at the first sight, one may guess the theory is unrenormalizable when the ordinary perturbation expansion in the coupling constant or loop expansion is carried out.[2] It is, however, not the case, bacause the exponential-type interaction provides a promissing way of constructing renormalizable massive YM theories, as pointed out by several authors.[5),6)]

We attack the renormalizability problem on the basis of nonpolynomial Lagrangian theories. Hereafter, we consider the case of SU(2), for simplicity. In this case, nonpolynomial parts of the interaction Lagrangian can be expressed as follows:

$$L^{\xi}_{int} = P^{ab}_{\Delta}\partial_\mu\xi^a\partial_\mu\xi^b \exp[it\ell(\lambda\xi)] \ , \tag{3}$$

$$L^{A}_{int} = mQ^{ab}_{\Delta}A^a_\mu\partial_\mu\xi^b \exp[it\ell(\lambda\xi)] \ . \tag{4}$$

Here, $(\lambda\xi) = \lambda^a\xi^a$ and the operator P^{ab}_{Δ} and Q^{ab}_{Δ} are defined by

$$P^{ab}_{\Delta} = \int_0^1 dt\, t^{\Delta}\int d\Omega\, \frac{(t-1)^2}{2t}\left(\frac{\partial^2}{\partial\lambda^a\partial\lambda^b} - \delta^{ab}\frac{\partial^2}{\partial\lambda^n\partial\lambda^n}\right) \ , \tag{5}$$

$$Q^{ab}_{\Delta} = \int_0^1 dt\, t^{\Delta}\int d\Omega\left[\frac{(t-1)}{2t}\left(\frac{\partial^2}{\partial\lambda^a\partial\lambda^b} - \delta^{ab}\frac{\partial^2}{\partial\lambda^n\partial\lambda^n}\right) - i\varepsilon^{abc}\frac{\partial}{\partial\lambda^c}\right], \tag{6}$$

where $d\Omega = \sin\theta d\theta d\phi/4\pi$, $\lambda^a = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$ and Δ is a positive number, which is introduced in order to guarantee commutativity of the t- and the other integrations in the S-matrix calculation and is taken to be zero finally.

We shall deal with the S-matrix elements by introducing so called superpropagators. Now, we consider only the case of L^{ξ}_{int} alone, for simplicity. The S-matrix is given by

$$S = 1 + \sum_{N\geq 1}\frac{i^N}{N!}\int dx^N S(x^N), \quad x^N = (x_1,\ldots,x_N) \ , \tag{7}$$

where $S(x^N)$ is expressed by using the Hori's formula[7] as follows:

$$\begin{aligned} S(x^N) = {} & \left[\prod_{i=1}^{N} P^{a_i b_i}_{\Delta_i}\right] \times \exp\left[-\frac{1}{2}\sum_{i,j}\partial_\mu\partial_\nu D_{ij}\frac{\partial^2}{\partial(\partial_\mu\xi^a_i)\partial(\partial_\nu\xi^a_j)}\right] \\ & \times \prod_{i=1}^{N}\left[\partial_\mu\xi^{a_i}_i + \sum_j (i\ell t_j\lambda^{a_i}_j)\partial_\mu D_{ij}\right]\left[\partial_\mu\xi^{b_i}_i + \sum_j (i\ell t_j\lambda^{b_i}_j)\partial_\mu D_{ij}\right] \\ & \times \exp\left[-\frac{1}{2}\sum_{i,j}\{\ell^2 t_i t_j(\lambda_i\lambda_j)D_{ij}\}\right] \times :\prod_{i=1}^{N}\exp[i\ell t_i(\lambda_i\xi_i)]:, \end{aligned} \tag{8}$$

where $\xi_i^a = \xi^a(x_i)$ and $\delta^{ab}D_{ij} = \delta^{ab} < 0|T\xi^a(x_i)\xi^b(x_j)|0>$. This is the sum of various kinds of N-th order (with respect to L_{int}^{ξ}) diagrams which are specified by the way how to contract $\partial_\mu\xi^a$ with the other fields. Each N-th order diagram has the following structure: Every two vertex-points are connected with a propagator of the forms $e^{\kappa D}$, $\partial_\mu De^{\kappa D}$, etc. called superpropagator (SP), in contrast with the usual Feynman propagator D. Thus an N-th order diagram is a supergraph expressed as the N(N-1)/2 products of SPs.

We are now in a position to construct all SPs appearing in general supergraphs. As is well known, $e^{\kappa D}$ is the typical SP in the case for an exponential interaction of scalar fields, and the exponential is expressed by the Sommerfeld-Watson transform:

$$e^{\kappa D} = \frac{i}{2}\int_\Gamma dz \frac{\kappa^z D^z}{\tan\pi z\Gamma(z+1)} + 1 , \qquad (9)$$

where the contour Γ encloses the positive real axis in the z plane. The contour is then opened out to lie parallel to the imaginary axis in $0 < \mathrm{Re}z < 1$; namely, $\Gamma \to C$ ($0 < \mathrm{Re}z < 1$; $z\epsilon C$). The Fourier transform (FT) of D^z is defined for $0 < \mathrm{Re}z < 2$ by[8)]

$$FT[D^z] = \int d^4xe^{ikx}[\frac{1}{4\pi^2(x^2+i\varepsilon)}]^z = \frac{i\pi(4\pi)^{2-2z}(k^2-i\varepsilon)^{z-2}}{\sin\pi z\Gamma(z)\Gamma(z-1)}. \qquad (10)$$

Substituting (10) into (9) with $\Gamma \to C$, we obtain the FT of $e^{\kappa D}$.

Due to the fact that (3) contains the derivatives $\partial_\mu\xi^a$, we need some more SPs. We express them symbolically as

$$f(\partial_\mu D,\partial_\mu\partial_\nu D)e^{\kappa D} = \frac{i}{2}\int_{C_f} dz\frac{\kappa^z f(\partial_\mu D,\partial_\mu\partial_\nu D)D^z}{\tan\pi z\Gamma(z+1)} , \qquad (11)$$

where $f(\partial_\mu D,\partial_\mu\partial_\nu D)$ stands for all possible factors, $\partial_\mu D$, $\partial_\mu\partial_\nu D$, $(\partial_\mu D)(\partial_\nu D)$, $(\partial_\mu\partial_\nu D)(\partial_\nu D)$, $(\partial_\mu D)(\partial_\nu D)^2$, $(\partial_\mu\partial_\nu D)^2$, $(\partial_\mu\partial_\nu D)(\partial_\mu D)(\partial_\nu D)$ and $(\partial_\mu D)^2(\partial_\nu D)^2$, and the contour $C_f(\mathrm{Re}z < 0)$ is taken in a range where the FT of $f(\partial_\mu D,\partial_\mu\partial_\nu D)D^z$ can be well defined.

The generalized function $f(\partial_\mu D,\partial_\mu\partial_\nu D)D^z$ is dealt with as follows: Consider $(\partial_\mu\partial_\nu D)e^{\kappa D}$, for example. Supposing D is an ordinary classical function, $D = (x^2)^{-1}$, we obtain the identity

$$(\partial_\mu\partial_\nu D)D^z = \frac{2}{(z+1)(z+2)}\{\partial_\mu\partial_\nu - \frac{\delta_{\mu\nu}}{4}\Box\}D^{z+1}, \quad (\Box=\partial_\mu\partial_\mu) \qquad (12)$$

which holds also for the generalized function D except in the neighborhood of $x_\mu = 0$. Evidently, (12) fails at $z = 0$, since the left hand side then becomes $\partial_\mu\partial_\nu D$ while the right hand side is not so because of the fact $\Box D=i\delta^4(x)$ [$\neq 0$] for the generalized function D. Information in the neighborhood of $x_\mu = 0$ lacks this point. Hence, (12) is to be elaborated so that the second term of the right hand side should vanish at $z = 0$, and then we are to have a valid formula for all x_μ in place of (12). Noting that $(\partial_\mu\partial_\nu D)D^z$ always appears as the integrand like (11), we adopt

$$(\partial_\mu\partial_\nu D)D^z = \lim_{\delta\to 0} \frac{2}{(z+1)(z+2)}\{\partial_\mu\partial_\nu - \frac{\delta_{\mu\nu}}{4}(\frac{z}{z+\delta})^N \Box\}D^{z+1} \qquad (13)$$

as the valid formula,[6] where N is an integer (≥ 1) and δ is such a positive number that the singularity at $z = -\delta$ lies to the left of the z-contour, and the limit $\delta \to 0$ is taken after the z-integration. Similar arguments lead to formulae for the other remaining $f(\partial_\mu D, \partial_\mu\partial_\nu D)D^z$, from which their FT together with the position of C_f are obtained by use of (10). Next, we note that the operator $P^{a_i b_i}_{\Delta_i}$ defined by (5) may yield extra singularities in (8) in the complex z_{ij} plane of several variables; in fact, the t_i-integration of $P^{a_i b_i}_{\Delta_i}$ produces factors like $(\sum z_{ij} + n + \Delta_i)^{-1}$ with n any positive integer. Such a singularity, however, can be avoided: Take the positive number Δ_i to be so large that the contours C_{ij} guarantee $(\sum \mathrm{Re} z_{ij} + n + \Delta_i) > 0$, then we can safely bend the contour C_{ij} back to Γ_{ij} in (8), since we can assure that the inequality $(\mathrm{Re} z_{ij} + n) \geq 1$ always holds at any pole-point with respect to z_{ij} when we count the residues of the z_{ij}-integrals. Hence, the limit $\Delta_i \to 0$ brings no singular result.

In this stage, we can grasp a power-counting scheme for supergraphs, which estimates the degree of ultraviolet divergences in their momentum integrations. In (11), we transform z, the real part of which is already in a certain range, into z' such that $z \to z'$ ($0 < \mathrm{Re} z' < 1$). We then find that the FT of $f(\partial_\mu D, \partial_\mu\partial_\nu D)e^{\kappa D}$ becomes proportional to $(k^2)^{\alpha(n)}$ with $\alpha(n) = z'-2+(n/2)$ according to n (= 0,1,2,3,4) being the number of derivatives in f. Hence, with respect to asymptotic behaviors as $k \to \infty$, the FTs of $fe^{\kappa D}$ can be classified into five types by n; denote each of them by S(k;n). Our power-counting scheme is as follows: Consider an N-th order supergraph, in which the numbers of S(k;n) are I_n. Then, it holds

$$I \equiv \sum_{n=0}^{4} I_n = N(N-1)/2 \ , \ I_1 + 2I_2 + 3I_3\ 4I_4 = 2N - E_{\partial\xi} \ , \qquad (14)$$

where $E_{\partial\xi}$ is the number of external $\partial_\mu\xi^a$-lines. The degree of divergence d is given in this case by

$$d = 4(I-N+1) - (4I_0+3I_1+2I_2+I_3) + 2\sum_{ij} \mathrm{Re}z'_{ij}$$

$$= 4 - 2N - E_{\partial\xi} + 2\sum_{ij} \mathrm{Re}z'_{ij} \qquad (15)$$

from (14). Since $\mathrm{Re}z'_{ij}$ can be taken to be as close to zero as possible, the condition for convergence of the supergraph ($d < 0$), $4 - 2N - E_{\partial\xi} < 0$, is surely satisfied for $N \geq 3$. For $N = 2$, we have a single SP which is of course well defined. Therefore, we can conclude that all S-matrix elements are finite for L_{int}^{ξ}.

Finally we discuss the case of the total interaction Lagrangian. The presence of L_{int}^{A} calls for four more kinds of SPs other than those for L_{int}^{ξ} alone: $D_{\mu\nu}e^{\kappa D}$, $(\partial_\mu D)D_{\mu\nu}e^{\kappa D}$, $(\partial_\mu\partial_\nu D)D_{\mu\nu}e^{\kappa D}$ and $(\partial_\mu D)(\partial_\nu D)D_{\mu\nu} \times e^{\kappa D}$. Here, we are interested in their asymptotic behaviors. Noting the asymptotic form of $D_{\mu\nu}$ being as $\sim(k^2)^{-1}$, we can approximate $D_{\mu\nu}$ in them by D and obtain their asymptotic forms. Therefore, our power-counting scheme tells us that the degree of divergence of an N-th order supergraph with respect to the total L_{int} is given by

$$d = 4 - E_A - E_{\partial\xi} - 2N - 2(I_0^A + I_1^A + I_2^A) + 2\sum_{ij}\mathrm{Re}z'_{ij} \ . \qquad (16)$$

Here, E_A denotes the number of external A_μ^a-lines and I_n^A are those of SPs containing $D_{\mu\nu}$ and n derivatives of ξ. Recognizing that the term $(4-E_A)$ comes from the renormalizable interactions among A_μ^a, $\bar{C}^a$ and C^a, we find that the divergence of any total supergraph becomes the less, as the more super-vertices are inserted.

Thus, we conclude that the massive YM theory can be dealt with as being renormalizable in view of nonpolynomial Lagrangian theories.

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