RENORMALIZATION AND SCALING OF NON-ABELIAN GAUGE FIELDS IN CURVED SPACE-TIME

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It is well known that in flat space-time non-abelian gauge field theories are renormalizable and asymptotically free. In view of the absence of a theorem stating that flat-space renormalizability implies renormalizability in more general space-times, one must examine each theory individually. This is a brief summary of work by my student Todd K. Leen showing that non-abelian gauge fields in curved space-time are renormalizable at the one-loop level; and that the property of asymptotic freedom is preserved.[1] Renormalizability of these theories has been proved independently by David J. Toms.[2]

In the present work, it is shown that gauge invariance, as expressed through the Taylor-Slavnov identities, insures that no curvature-dependent divergences occur in the vector two-point function. The divergences in the ghost two-point function and the ghost-vector-ghost vertex are extracted using local momentum space expansions for the propagators. As with the vector two-point function, no curvature-dependent divergences are found. Thus, these three Green functions are rendered finite by the usual Minkowski space counterterms. It then follows from gauge invariance, that the three- and four-vector vertex functions are finite. Finally, renormalization group arguments are used to show that the theory remains asymptotically free in curved space-time.

We first establish notation. The gauge group is denoted Ω and its associated Lie algebra T_Ω. The gauge covariant derivative is written

$$D_\mu \equiv \nabla_\mu - gA_\mu \tag{1}$$

where ∇_μ is the covariant derivative on the space-time and g the coupling constant of the theory. The field strength (or curvature) tensor is given by

$$F_{\mu\nu} \equiv \nabla_\nu A_\mu - \nabla_\mu A_\nu + g[A_\mu, A_\nu] \ . \tag{2}$$

The potentials are decomposed along a basis $\{T^a\}$ on T_Ω as

$$A_\mu = A^a_\mu T_a \ . \tag{3}$$

The gauge invariant classical action is then written as

$$S = \int d\tau(x) \ (- \tfrac{1}{4} F^{\mu\nu}_a F^a_{\mu\nu}) \tag{4}$$

where $d\tau(x)$ is the covariant volume element on the space-time. The theory is quantized via the path integral formalism. The procedure of Faddeev and Popov[3] allows the elimination of the redundancy due to integration over gauge related field configurations. The resulting generating functional with ghost fields $\overline{C}$ and C is written

$$W[j,\overline{\xi},\xi] = \int D[A]D[\overline{C}]D[C] \exp i\int d\tau(x)\Big\{-\frac{1}{4} F_a^{\mu\nu}F_{\mu\nu}^a - \frac{1}{2\alpha}(\nabla_\mu A^\mu)^2$$

$$- (\nabla_\mu \overline{C}^a)(D^\mu C)_a + j_a^\mu A_\mu^a + \overline{\xi}^a C_a + \overline{C}^a \xi_a \quad . \tag{5}$$

The perturbative expansion of this generating functional and the extraction of Green functions leads to a diagrammatic expansion as in flat space-time. The bare propagators for the ghost and vector fields satisfy, respectively

$$\Box D_{ab}(x,x') = -\delta_{ab}\,\delta(x,x') \tag{6}$$

and

$$\Box D_{\nu\sigma}^{ab}(x,x') - (1 - \frac{1}{\alpha})\nabla_\nu \nabla^\mu D_{\mu\sigma}^{ab}(x,x') - R_\nu{}^\mu D_{\mu\sigma}^{ab}(x,x') = -g_{\mu\sigma}\delta^{ab}\delta(x,x') \tag{7}$$

where $\delta(x,x')$ is the covariant delta function and the derivatives and Ricci tensor are at the point x.

The gauge invariance of the classical action Eq. (4) is reflected in relations between the Green functions of the theory revealed in the Taylor-Slavnov identities. For the corrected vector propagator one recovers the relation ($\alpha = 1$ hereafter)

$$\nabla'_\nu \nabla_\mu \tilde{D}_{ab}^{\mu\nu}(x,x') = \delta_{ab}\delta(x,x') \quad . \tag{8}$$

In lowest order we write the corrected vector propagator as

$$\tilde{D}^{\mu\nu}(x,x') = D^{\mu\nu}(x,x') + \int d\tau(y)d\tau(y')D^{\mu\sigma}(x,y)\pi_{\sigma\rho}(y,y')D^{\rho\nu}(y',x') \tag{9}$$

where the vacuum polarization tensor $\pi_{\sigma\rho}(y,y')$ contains one-loop contributions only. Making use of Eqs. (8) and (7) we recover

$$\nabla_\mu \nabla'_\nu \pi^{\mu\nu}(x,x') = 0 \quad . \tag{10}$$

Thus the vacuum polarization (to lowest order) remains transverse as in Minkowski space. The pole part of $\pi^{\mu\nu}$ may be written as

$$\pi^{\mu\nu}(x,x')\Big|_{\text{divergent}} = Ag^{\mu\nu}\delta(x,x') + B\nabla^\mu \nabla^{\nu'}\delta(x,x') + E\,\Box g^{\mu\nu}\delta(x,x')$$

$$+ (aR^{\mu\nu} + bg^{\mu\nu}R)\delta(x,x') \tag{11}$$

where A represents the quadratic divergence while B, E, a and b carry logarithmic divergences. This may be understood as follows. The divergences in $\pi^{\mu\nu}(x,x')$ arise in the coincidence limit of the arguments, hence the delta functions. The $\nabla^\mu \nabla^{\nu'}$, $\Box$, $R^{\mu\nu}$ and $g^{\mu\nu}R$ factors carry dimension $(\text{length})^{-2}$ so that their corresponding coefficients must carry two additional powers of momentum in the denominator of

Feynman integrals. Since A carries the leading quadratic divergence, the remaining terms are, at most, logarithmically divergent. The transversality of the polarization Eq. (10) determines relations between the coefficients in Eq. (11). One finds

$$\pi^{\mu\nu}(x,x')\Big|_{\text{div.}} = B(g^{\mu\nu}\Box\delta(x,x') + \nabla^{\mu}\nabla^{\nu'}\delta(x,x') - R^{\mu\nu}\delta(x,x')) \ . \tag{12}$$

Substituting this form for the polarization tensor into Eq. (9) for the corrected vector propagator leaves

$$\tilde{D}^{\mu\nu}(x,x') = (1-B)D^{\mu\nu}(x,x') - B\int d\tau(y)(\nabla'_{\rho_y}\nabla_{\sigma_y}D^{\mu\sigma}(x,y))D^{\rho\nu}(y,x') \tag{13}$$

Since B is a curvature-independent space-time constant, we see that curvature-dependent divergences in the vector propagator are absent. The divergence in the first term of Eq. (13) is removed by wavefunction renormalization while that in the second term is removed by renormalizing the gauge fixing parameter. We define renormalized quantities via

$$A^{\mu} = Z_3^{1/2}\ A_R^{\mu} \tag{14}$$

$$\alpha = Z_3\ \alpha_R \ . \tag{15}$$

The renormalization constant is identical to that in Minkowski space

$$Z_3 = 1 + \frac{g^2C_2}{16\pi^2}\left(\frac{5}{3}\right)\frac{2}{\varepsilon} \tag{16}$$

where $\varepsilon = \frac{-N}{2} + 2$ is the dimensional parameter appearing in the regularization and C_2 is the value of the quadratic Casimir operator for Ω.

We have seen that the gauge invariance is sufficient to insure that no divergences not present in Minkowski space arise in a general curved space-time. The appearance of explicitly curvature-dependent divergent corrections to the vector two-point function would have necessitated the introduction of renormalized couplings between the gauge field and the background. Such couplings would spoil the gauge properties of the theory.

The divergences in the ghost propagator are handled using the local momentum space expressions of Refs. (4,5,6) for the bare propagators. Here again we find no divergences not present in Minkowski space. The latter are removed by defining renormalized ghost fields

$$C = \tilde{Z}_3^{1/2}\ C_R \tag{17}$$

with the usual renormalization constant

$$\tilde{Z}_3 = \left(1 + \frac{g^2C_2}{16\pi^2}\frac{1}{\varepsilon}\right) \ . \tag{18}$$

Finally, the vector-ghost-ghost vertex is computed using the momentum space expansions. Again we find no curvature-dependent divergences. This vertex is thus renormalized as in flat space-time leading to the definition of the renormalized coupling constant

$$g = \frac{\tilde{Z}_1}{\tilde{Z}_3 Z_3^{1/2}} g_R \mu^{\varepsilon/2} \tag{19}$$

where μ is the mass parameter required to maintain proper dimensions during the regularization. The constant $\tilde{Z}_1$ is as in Minkowski space

$$\tilde{Z}_1 = 1 - \frac{g^2 C_2}{16\pi^2} \frac{1}{\varepsilon} . \tag{20}$$

Gauge invariance insures that the above renormalizations are sufficient to render finite the three- and four-vector vertices.

We have seen that the divergences in the theory are removed by the same counter-terms that render the flat space theory finite. This suggests that the remarkable feature of asymptotic freedom remains in the presence of space-time curvature. To verify this we derive a renormalization group equation and exhibit the behavior of the effective coupling constant. The renormalized one particle irreducible Green functions are given by

$$\Gamma^{(n)}_{\mu,\dots}(x_1,\dots x_n;\ g_R,\alpha_R,\mu,g_{\alpha\beta}) = Z_3^{(n/2)} \Gamma^{UN}_{\mu,\dots}(x_1,\dots x_n;\ g,\alpha,g_{\alpha\beta}) \tag{21}$$

where Γ^{UN} are the unrenormalized n-point functions. Differentiating the above with respect to μ and multiplying by μ leaves

$$(\mu\ \partial/\partial_\mu + \beta\ \partial/\partial g_R - \gamma(\alpha_R\ \partial/\partial\alpha_R + \tfrac{n}{2}))\Gamma^{(n)}_{\mu,\dots}(x_1\dots x_n;\ g_R,\alpha_R,\mu,g_{\alpha\beta}) = 0 \tag{22}$$

where

$$\beta \equiv \mu \frac{\partial g_R}{\partial\mu} \tag{23}$$

$$\gamma \equiv \mu \frac{\partial \ln Z_3}{\partial\mu}$$

The usual procedure is to eliminate $\mu\ \partial/\partial\mu$ by scaling the coordinates x_i or the momenta p_i of the external legs. In curved space-times the natural approach is to scale the metric tensor.[7] We consider a one-parameter family of metrics $g_{\alpha\beta}/K^2$ and scale the parameter K. The resulting renormalization group equation reads.

$$[(D - \tfrac{n}{2}\gamma) - K\ \partial/\partial K + \beta\ \partial/\partial g_R - \gamma\alpha_R\ \partial/\partial\alpha_R]\Gamma^{(n)}_{\mu,\dots}(x_1\dots;\ g_R,\alpha_R,\mu,g_{\alpha\beta}/K^2) = 0. \tag{24}$$

where D is the mass dimension of Γ.

Since, in lowest order, the β function is independent of α,[8,9] we are free to discuss the solution to this equation for $\alpha = 0$; and calculate the β function using our previous results (calculated using $\alpha = 1$) for the renormalization constants. The solution to Eq. (24) is

$$\Gamma^{(n)}_{\mu,\ldots}(x_1,\ldots x_n;g_R,\alpha_R=0,\mu,g_{\alpha\beta}/K^2)$$

$$= K^D\Gamma^{(n)}_{\mu,\ldots}(x_1\ldots x_n;\ g(K),\mu,g_{\alpha\beta})\exp - \frac{n}{2}\int_1^K \frac{dK'}{K'}\,\gamma(g(K')) \qquad (25)$$

with $$K\,\frac{\partial g}{\partial K} = \beta(g(K)) \qquad g(K=1) = g_R \qquad (26)$$

Using Eqs. (16), (18), (19) and (20) we find

$$\beta = -\,\frac{11}{3}\,\frac{C_2}{16\pi^2}\,g^3(K) \qquad (27)$$

Because β is negative this indicates that the theory remains asymptotically free in curved space-times.

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