TWO-POINT FUNCTIONS AND RENORMALIZED OBSERVABLES

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This is a victory declaration in the theory of a quantized field propagating in a given curved background space-time. We now have an unambiguous, internally consistent quantum theory of such a system, in which any physical quantity can in principle be calculated. Whether this kind of model is relevant to the real world is a separate question. I shall present the theory rather dogmatically.

Doctrine 1: The physical interpretation of a quantum field theory in curved space must be sought in the stress-energy-momentum tensor and other local field observables. (Particle observables are not meaningful, in general.)

As a concrete example, let's keep in mind the minimally coupled neutral scalar field, whose stress tensor is

$$T_{\mu\nu}(x) = \nabla_\mu \phi(x) \nabla_\nu \phi(x) - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi + \frac{1}{2} m^2 g_{\mu\nu}(x) \phi(x)^2. \qquad (1)$$

For a field satisfying a linear wave equation, there is little difficulty in defining Heisenberg-picture field operators rigorously. Making sense of the stress tensor, as a quantum operator, is much more of a problem. In my opinion, the most profound clarification of this problem came in a series of papers by Wald (1977, 1978a,b). The ingredients of the solution go back to the work of Utiyama and DeWitt (1962). The bibliography lists many (but not all) other papers which have contributed to our present understanding.

Doctrine 2: The stress tensor is conserved [$\nabla^\mu T_{\mu\nu} = 0$], it depends causally on the metric, and the difference of its expectation values with respect to any two quantum states can be correctly calculated from the classical formula (1). ("The divergent part of $T_{\mu\nu}$ is a c-number.")

(These are three of Wald's five axioms. The other two have been supplanted by an improved understanding of the renormalization problem. See Wald's papers for a definition of "causally." I leave as an exercise the equivalence of Wald's axiom about orthogonal matrix elements to mine about differences of expectation values.)

Theorem 1 (Wald): The stress tensor operator of a given field theory is uniquely determined by these requirements, up to c-number terms proportional to conserved, covariant, local, polynomial functionals of the metric and curvature tensors, such as

$$c_1 g_{\mu\nu} + c_2(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}) + c_3 g^{-1/2}\frac{\delta}{\delta g^{\mu\nu}}\int R^2 g^{1/2}dx + c_4 g^{-1/2}\frac{\delta}{\delta g^{\mu\nu}}\int C^2 g^{1/2}dx. \tag{2}$$

(C^2 is the square of the Weyl tensor).

Wald observed that the procedure of "point-splitting" seemed to provide a $T_{\mu\nu}$ satisfying his axioms. The evidence (e.g., from calculations in simple models) suggested: (1) The two-point function

$$G(x,y) = \langle\psi|\phi(x)\phi(y)|\psi\rangle$$

is well-defined as a distribution for a large class of states ψ. (2) As $y \to x$, G has an asymptotic expansion consisting of terms which are singular where the geodesic separation between x and y is null (lightlike). These singular terms are (a) c-numbers (have the same value in all quantum states) and (b) purely local and polynomial in their dependence on the geometry. For example, a typical singular term in $G(x,y)$ is

$$\frac{R_{\mu\nu}(x)\sigma^\mu\sigma^\nu}{g_{\alpha\beta}(x)\sigma^\alpha\sigma^\beta} \quad ,$$

where $-\sigma^\mu(x,y)$ is the vector at x tangent to the geodesic from x to y, with length equal to the geodesic separation (in other words, the Riemann normal coordinates of y relative to x). [A similar description applies to an expectation value of the point-split stress tensor, $T_{\mu\nu}(x,y)$, obtained by applying a suitable differential operator to $G(x,y)$, so that $T_{\mu\nu}(x)$ [Eq. (1)] is formally recovered when $y = x$:

$$T_{\mu\nu}(x,y) = e_\nu{}^{\nu'}(x,y)\nabla_\mu\phi(x)\nabla_{\nu'}\phi(y) + \ldots ,$$
$$\langle\psi|T_{\mu\nu}(x,y)|\psi\rangle = e_\nu{}^{\nu'}\nabla_\mu\nabla_{\nu'}G(x,y) + \ldots ,$$

where $e_\nu{}^{\nu'}$ is a parallel-transport matrix. See Christensen (1976) for details.] (3) The symmetric part (under the interchange $x \leftrightarrow y$) of this singular series coincides with what is called the Hadamard solution of the hyperbolic field equation ($\nabla^\mu\nabla_\mu + m^2\phi = 0$ in our example). For the definition of this object, see DeWitt and Brehme (1960), Hadamard (1952), Garabedian (1964), Friedlander (1975), Adler et al. (1977). (The antisymmetric part of G is the familiar commutator distribution, which is entirely c-number and local and hence can be disregarded in the rest of our discussion.

I emphasize that the remainder in the symmetrized G (which can be made arbitrarily smooth by subtracting off enough terms of the series) is not a local functional of the metric and is not a c-number -- it depends on the quantum state ψ. Precisely for this reason it contains the most interesting physics in any concrete problem.

For consistency of the theory it was necessary to prove that the picture I've just described holds in general. In 1978, Sweeny, Wald, and I proved:

Theorem 2: If a two-point function, $\langle\psi|\phi(x)\phi(y)|\psi\rangle$, is a distribution of Hadamard form at one instant of time, then it remains so for all time.

In 1981, Narcowich, Wald, and I closed the remaining hole by proving:

Theorem 3: In a static background geometry, if ψ is the "natural" vacuum state and the mass is positive, then $G(x,y)$ is a distribution of the Hadamard form.

Thus it now makes sense to state:

Doctrine 3: A "physically reasonable" state ψ of a quantum field system in curved space is one for which the two-point function G is a distribution with singularity of the Hadamard form.

For, as a corollary of Theorems 2 and 3, we have:

Theorem 4: In an arbitrary globally hyperbolic space-time (i.e., one where the Cauchy problem is well-posed) there exist many "physically reasonable" states. (They form a dense subspace of a Hilbert space.)

If $m > 0$, the vacuum in a static background $g_{\mu\nu}$ is in this class of states. If $m = 0$, strangely enough, the traditional vacuum sometimes does not qualify as physically reasonable because its two-point function does not exist, as a distribution. In that case the "good" states contain, in some sense, lots of particles in the infrared modes.

The reason why the Hadamard states are regarded as "physically reasonable" is that they yield finite expectation values for physical observables after renormalization:

Doctrine 4: The renormalized stress tensor, $T^{ren}_{\mu\nu}(x)$, is obtained by subtracting from $T_{\mu\nu}(x,y)$ a c-number equal to sufficiently many terms of its Hadamard series, and taking the limit $y \rightarrow x$. The ambiguity in this prescription must be reduced by requiring that (1) $T^{ren}_{\mu\nu}$ is conserved, and (2) terms involving derivatives of $g_{\mu\nu}$ of degree higher than fourth are not to be subtracted.

This procedure is manifestly covariant, since the Hadamard expansion is. The result is ambiguous since the terms to be subtracted could be changed by any covariant, local, polynomial functional of the curvature tensor at x. Even after requirements (1) and (2) are imposed, some ambiguity remains (cf. Theorem 1).

Doctrine 5: The ambiguous terms in $T^{ren}_{\mu\nu}$ can be absorbed into the coupling constants in the gravitational field's equation of motion. In other words, they represent the nonexistence of any clear division of the physical energy density into matter energy and gravitational energy.

Indeed, the ambiguous terms are precisely those listed in Eq. (2); c_1 renormalizes the cosmological constant, c_2 renormalizes the gravitational constant, and c_3 and c_4 renormalize the coefficients of terms involving fourth derivatives of the metric tensor. These coupling constants must be determined by experiment. The appearance of fourth-order terms is one aspect of the nonrenormalizability of the gravitational interaction.

Theorem 5 (Wald): In a scale-invariant theory, it is meaningless to require a priori that c_3 and c_4 be 0.

The point is that c_3 and c_4 appear in contexts of the general nature of

$$c_3 + \ln(L^2R).$$

The value of c_3 depends on the arbitrary length L. We could define L so that $c_3 = 0$, but then L would be a new fundamental constant with units of length. It could be determined (in principle) by experiment, but it cannot be predicted by a scale-invariant theory. Similar phenomena are known to particle physicists under the headings of "renormalization group" and "dimensional transmutation."

The point I have tried to make by this show of orthodoxy is this: Within its own terms, the theory is completely well defined, and it is uniquely determined, I think, by accepted physical principles (except for the numerical values of coupling constants). The calculation of the stress tensor, or some other observable, in any given quantum state is today a problem of ordinary applied mathematics, not a matter of the individual investigator's choice of ad hoc mumbo-jumbo as was the case seven years ago.

However, I do not regard it as a substitute for a full quantum theory of gravity. The physical circumstances under which such an external-field model is a valid approximation to reality have not yet been established; they are the subject of active investigation and debate [Horowitz (1981), Duff (1981), Kay (1981), Ford (1982), Fulling (1983)].

BIBLIOGRAPHY

Adler, S. L., Lieberman, J., and Ng, Y. J., Ann. Phys. (N.Y.) 106, 279, (1977).

Bunch, T. S., Phys. Rev. D 18, 1844 (1978).

Bunch, T. S., Christensen, S. M., and Fulling, S. A., Phys. Rev. D 18, 4435 (1978).

Christensen, S. M., Phys. Rev. D 14, 2490 (1976).

Christensen, S. M., and Fulling, S. A., Phys. Rev. D 15, 2088 (1977).

Davies, P. C. W., Proc. Roy. Soc. A 354, 529 (1977a).

Davies, P. C. W., Phys. Lett. B 68, 402 (1977b).

Davies, P. C. W., and Fulling, S. A., Proc. Roy. Soc. A 354, 59 (1977).

Davies, P. C. W., Fulling, S. A., Christensen, S. M., and Bunch, T. S., Ann. Phys. (N.Y.) 109, 108 (1977).

Davies, P. C. W., Fulling, S. A., and Unruh, W. G., Phys. Rev. D 13, 2720 (1976).

Davies, P. C. W., and Unruh, W. G., Phys. Rev. D 20, 388 (1979).

DeWitt, B. S., Dynamical Theory of Groups and Fields, Gordon and Breach, New York (1965).

DeWitt, B. S., Phys. Reports 19, 295 (1975).

DeWitt, B. S., and Brehme, R. W., Ann. Phys. (N.Y.) 9, 220 (1960).

Dimock, J., Commun. Math. Phys. 77, 219 (1980).

Dowker, J. S., and Critchley, R., Phys. Rev. D 15, 1484 (1977).

Duff, M. J., in Quantum Gravity 2: A Second Oxford Symposium, ed. by C. J. Isham, R. Penrose, and D. W. Sciama, Oxford University Press, Oxford (1981), pp. 81-105.

Ford, L. H., Phys. Rev. D 11, 3370 (1975).

Ford, L. H., "Gravitational Radiation by Quantum Systems," to appear (1982).

Friedlander, F. G., The Wave Equation on a Curved Space-Time, Cambridge University Press, Cambridge (1975).

Fulling, S. A., in Quantum Theory and Gravitation, ed. by A. R. Marlow, Academic Press, New York (1980), pp. 187-197.

Fulling, S. A., in a volume in honor of B. S. DeWitt, ed. by S. M. Christensen, Institute of Physics, to appear (1983).

Fulling, S. A., Narcowich, F. J., and Wald, R. W., Ann. Phys. (N.Y.) 136, 243 (1981).

Fulling, S. A., and Parker, L., Ann. Phys. (N.Y.) 87, 176 (1974).

Fulling, S. A., Sweeny, M., and Wald, R. W., Commun. Math. Phys. 63, 257 (1978).

Garabedian, P. R., Partial Differential Equations, Wiley, New York (1964).

Hadamard, J., Lectures on Cauchy's Problem in Linear Differential Equations, Dover, New York (1952).

Horowitz, G. T., in Quantum Gravity 2: A Second Oxford Symposium, ed. by C. J. Isham, R. Penrose, and D. W. Sciama, Oxford University Press, Oxford (1981), pp. 106-130.

Horowitz, G. T., and Wald, R. M., Phys. Rev. D 17, 414 (1978).

Horowitz, G. T., and Wald, R. M., Phys. Rev. D 25, 3408 (1982).

Isham, C. J., in Differential Geometrical Methods in Mathematical Physics II, ed. by K. Bleuler, H. R. Petry, and A. Reetz, Springer, Berlin (1978), pp. 459-512.

Kay, B. S., Commun. Math. Phys. 62, 55 (1978).

Kay, B. S., Phys. Lett. B 101, 241 (1981).

Nariai, H., Prog. Theor. Phys. 46, 433 (1971).

Parker, L., Phys. Rev. 183, 1057 (1969).

Parker, L., and Fulling, S. A., Phys. Rev. D 9, 341 (1974).

Sakharov, A. D., Dokl. Akad. Nauk SSSR 177, 70 (1967) [Sov. Phys. --Dokl. 12, 1040].

Sakharov, A. D., Teor. Mat. Fiz. 23, 178 (1975) [Theor. Math. Phys. 23, 435].

Schwinger, J., Phys. Rev. 82, 664 (1951).

Utiyama, R., and DeWitt, B. S., J. Math. Phys. 3, 608 (1962).

Vilenkin, A., Nuovo Cim. A 44, 441 (1978).

Wald, R. M., Commun. Math. Phys. 54, 1 (1977).

Wald, R. M., Ann. Phys. (N.Y.) 110, 472 (1978a).

Wald, R. M., Phys. Rev. D 17, 1477 (1978b).