The geometry of the configuration space of non abelian gauge theories

C.M. Viallet

Laboratoire de Physique Théorique et Hautes Energies
Université Pierre et Marie Curie
Tour 16 - 1er étage
4 place Jussieu
75230 PARIS CEDEX 05

We present here some results on the geometry of the configuration space of non abelian gauge theories [1, 2, 3]. It is on this space that a Schrödinger equation (or equivalently a path integral) is to be defined. The study of the geometry of the configuration space is a necessary step for a proper non perturbative quantization of the theory. Two basic ingredients enter our study : (1) a volume cut-off (space is compact without boundary e.g. a sphere or a torus) and (2) the theory is non abelian.

I. SOME NOTATIONS [4, 5]:

The basic objects of the theory are gauge potentials (connections) on which acts a group of gauge transformations. Let $\mathcal{C}$ be the space of connections on a principal fibre bundle $P(M, G)$ where $\mathcal{M}$ = compact metric space without boundary and G = compact semisimple group.

On $\mathcal{C}$ acts the group of gauge transformations $\mathcal{G}$. Locally

$$A_\mu \longrightarrow {}^{g}A_\mu = g^{-1}\partial_\mu g + g^{-1} A_\mu g$$

To any ω in $\mathcal{C}$ is associated a covariant derivative ∇_ω acting on covariant objects. In a small gauge transformation $g = \exp \xi \approx 1 + \xi$, we have $\delta\omega = \nabla_\omega \xi$.

$\mathcal{C}$ is an affine space.

$\mathcal{C}$ is equipped with a gauge invariant scalar product $(\ , \ )$:

$$(\tau, \eta) = \int_{\mathcal{M}} dv \ \ \mathrm{tr}(\tau_\mu \eta^\mu).$$

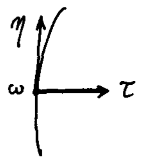
II. LOCAL STRUCTURE OF $\mathcal{C}$:

Through a point ω we may draw the orbit formed with all gauge related points in $\mathcal{C}$. Tangent vectors to this orbit at ω will be called <u>vertical</u> at ω (all vertical vectors are of the form $\nabla_\omega \xi$). By definition, a vector at ω is said to be <u>horizontal</u> if it is perpendicular to all vertical vectors at ω. If ∇_ω^* is the adjoint of ∇_ω with respect to $(\ , \ )$, horizontal vectors verify :

$$\nabla_\omega^* \tau = 0.$$

Generically the covariant laplacian $\Box_\omega = \nabla_\omega^* \nabla_\omega$ has trivial kernel. We denote by G_ω its inverse.

The local structure of $\mathcal{C}$ around a generic point may then be described by the splitting of the tangent space at ω into vertical and horizontal space.



$T_\omega(\mathcal{E}) = H_\omega \oplus V_\omega$.

V_ω = space of vertical vectors at ω = tangent through the orbit through ω

H_ω = space of horizontal vectors.

For generic points, there is a projection operator $\Pi_\omega = \mathbb{1} - \nabla_\omega G_\omega \nabla_\omega^*$ (orthogonal projection on H_ω along V_ω).

III. GAUGE CONDITION

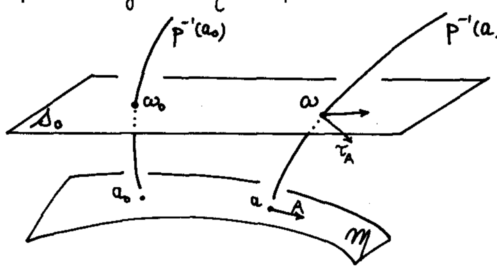
To fix the gauge is to cut all orbits once. Around a point ω_0 define the set $\Delta_0 = \{\omega \in \mathcal{E} \mid \tau = \omega - \omega_0 \text{ is horizontal at } \omega_0\}$. For generic connections this is locally a good gauge condition [6]. We then have a local coordinate system around ω_0 for the quotient space $\mathcal{E}/\mathcal{G}$, given by the covariant background gauge condition at ω_0.

IV. ORBIT SPACE. METRIC ON THE ORBIT SPACE

Modulo certain restrictions on the connections and the gauge transformations (taking some away and imposing some regularity conditions) the quotient space $\mathcal{E}/\mathcal{G}$ is a C^∞ manifold of infinite dimension and the projection $p : \mathcal{E} \longrightarrow \mathcal{M} = \mathcal{E}/\mathcal{G}$ is a principal fibration [1, 2, 7].

Notice that the horizontality condition introduced above in $\mathcal{E}$ yields a connection in $\mathcal{E}$ with connection form $\chi_\omega = G_\omega \nabla_\omega^*$, which is not flat : we cannot construct a horizontal section of $\mathcal{E}$.

Since $\mathcal{E}$ is equipped with a gauge invariant scalar product, there is an induced scalar product g on $\mathcal{M}$ computed as follows :



Suppose A and B are vectors tangent to $\mathcal{M}$ at a. Let ω be a point of $p^{-1}(a)$. The vectors A and B have horizontal lifts τ_A and τ_B at ω. By definition $g(A, B) = (\tau_A, \tau_B)$. In the coordinate system defined by $\omega_0 \in p^{-1}(a_0)$, (supposing a is not too far from a_0) the metric is given by $g = \Pi_0 \Pi_\omega \Pi_0$.

V. NON SINGULAR LAGRANGIAN OF GAUGE THEORIES [8, 9].

M = 3-dimensional space, e.g. S^3, T^3 ... On $P(M, G)$ are defined time dependant potentials $A(t)$. The Lagrangian of the theory is $L = \frac{1}{2}(\dot{A} - \nabla A_0, \dot{A} - \nabla A_0) - V(A)$, where $\dot{A} = \frac{\partial A_i}{\partial t} dx^i$ and $V = \frac{1}{4}\int_M dv \; \mathrm{tr}\, F_{ij}^2 \quad (i,j = 1,2,3)$, $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$.

This lagrangian is singular. The use of Dirac's analysis for singular lagrangians yields the definition of the proper configuration space, which is nothing but the orbit-space $\mathcal{E}/\mathcal{G} = \mathcal{M}$. The non singular lagrangian we get is just $\mathcal{L} = \frac{1}{2}(\pi_A \dot{A}, \pi_A \dot{A}) - V$.

. In other words if a is a point on $\mathcal{M}$ and $\dot{a} = \frac{da}{dt}$, we have $\mathcal{L} = \frac{1}{2} g(\dot{a}, \dot{a}) - V(a)$, where g is just the natural metric introduced earlier [10].

VI. METRIC AND FADDEEV-POPOV DETERMINANT

In a given system of coordinates (say the covariant background gauge condition at ω_0) we may compare det g and det γ where γ = Faddeev Popov operator = $\nabla_0^* \nabla_\omega$, and $g = \Pi_0 \Pi_\omega \Pi_0$. Formally we have [10] .

$$\sqrt{\det g} = \frac{\det \gamma}{\sqrt{\det \Box_0}\,\sqrt{\det \Box_\omega}} \,.$$

VII. RIEMANNIAN CALCULUS ON $\mathcal{M}$.

We may write down covariant derivative, curvature, equation for geodesics. If we use the background gauge condition, everything can be expressed in terms of simple operators.

Set $P = \mathbb{1} - \nabla_0 \gamma^{-1} \nabla_\omega^*$ (resp. $P^* = \mathbb{1} - \nabla_\omega \gamma^{-1} \nabla_0^*$)

$K_\tau : \xi \to K_\tau(\xi) = [\tau, \xi]$, and K_τ^* its adjoint

$\chi_\omega = G_\omega \nabla_\omega^*$ and χ_ω^* its adjoint.

Suppose X , Y, Z denote vector fields.

The covariant derivative is

$$D_X Z = \frac{1}{2} P^* P \left(-\chi_\omega^* K_X^* \pi_\omega Z - \pi_\omega K_X \chi_\omega Z - \chi_\omega^* K_Z^* \pi_\omega X - \pi_\omega K_Z \chi_\omega X + [\chi_\omega X, \pi_\omega Z] + [\chi_\omega Z, \pi_\omega X] \right).$$

The Riemannian curvature tensor $R(X,Y) = [D_X, D_Y] - D_{[X,Y]}$ is

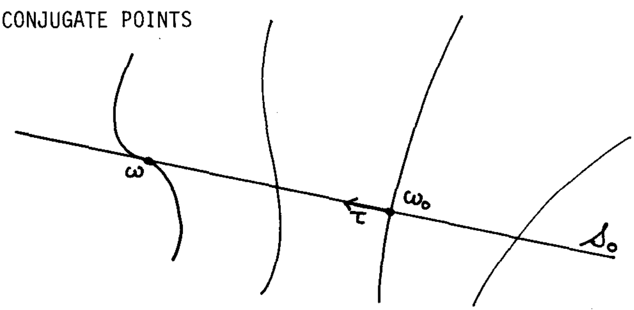
$$R(X,Y)Z = \Pi_0 \left(-2\, K_Z\, G_\omega\, K_X^*(Y) - K_Y\, G_\omega\, K_X^*(Z) + K_X\, G_\omega\, K_Y^*(Z) \right)$$

The sectional curvature in the 2-plane generated by two orthogonal vectors X, Y is $\mathcal{H}(X,Y) = g(R(X,Y)Y, X) = 3\,(K_X^*(Y), G_\omega K_X^*(Y))$.

<u>The sectional curvature is everywhere non negative</u>

Notice that any straight line in $\mathcal{E}$ which cuts one orbit perpendicularly cut all orbits it meets perpendicularly. In other words there are horizontal straight lines. They project on geodesics i.e. <u>the background gauge condition at ω_0 gives normal coordinates at ω_0</u> .

VIII. CONJUGATE POINTS



Starting from ω_0 along the horizontal line $\omega_0 + \lambda\tau$, we reach a conjugate point ω of ω_0 when some vector which is vertical at ω, verifies the gauge condition. This is equivalent to saying that there exists ξ such that $\nabla_0^* \nabla_\omega \xi = \gamma \xi = 0$ i.e. the Faddeev-Popov operator has non trivial kernel. For any ω_0 and τ, there is a finite λ for which this happens [1]. There are conjugate points at finite distance in all directions. Moreover if we consider the region Ω around ω_0 where γ is a positive operator, this region is convex and has the "Gribov horizon" as a boundary [11].

We thus get to the following conclusion : the configuration space of gauge theories is an infinite dimensional space, but the volume cut-off and the non abelian character of the theory makes it look like a "sphere" (positive curvature, possibly finite diameter). On that space is defined the potential term coming from the magnetic part of the Lagrangian. The next step toward quantization will be to write down a Schrödinger equation, which includes both, and control the removing of necessary cut-offs. Some hope is reasonable for 2 + 1 dimensions especially about about the existence of a mass gap. The 3 + 1 dimensional case is still out of reach, but in any case, the non trivial geometry of the configuration space is a saliant feature of non abelian pure gauge theory, and it will matter for any non perturbative result.

References

[1] I.M. Singer, Com. Math. Phys. 60, 7 (1978) and Physica Scripta 24, 817 (1981).
[2] M.S. Narasimhan, T.R. Ramadas, Com. Math. Phys. 67, 21 (1979).
[3] O. Babelon, C.M. Viallet, Com. Math. Phys. 81, 515 (1981).
[4] M. Daniel, C.M. Viallet, Rev. Mod. Phys. 52, 175 (1980).
[5] T. Eguchi, P.B. Gilkey, A. Hanson, Phys. Rep. 66, 6 (1980).
[6] M. Daniel, C.M. Viallet, Phys. Lett. 76B, 458 (1978).
[7] P.K. Mitter, C.M. Viallet, Com. Math. Phys. 79, 457 (1981).
[8] P.A.M. Dirac in "Lectures on Quantum Mechanics" Belfer Series (1964).
[9] L.D. Faddeev, Theor. Math. Phys. 1, 3 (1969).
[10] O. Babelon, C.M. Viallet, Phys. Lett. 85B, 246 (1979).
[11] D. Zwanziger, Preprint. NY U/TR 5/82.