Recent Developments in the Theory of Large N Gauge Fields

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It has been known for some time that quantum field theories with global or local U(N) (SU(N), O(N),,,) symmetry greatly simplify in the limit $N \to \infty$. The well-known example is the so-called O(N) vector model which is a theory of massless scalar fields ϕ_i (i=1,,N) taking values on S^{N-1}. In two dimensions lagrangian is given by

$$\mathcal{L} = \int \frac{1}{2} \sum_{i=1}^{N} (\partial_\mu \phi_i)^2 d^2x, \qquad \sum_{i=1}^{N} \phi_i^{\,2} = 1. \qquad (1)$$

It is known that the interaction caused by the constraint $\sum \phi_i^{\,2}=1$ creates a finite mass gap in this system ($N \geq 3$). In fact the model becomes exactly soluble in the large N limit and becomes a free theory of massive scalar particles. When N is finite, these massive fields begin to interact weakly with the strength 1/N. Thus the $N = \infty$ limit yields an exact solution which gives a qualitatively correct description of the system also for finite N.

In the case of gauge theories it is also known that a considerable simplification takes place in the limit of large gauge group. Both in the continuum and the lattice formulation of gauge theory the following characteristic properties have been known of the $N = \infty$ gauge fields.

1. Dominance of planar Feynman diagrams[1] (planar surfaces) in the weak coupling (strong coupling) perturbation theory;
2. The factorization property of Wilson loop amplitudes.[2]

These properties are derived from a power-counting analysis of Feynman (strong coupling) diagrams. In the case of weak coupling perturbation theory a simple combinatorial analysis shows that the weight of a Feynman diagram becomes N^χ where χ is the Euler characteristic of the diagram (interaction vertices, propagators and color loops are regarded as vertices, edges and faces of a polyhedron, respectively) when we let g^2N to be independent of N (or let $g^2=O(1/N)$). Thus in the limit of $N = \infty$ with g^2N fixed graphs with the heighest Euler number give the dominant contributions. In the case of a Wilson loop amplitude, for instance,

$$\langle \exp i \int_C A_\mu dx_\mu \rangle \approx O(N) \tag{2}$$

the leading contributions come from the planar graphs with the topology of a disc while non-planar graphs with h handles are down by N^{-2h}.

Moreover, when we consider the case of more than one quark loop, say C_1 and C_2, the leading contribution to the correlation function

$$\langle \exp i \int_{C_1} A_\mu dx_\mu \; \exp i \int_{C_2} A_\mu dx_\mu \rangle \tag{3}$$

come from the disconnected piece,

$$\langle \exp i \int_{C_1} A_\mu dx_\mu \rangle \langle \exp i \int_{C_2} A_\mu dx_\mu \rangle \approx O(N^2). \tag{4}$$

This is because when there exist gluon exchanges between C_1 and C_2 we obtain a topology of an annulus and hence of order only $O(N^0)$.

In this way in the large N limit with g^2N fixed there exist no correlation between quark loops and the amplitudes always factor

$$\langle \prod_i \exp i \int_{C_i} A_\mu dx_\mu \rangle = \prod_i \langle \exp i \int_{C_i} A_\mu dx_\mu \rangle \tag{5}$$

Since a Wilson loop amplitude may be interpreted as a meson propagator, factorization implies that at $N = \infty$ gauge interactions are exhausted to form bound states and mesons do not scatter from each other. In this respect $N = \infty$ gauge theory is analogous to O(N) vector model where interactions are also exhausted at $N = \infty$ in creating a dynamical mass to scalar fields. The dominance of planar surfaces and factorization are also shown order by order in the strong coupling lattice perturbation theory.

Reduction

Now the recent developments on the theory of large N gauge and spin system[3,4,5,6] have uncovered a further property of $N = \infty$ gauge fields;

3. Reduction of dynamical degrees of freedom.

By means of reduction one may replace the $N = \infty$ gauge field theory by a much simpler system, a model with only a finite number (=space-time dimensionality) of U(N) (or N×N hermitian) matrices, without loosing information of the original theory. This is a remarkable result in the sense that a quantum field theory may be reduced to a kind of dynamical system with a finite number of dynamical variables.
In the lattice formulation of gauge fields the argument goes as follows[3]; we start from the standard Wilson theory defined by the partition function

$$Z = \prod_{y} \prod_{\rho=1}^{d} dU_{y,\rho} \exp\{\beta \sum_{y} \sum_{\rho\neq\sigma=1}^{d} \mathrm{tr}\, U_{y,\rho} U_{y+\rho,\sigma} U^{+}_{y+\sigma,\rho} U^{+}_{y,\sigma}\} \qquad (6)$$

where $U_{y,\rho}$ is an U(N) matrix lying on a link connecting the lattice sites y and $y+\rho$ (ρ is the unit vector in the ρ-direction) and d is the dimensionality of space-time.
The Wilson loop amplitude is defined by

$$W(C) = \langle \mathrm{tr}\, U_{x,\mu} U_{x+\mu,\nu} U_{x+\mu+\nu,\lambda} \cdots U_{x-\sigma,\sigma} \rangle \qquad (7)$$

for a contour C which connects lattice sites $x, x+\mu, x+\mu+\nu, x+\mu+\nu+\lambda, , , x-\sigma$, x successively. We then reduce the model by identifying all link variables in the same direction

$$U_{y,\rho} \Longrightarrow U_{\rho} \qquad (8)$$

and thereby shrink the entire space-time lattice to a single hypercube. The reduced model is then defined in terms of matrices $U_1, U_2, , , \cdot\cdot, U_d$ and its action is given by

$$S_r = \sum_{\rho\neq\sigma=1}^{d} \mathrm{tr}(U_\rho U_\sigma U^{+}_\rho U^{+}_\sigma). \qquad (9)$$

The analogue of the Wilson loop amplitude is defined by

$$W_r(C) = \langle \mathrm{tr}(U_\mu U_\nu U_\lambda \cdots U_\sigma) \rangle_r \qquad (10)$$

where the averaging is taken with respect to the weight Eq.(9) and we have identified the contour C with the sequence of directions $(\mu,\nu,\lambda,,,\sigma)$

$$C\ ;\ (x,x+\mu,x+\mu+\nu,x+\mu+\nu+\lambda,\cdots,x-\sigma,x) \approx (\mu,\nu,\lambda,\cdots,\sigma). \qquad (11)$$

The above correspondence is one to one when we ignore over-all translations.
We remark that the reduced model Eq.(9) is ivariant under the phase transformation

$$U_\rho \longrightarrow e^{i\theta}\, U_\rho \qquad (12)$$

and this symmetry implies

$$W_r(C) = \langle \mathrm{tr}\, U_\mu U_\nu U_\lambda \cdots U_\sigma \rangle = 0 \qquad (13)$$

for every open contour C. This is because in the case of an open contour there exists at least one direction ρ for which U_ρ and U_ρ^+ appear different number of times and $W_r(C)$ has to vanish. It was noticed[3] that if the symmetry Eq.(12) is left intact, i.e. not spontaneously broken, the equation of motion for the Wilson loop amplitudes in the original and reduced models become identical in the limit $N = \infty$ with g^2N fixed. Consequently the Wilson loop amplitudes agree

$$W(C) = W_r(C). \qquad (14)$$

Also the free-energy per unit volume of the original theory agrees with the free-energy of the reduced model. Thus the infinite-volume Wilson theory Eq.(6) and the one-site reduced model Eq.(9) become equivalent to each other in the large N limit.
In order to elucidate the physical meaning of the reduction let us consider a "partial reduction" of $N = \infty$ gauge field where we shrink the size of one particular direction, say, the time-direction, of the d-dimensional lattice to a unit distance while keeping the size of the other directions unchanged. We compare the free-energy of the original and partially reduced systems,

$$\begin{aligned} Z &= \mathrm{Tr}\ T^L = e^{-LF} \\ Z_r &= \mathrm{Tr}\ T = e^{-F_r} \end{aligned} \qquad (15)$$

where we have introduced the transfer matrix and L is the original size of the time-direction. Making use of eigenstates of the transfer matrix with eigenvalues E_i (i=0,1,2,..) we obtain

$$F = E_0 \ ,$$
$$F_r = E_0 - \ln\left(1 + \sum_i e^{-(E_i - E_0)}\right). \qquad (16)$$

If the system confines, there exist only color-singlet excitations and the sum over i will not generate N dependent factors. Thus

$$F_r = E_0 + O(N^0). \qquad (17)$$

On the other hand we know that the vacuum energy E_0 is $O(N^2)$ and consequently

$$F_r = F, \qquad N = \infty. \qquad (18)$$

Therefore the free-energy of the $N = \infty$ gauge field is independent of the lattice size and we may reduce the theory so far as the system confines.

U(1) symmetry

In the partial reduction of the lattice we have squeezed the time-direction to a unit distance. This creates a finite-temperature situation since the periodic boundary condition is imposed in the reduced model. Thus the reduction effectively heats up the gauge system. It is known that gauge fields, when heated, will undergo a deconfining transition into a plasma phase at a certain critical temperature T_c via the spontaneous violation of the invariance under the center of the gauge group.[7,8] Therefore if this transition temperature stays finite at $N = \infty$ gauge fields, we would expect that there exists a critical coupling λ_c in the reduced model such that for λ $(= N/\beta) > \lambda_c$ U(1) (the center of U(N)) symmetry Eq.(12) is unbroken, however, it will break spontaneously below λ_c (T_c and λ_c are related as $1/T_c = a(\lambda_c)$ where a is the lattice constant dependent upon the coupling strength λ). Monte Carlo simulation of the reduced model[4,9] in fact shows the spontaneous violation of U(1) symmetry below $\lambda_c \simeq 2$. This is signaled by the non-zero expectation value for open loops. Below the transition point the eigenvalues of the reduced link variables U_ρ's are no longer

uniformly distributed but are concentrated around an arbitrary point on the unit circle. The transition is smooth and appears to be 2nd order. This is consistent with our interpretation of its being the finite-temperature deconfining transition.
Thus the original reduced model is equivalent to the standard Wilson theory only in the strong coupling regime $\lambda > \lambda_c$ and the equivalence will be lost below λ_c.

Quenching
In this situation it is possible to restore the broken symmetry by integrating over the location of the concentration of the eigenvalues In the quenching procedure of Bhanot-Heller-Neuberger link variables are diagonalized as

$$U_\rho = V_\rho D_\rho V_\rho^+ \qquad (D_\rho)_{ij} = e^{i\theta_\rho^j}\delta_{ij}. \tag{19}$$

The angular variables θ_ρ^j's are held fixed when we first average over marices V_ρ's

$$W(C;\theta) = \frac{\int \prod_\rho dV_\rho \; tr(V_\mu D_\mu V_\mu^+ V_\nu D_\nu V_\nu^+ \cdots V_\sigma D_\sigma V_\sigma^+) \; e^{+\beta S(V,\theta)}}{\int \prod_\rho dV_\rho \; e^{+\beta S(V,\theta)}} . \tag{20}$$

We then take an averaging over θ's

$$W_q(C) = \int d\mu(\theta) \quad W(C;\theta) \tag{21}$$

with a suitable measure μ. Using the method of Parisi[5] it was shown[6] that when one makes a change of variable

$$V_\mu D_\mu V_\mu^+ = D_\mu W_\mu . \tag{22}$$

and expands

$$W_\mu = \exp iag A_\mu \tag{23}$$

in terms of g, the quenched model reproduces the planar perturbation theory of the continuum gauge fields. Here the eigenvalue $\theta_\mu^i - \theta_\mu^j$ has the meaning of the momentum carried by a gluon line with a pair of

color indices (i,j).
Thus the quenched model reproduces the weak coupling expansion of the continuum gauge theory for small λ while it agrees with the original reduced model for large λ. Hence it is now believed to be equivalent to the standard theory throughout the range of coupling constants. Here the possible trouble is that the quenching procedure is justified only with recourse to the weak coupling perturbation theory and it is not completely clear if the non-perturbative information of the theory is coded correctly into the quenched model.

Twist

Another promissing method of avoiding the problem of the degeneracy of the eigenvalues of U_ρ is to introduce a twist to the system.[10] For instance, we introduce a phase factor $e^{i2\pi/N}$ into the action. Then the minimum energy configuration for a plaquette

$$e^{i2\pi/N}\,\mathrm{tr}(U_\rho U_\sigma U_\rho^+ U_\sigma^+) + \mathrm{h.c.} \qquad (24)$$

is no longer given by $U_\rho = U_\sigma = 1$ but by

$$\begin{aligned} &U_\rho = P, \quad U_\sigma = Q \\ &PQ = QP\, e^{i2\pi/N} \end{aligned} \qquad (25)$$

where P, Q are matrices of 't Hooft[11]

$$P = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & e^{2\pi i/N} & & 0 \\ & & \ddots & \\ 0 & & & e^{2\pi i(N-1)/N} \end{pmatrix} \qquad (26)$$

Eigenvalues of P and Q are uniformly distributed on the circle and thus U(1) symmetric. Therefore the introduction of a twist lifts the degeneracy of eigenvalues and would restore U(1) symmetry. In fact there exists some numerical indication[10] that the original reduced model has no U(1) symmetry breaking in the region of negative coupling constant λ and agrees with the Wilson theory.

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