Topological Excitations on a Lattice

Y. Iwasaki and T. Yoshiè
Institute of Physics, University of Tsukuba, Ibaraki 305, JAPAN

We study by numerical methods the existence (or the non-existence) of instantons and the role played by instantons on a lattice, by taking the CP^1 model in two dimensions as an example.

The CP^1 (non-linear O(3) σ) model is defined by

$$L = \frac{\beta}{2} \sum_{\mu=1}^{2} \sum_{i=1}^{3} (\partial_\mu \sigma_i)(\partial_\mu \sigma_i) \ : \qquad \Sigma\, {\sigma_i}^2 = 1 \tag{1}$$

An O(3) invariant regularization of this model gives the classical O(3) Heisenberg model defined by

$$H = \sum_n \sum_{\hat\mu} \vec{S}(n)\vec{S}(n+\hat\mu) \ , \tag{2}$$

where $\vec{S}$ is a 3-component unit vector, n is a lattice site and $\hat\mu$ is a unit vector on the lattice. This model will be referred to as the standard model.

However there are infinitely many other choices for Hamiltonians which give the same naive (classical) continuum limit. For example, taking three couplings (nearest, next-nearest, and third-nearest neighbor), we write[1] in the form

$$\begin{aligned} H = \Sigma\ \{&\alpha_1(\nabla_\mu\vec{S}(n)\nabla_\mu\vec{S}(n)) + \alpha_2(\nabla_\mu\nabla_\nu\vec{S}(n))(\nabla_\mu\nabla_\nu\vec{S}(n)) \\ &+ \alpha_3[(\nabla_x^2\vec{S}(n))(\nabla_x^2\vec{S}(n)) + (\nabla_y^2\vec{S}(n))(\nabla_y^2\vec{S}(n))]\} \end{aligned} \tag{3}$$

where

$$\nabla_\mu f(n) = f(n+\hat\mu) - f(n) \tag{4}$$

The correlation functions are given by

$$\langle F(S)\rangle = \frac{1}{Z}\int \prod_n d\vec{S}(n)\,\delta(S^2(n) - 1)F(S)\exp(-\beta H) \ . \tag{5}$$

If we take $\alpha_1 = 1/2$, any Hamiltonian given by eq.(3) reduces to the non-linear O(3) σ model defined by eq.(1) in the classical continuum limit.

Let us investigate topological properties of the models. Berg and Lüscher[2] have provided a suitable definition of topological charge on the lattice in the form

$$Q = \sum_{n^*} q(n^*) \tag{6}$$

where n* is a dual lattice site. They have found that the topological susceptibility

$$\chi_t = \sum_{n^*} \langle q(0)q(n^*)\rangle \tag{7}$$

does not scale as a renormalization invariant $(\text{mass})^2$ in the case of the standard model. Further Lüscher[3] has explained this phenomenon by pointing out that there exist short range fluctuations of the topological charge with such a small action that they overwhelm the contribution of the slowly varying fields, which otherwise dominate in the continuum limit. The point is that the minimum energy configurations with $Q = 1$ are not the solutions of the lattice field equation. The spin configurations are planar. They are at the boundary of configurations with $|Q| = 1$ and those with $Q = 0$; they are "exceptional" configurations due to the terminology in ref.2). We call these exceptional configurations "dislocations", following Berg.[4),5)] These dislocations dominate the topological susceptibility χ_t at low temperatures.

Up to this point they are all well known. Now, let us take the Hamiltonians (3) instead of the standard one. With positive α_2 and α_3, the energy of a short range fluctuation becomes large. However, the energy of a slowing varying field does not change much. Therefore by increasing the parameter α_2 and/or α_3, we can make the energy of a dislocation much larger than 4π, the energy of an instanton of the continuum theory.

If the energy of a dislocation is much larger than 4π, we naturally expect that some configurations with $Q = 1$ are solutions of the lattice field equation. We indeed find such solutions by increasing α_2 and/or α_3. We find them by two ways: One way is to start from random spin configurations and to lower systematically the energy of spin configurations until a solution of the classical lattice field equation is obtained. The other way is to start from a discretized-instanton

$$\omega(n) = \frac{S_1(n) + iS_2(n)}{1 + S_3(n)} = c\,\frac{z-a}{z-b} \tag{8}$$

and to lower systematically the energy. Here $z = n_1 + in_2$ and a, b, c are complex numbers.

For a 25×25 lattice, with $c = 1$, $a = 7.5 + 11.5i$, $b = 16.5 + 11.5i$, the energy of the discretized-instanton is systematically lowered by replacing a spin in such a way to minimize its local energy. Even for $\alpha_2 = 0.1$ and $\alpha_3 = 0.0$ where the energy of the dislocation is smaller than 4π, we obtain a stable instanton. For $\alpha_2 = 0.25$ and $\alpha_3 = 0.25$ (these numbers have no special meanings. We choose them arbitrary) where the energy of the dislocation is about twice that of the continuum instanton, we certainly obtain a stable instanton.

Thus we conclude that the existence (or the non-existence) of instantons depends on the form of lattice action. Among theories which are identical in the naive continuum limit, some theories where short range fluctuations are suppressed prossess stable instanton solutions, while others do not.

Because the Hamiltonians (3) reduce to the same non-linear O(3) σ model in the naive (classical) continuum limit irrespective of α_2 and α_3, all of them are equivalent in perturbative theory, only by redefining the coupling constant β. However, they may give inequivalent non-perturbative effects. We will discuss it below.

First let us give the relation between various coupling constants in the framework of perturbation theory. We follow the method which was first used by Parisi[6] when deriving the relation between the coupling constant of the standard model and that of the continuum theory.

The relation may be given in terms of the scale parameter

$$\Lambda = \frac{\beta}{a} \exp(-2\pi\beta) \tag{9}$$

After some calculation we obtain

$$\begin{aligned} &\Lambda(\alpha_2{=}0.1,\ \alpha_3{=}\ 0) = 8.9\ \Lambda(\alpha_2{=}0,\ \alpha_3{=}0) \\ &\Lambda(\alpha_2{=}0.25,\ \alpha_3{=}0.25) = 124.3\ \Lambda(\alpha_2{=}0,\ \alpha_3{=}0) \end{aligned} \tag{10}$$

If we neglect β in the numerator in eq.(9) we have rough relations from eq.(10)

$$\begin{aligned} &\beta(\alpha_2{=}0.1,\ \alpha_3{=}0) \cong \beta(\alpha_2{=}0,\ \alpha_3{=}0) - 0.35 \\ &\beta(\alpha_2{=}0.25,\ \alpha_3{=}0.25) \cong \beta(\alpha_2{=}0,\ \alpha_3{=}0) - 0.77 \end{aligned} \tag{11}$$

At any rate eqs.(10) hold for $\beta >> 1$ and therefore eqs.(11) are enough for our later use. Thus the inverse-temperature $\beta = 1.3$ in the standard model corresponds to $\beta \cong 0.85$ when $\alpha_2 = 0.1$, $\alpha_3 = 0$ and $\beta \cong 0.53$ when $\alpha_2 = 0.25$, $\alpha_3 = 0.25$.

We will mainly discuss two theories with $\alpha_2 = 0$, $\alpha_3 = 0$ and with $\alpha_2 = 0.25$, $\alpha_3 = 0.25$. The case with $\alpha_2 = 0$, $\alpha_3 = 0$ corresponds to the case where an instanton is unstable, while the cases with $\alpha_2 = 0.25$, $\alpha_3 = 0.25$ to the case where an instanton is stable.

Next we measure χ_t defined by eq.(7) by Monte Carlo simulations on 50×50 spins. In the case of $\alpha_2 = \alpha_3 = 0$, Berg and Lüscher[2] have already measured χ_t for a 100×100 lattice to find that χ_t does not scale as expected. We find that for $\alpha_2 = \alpha_3 = 0.25$, χ_t is consistent with the scaling for $0.4 \leq \beta \leq 0.6$ [where the correlation length ξ varies from 2 to 6]. This is expected because the energy of an instanton is much lower than that of the dislocation. The reason why we have considered χ_t for the range of β, $0.4 \leq \beta \leq 0.6$, is the following; for $\beta < 0.4$ the correlation length is too small to expect the scaling behavior of χ_t and for $\beta > 0.6$ finite size effect becomes significant.

Qualitative differences between the two theories in physical quantities which are connected with the topological charge are naturally expected. However, the differences between the two theories may be more deep. If instantons are stable at $\beta = \infty$, their effects will remain in the limit $\beta \to \infty$, while if instantons are unstable at $\beta = \infty$, their effects will become weak in the limit $\beta \to \infty$.

As already noticed by Berg and Lüscher[2], and Martinelli, Parisi and Petronzio[7], the magnetic susceptibility χ_m itself does not scale as expected, although the deviation from the scaling is not so large as in χ_t. Our results are consistent with the previous results for $\alpha_2 = \alpha_3 = 0$ and the data for $\alpha_2 = \alpha_3 = 0.25$ are consistent with RG.

The deviation from RG for the standard model might be due to the higher order power corrections. However, we rather interpret the fact that χ_m does not scale as consistent with RG for $\alpha_2 = \alpha_3 = 0$, while it does scale for $\alpha_2 = \alpha_3 = 0.25$, as results from the fact that short range fluctuations dominate for $\alpha_2 = \alpha_3 = 0$ at low temperature, while slowly varing fields (instantons) dominate for $\alpha_2 = \alpha_3 = 0.25$.

If our interpretation is correct, it means that in the continuum limit for $\alpha_2 = \alpha_3 = 0.25$ χ_t and χ_m have their limits, for $\alpha_2 = \alpha_3 = 0$ they do not have non-trivial limits. This further implies that theories which are equivalent in perturbation theory do not necessarily give equivalent non-perturbative effects.

Now let us try to find indications of the topological symmetry breaking proposed previously by the present authors: In previous papers[8] we have conjectured that the vacuum of the non-linear O(3) σ model (CP^1 model) is two-fold degenerate and that a spontaneous symmetry breaking occurs from the requirement of the cluster property of the vacuum. We add an external source term proportional to Q

$$- 4\pi\theta Q \tag{12}$$

to the Hamiltonian (3). We use the Metropolis method to measure hysteresis effects by changing θ. We measure hysteresis curves for $\alpha_2 = \alpha_3 = 0$ and for $\alpha_2 = \alpha_3 = 0.25$. Two cases show completely different patterns. In the case of $\alpha_2 = \alpha_3 = 0$, at $\theta = 0$ the residual topological charge is zero, while in the case of $\alpha_2 = \alpha_3 = 0.25$, at $\theta = 0$ the residual topological charge is about ten. We make the same measurement by changing random numbers as well as the number of steps. In every cases the patterns of hysteresis curves are the same, only the residual topological charge at $\theta = 0$ for $\alpha_2 = \alpha_3 = 0.25$ changes slightly. This behavior is consistent with our assertion that the topological symmetry breaking occurs for $\alpha_2 = \alpha_3 = 0.25$.

We also investigate the size dependence of χ_t for both cases with $\alpha_2 = \alpha_3 = 0$

and $\alpha_2 = \alpha_3 = 0.25$. If a system has the spontaneous topological symmetry breaking, the expectation value of the topological charge density is not zero;

$$\langle q(i) \rangle = q \tag{13}$$

Therefore the topological susceptivility χ_t has a size dependence $\chi_t \sim N^2 q^2$ where the N^2 is the size of the lattice. This is analogue of the spontaneous magnetization. On the other hand, if a system has no spontaneous topological symmetry breaking, χ_t is expected to be size independent.

It should be noted that χ_t is dominated by dislocations in the standard model. Dislocations are short range fluctuations and therefore they are not influenced each other; a dislocation with positive topological charge and a dislocation with negative topological charge can coexist. The situation is similar to that of the dilute instanton gas picture. Contrary to the standard model, in the model with $\alpha_2 = \alpha_3 =$ 0.25, instantons dominate χ_t; instantons are slowly varing fields and they influence each other. In the continuum limit we cannot apply the dilute instanton gas picture to this system due to the analysis of the continuum theory.[9),10)]

AT $\beta = 1.3$ with $\alpha_2 = \alpha_3 = 0$, the topological susceptibility χ_t does not change even if the size of the lattice is changed from 25×25 to 100×100 (via 50×50). On the other hand, at $\beta = 0.6$ with $\alpha_2 = \alpha_3 = 0.25$, χ_t increases if the size of the lattice changed from 25×25 to 50×50. For a 100×100 lattice we have also a preliminary result for χ_t; it shows also a tendency to increase. These results strongly support our assertion.

More details are discussed elsewhere.[11)]

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