BPS Transformation and Color Confinement*)

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Interpretation of hadrons in terms of quarks and antiquarks has been so successful that one can no longer think of its substitute. The hadron spectrum and high energy hadron reactions are believed to be described by means of quantum chromodynamics (QCD). Thus, we feel the existence of quarks so real on one hand, but we have never detected isolated quarks on the other hand. In this way the explanation of the confinement of quarks and also of gluons became one of the central problems in QCD.

In the lattice gauge theory the condition for quark confinement is given by the area law for the Wilson loop [1]. In the present paper we shall look for the corresponding condition within the framework of the conventional continuum field theory. As we shall see later this condition is given by the existence of certain bound states between a pair of Faddeev-Popov ghosts.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}\cdot F_{\mu\nu} + A_\mu\cdot\partial_\mu B + \frac{\alpha}{2}B\cdot B$$
$$+ i\partial_\mu\bar{c}\cdot D_\mu c - \psi(\gamma_\mu D_\mu+m)\psi, \qquad (1)$$

where covariant derivatives D_μ are defined by

$$D_\mu c = \partial_\mu c + gA_\mu \times c,$$
$$D_\mu\psi = (\partial_\mu - igT\cdot A_\mu)\psi, \qquad (2)$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + gA_\mu \times A_\nu.$$

We have made use of the abbreviations, $S\cdot T = S^aT^a$ and $(S\times T)^a = f_{abc}S^bT^c$.

Next we introduce the Becchi-Rouet-Stora (BRS) transformtion of Heisenberg-fields [2].

$$\delta A_\mu = D_\mu c,$$
$$\delta B = 0,$$
$$\delta c = -\frac{1}{2}g\, c \times c, \qquad (3)$$

$$\delta\bar{c} = iB,$$
$$\delta\psi = ig(c\cdot T)\psi.$$

This supersymmetric transformation can be expressed in terms of its generator Q_B as

$$\delta O = i[Q_B, O]_{\mp}, \tag{4}$$

where we choose the $-(+)$ sign when O involves an even (odd) number of the hermitian anticommuting ghost fields c and $\bar{c}$. Kugo and Ojima [3] have introduced another charge Q_c satisfying

$$i[Q_c, c(x)] = c(x), \; i[Q_c, \bar{c}(x)] = -\bar{c}(x). \tag{5}$$

It commutes with all other fields, and it defines the ghost number, namely +1 for c and -1 for $\bar{c}$. These two charges satisfy the relations

$$i[Q_c, Q_B] = Q_B, \; Q_B^2 = 0. \tag{6}$$

The second relation implies that the BRS transformation is nilpotent, namely, $\delta^2 = 0$.

Then we shall introduce asymptotic fields and their BRS transformation. Because of infrared singularities in QCD the existence of asymptotic fields might be doubtful, nevertheless we shall simply assume it in the present paper. Then the BRS transformation for the asymptotic fields is linear. When $\delta a(x) = b(x) \neq 0$ so that $\delta^2 a(x) = 0$, $\{a(x), b(x)\}$ is called a BRS doublet. When $\delta a(x) = 0$ but its parent $f(x)$, satisfying $\delta f(x) = a(x)$, does not exist, $a(x)$ is called a BRS singlet. Doublets and singlets are the only irreducible representations of the BRS transformation.

By extending the assumed existence of asymptotic fields we shall further postulate the asymptotic completeness. The state vector space spanned by asymptotic fields in QCD will be denoted by $\mathcal{V}$. Kugo and Ojima [3] introduced a physical subspace $\mathcal{V}_{phys}$ by

$$\mathcal{V}_{phys} = \{|\ \rangle \,\big|\, |\ \rangle \in \mathcal{V},\ Q_B|\ \rangle = 0\} \tag{7}$$

Then, by applying only the singlet asymptotic fields to the vacuum state a subspace of $\mathcal{V}$, denoted by $\mathcal{V}_s$, is generated. Obviously we have

$$\mathcal{V} \supset \mathcal{V}_{\text{phys}} \supset \mathcal{V}_S. \qquad (8)$$

The S matrix exists as a consequence of the asymptotic completeness and commutes with Q_B. When $|\alpha\rangle$ and $|\beta\rangle$ belong to $\mathcal{V}_S$, the unitarity condition of the S matrix can be expressed as

$$\langle\beta|\alpha\rangle = \langle\beta|S^+S|\alpha\rangle = \langle\beta|S^+P(\mathcal{V}_S)S|\alpha\rangle, \qquad (9)$$

and similarly for $SS^\dagger$. This relation is a consequence of the Kugo-Ojima theorem [3]. $P(\mathcal{V}_S)$ stands for the projection operator to the subspace $\mathcal{V}_S$, so that no doublets show up in the intermediate states of the unitarity condition. In this sense, doublets in QCD are analogous to longitudinal and scalar photons in QED and are confined in the unphysical state vector space. Interpreting that singlets represent hadrons, the problem of color confinement reduces to that of demonstrating that both quarks and gluons are BRS doublets.

We have already assumed that the vacuum state $|0\rangle$ belongs to $\mathcal{V}_S$ and hence to $\mathcal{V}_{\text{phys}}$. Thus we have $Q_B|0\rangle = 0$ and consequently the BRS identity

$$\langle 0|\delta T(\cdots)|0\rangle = 0. \qquad (10)$$

In what follows we shall abbreviate $\langle 0|T(\cdots)|0\rangle$ as $\langle\cdots\rangle$. Then, by making use of the BRS identities we find the following Ward-Takahashi (W-T) identities [4]:

$$\partial_\lambda\langle(D_\lambda\bar{c})^a(x),\ \delta\psi(y),\ \bar{\psi}(z)\rangle + \partial_\lambda\langle(D_\lambda\bar{c})^a(x),\ \psi(y),\ \delta\bar{\psi}(z)\rangle$$

$$= ig\ T^a(\delta^4(x-y) - \delta^4(x-z))S_F(y-z), \qquad (11)$$

$$\partial_\lambda\langle(D_\lambda\bar{c})^a(x),\ \delta A_\mu^b(y),\ A_\nu^c(z)\rangle + \partial_\lambda\langle(D_\lambda\bar{c})^a(x),\ A_\mu^b(y),\ \delta A_\nu^c(z)\rangle$$

$$= igM^a_{bc}(\delta^4(x-y) - \delta^4(x-z))D_{F\mu\nu}(y-z), \qquad (12)$$

where $M^a{}_{bc} = if_{bac}$, and S_F and $D_{F\mu\nu}$ denote propagators of the quark and gluon fields, respectively. We shall write

$$\langle(D_\lambda\bar{c})^a(x),\ \delta\psi(y),\ \bar{\psi}(z)\rangle = g\int d^4z'G_\lambda^a(yz':x)S_F(z'-z), \qquad (13)$$

$$\langle(D_\lambda\bar{c})^a(x),\ \psi(y),\ \delta\bar{\psi}(z)\rangle = g\int d^4y'S_F(y-y')\bar{G}_\lambda^a(y'z:x). \qquad (14)$$

The Fourier-transform of Eq. (11) may be expressed as

$$(p-q)_\lambda \cdot G^a_\lambda(p, q) S_F(q) + S_F(p)(p-q)_\lambda \cdot \bar{G}^a_\lambda(p, q)$$

$$= iT^a(S_F(p)-S_F(q)). \qquad (15)$$

In this equation we may replace G^a_λ and $\bar{G}^a_\lambda$ by their spin zero projection defined by

$$G^a_\lambda(p, q)^{(0)} = \frac{(p-q)_\lambda (p-q)_\mu}{(p-q)^2} G^a_\mu(p, q). \qquad (16)$$

In order to simplify our argument we shall choose the Landau gauge ($\alpha = 0$) in what follows. In this gauge we have $\partial_\lambda D_\lambda \bar{c} = 0$, and possible poles in G_λ due to massless vector particles will disappear in this projection. A pole due to the massless scalar particle is still present as can clearly be seen from the W-T identity

$$<(D_\lambda \bar{c})^a(x), c^b(y)> = i\delta_{ab}\partial_\lambda D_F(x-y), \qquad (17)$$

where D_F denotes the free massless propagator. This equation shows that $D_\lambda \bar{c}$ generates a massless scalar particle, and we introduce asymptotic fields corresponding to this massless scalar particle as

$$D_\lambda \bar{c} \to \partial_\lambda \bar{\Gamma}, \quad c \to \Gamma. \qquad (18)$$

We then replace $D_\lambda \bar{c}$ by $D_\lambda \bar{c} - \partial_\lambda \bar{\Gamma}$ and write F and $\bar{F}$ for G and $\bar{G}$ in Eqs. (13) and (14). The functions $F_\lambda{}^a(p, q)^{(0)}$ and $\bar{F}_\lambda{}^a(p, q)^{(0)}$ so defined are free of the poles at $(p-q)^2 = 0$ except for the projection operator in Eq. (16).

According to Nakanishi's theorem [5] the asymptotic field $\bar{\Gamma}$ carrying the ghost number (-1) cannot be a BRS singlet, but it must be a member of a BRS doublet. Confinement is realized when $\bar{\Gamma}$ is the second generation of the doublet expressible as

$$\delta\bar{d}(x) = \bar{\Gamma}(x). \qquad (19)$$

Then the BRS identity leads to

$$<\partial_\lambda \bar{\Gamma}(x), \delta\psi(y), \bar{\psi}(z)> + <\partial_\lambda \bar{\Gamma}(x), \psi(y), \delta\bar{\psi}(z)> = 0, \qquad (20)$$

and by subtracting Eq. (20) from Eq. (11) we find

$$(p-q)_\lambda \cdot F_\lambda{}^a(p,\ q)^{(0)} S_F(q) + S_F(p)(p-q)_\lambda \cdot \bar{F}_\lambda{}^a(p,\ q)^{(0)}$$

$$= iT^a(S_F(p) - S_F(q)). \tag{21}$$

We then put $p - q = \varepsilon P$ with $P^2 \neq 0$, and apply the limiting procedure $\lim_{\varepsilon\to 0} \partial/\partial\varepsilon$ to Eq. (21). Since the individual terms on the ℓ. h. s. of Eq. (21) are of the order of ε because of the absence of poles at $(p - q)^2 = 0$, we obtain

$$P_\lambda \cdot F_\lambda{}^a(p,\ p\colon P)^{(0)} S_F(p) + S_F(p) P_\lambda \cdot \bar{F}_\lambda{}^a(p,\ p\colon P)^{(0)}$$

$$= iT^a P_\lambda \cdot \frac{\partial}{\partial p_\lambda} S_F(p). \tag{22}$$

F_λ and $\bar{F}_\lambda$ gain a possible dependence on the direction of P through the factor $P_\lambda P_\mu/P^2$ originated from the projection operator in Eq. (16). Eq. (22) shows that F_λ and/or $\bar{F}_\lambda$ must share a pole with $S_F(p)$ at $ip\gamma + m = 0$. For the symmetry reason both must have this pole implying that both $\delta\psi$ and $\delta\bar{\psi}$ generate a pole at the quark mass.

When $\bar{\Gamma}$ is the first generation of the doublet contrary to Eq. (19), however, the BRS identity (20) does not hold. Then we have to go back to Eq. (15) because Eq. (21) does not follow. Since the functions $G_\lambda{}^a$ and $\bar{G}_\lambda{}^a$ are not free of the pole at $(p-q)^2 = 0$, the individual terms on the ℓ. h. s. of Eq. (15) are generally of the order of 1 and only the sum of the two terms is of the order of ε. Then application of the limiting procedure mentioned above leads to an equation in which derivatives of S_F appear not only on the r. h. s. but also on the ℓ. h. s. in a sharp contrast to Eq. (22) in which the S_F on the ℓ. h. s. is not differentiated. In such a case, however, we cannot conclude that G_λ and $\bar{G}_\lambda$ must have a pole at the quark mass. Perturbation theory falls into this category.

Thus, when Eq. (19) holds, $\{\psi^{in},\ \delta\psi^{in}\}$ represents a BRS doublet and quarks are confined. A similar argument starting from Eq. (12) shows that gluons are also confined under the same condition.

We shall reexpress the condition (19) in a more convenient form by using the BRS identity.

$$0 = \langle\bar{\Gamma}(x),\ \Gamma(y)\rangle$$

$$= \langle\delta\bar{d}(x),\ \Gamma(y)\rangle$$

$$= -\ \langle\bar{d}(x),\ \delta\Gamma(y)\rangle. \tag{23}$$

This implies the existence of $\delta\Gamma$. Since Γ is the asymptotic field of c and $\delta c \sim c \times c$, there must exist the asymptotic field of $c \times c$ carrying the same set of quantum numbers as that of c but for the ghost number. Conversely, when $\delta\Gamma$ exists, there must be an asymptotic field $\bar{d}$ for which $\langle \bar{d}, \delta\Gamma \rangle \neq 0$. Then we can replace $D_\lambda \bar{c}$ in Eqs. (13) and (14) by $D_\lambda c - \partial_\lambda \delta\bar{d}$ to introduce the poleless vertex functions F_λ and $\bar{F}_\lambda$. After that we can repeat the same argument leading to the quark confinement.

Thus the existence of the asymptotic field for $c \times c$ is a sufficient condition for color confinement. Quarks and gluons are confined when they form bound states with the ghost c as is clear from the explicit expressions for $\delta\psi$ and δA_μ in Eq. (3). When the ghost c itself forms a bound state with another ghost, the ability of forming a bound state with the ghost is communicated to other colored particles through the BRS identities.

The Bethe-Salpeter equation for the bound states between a pair of Faddeev-Popov ghosts can be solved exactly in the ladder approximation, but the normalization integral is not convergent. It can be shown, however, that introduction of a parameter of the dimension of mass is necessary for the convergence of the normalization integral. A promising way of improving the approximation to make the normalization integral convergent is to exploit the renormalization group method in which a mass parameter enters as a renormalization point.

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