COVARIANT OPERATOR FORMALISM OF GAUGE THEORIES AND ITS EXTENSION TO FINITE TEMPERATURE

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On the basis of "thermo field dynamics" allowing the application of the Feynman diagram method to real-time Green's functions at $T\neq 0°K$, a field-theoretical formulation of finite-temperature gauge theory is presented. It is an extension of the covariant operator formalism of gauge theory based upon the BRS invariance: The subsidiary condition specifying physical states, the notion of observables, and the structure of the physical subspace at finite temperatures are clarified together with the key formula characterizing the temperature-dependent "vacuum".

1. Introduction

Although thermodynamic aspects of gauge theory are currently discussed in the so-called imaginary-time formulation of QFT at finite temperatures, which is believed to be the only choice pertaining to the Feynman diagram method, this belief is not correct as shown by Takahashi and Umezawa [1]. They proposed a real-time formulation named "thermo field dynamics" in which statistical averages are expressed in the form of temperature-dependent "vacuum" expectation values and in which the Feynman diagram method can be applied to real-time causal Green's functions at a finite temperature (i.e. statistical averages of time-ordered products). In contrast to the imaginary-time formulation having no time variable, we have here both a temperature and a time variable without being bothered by the cumbersome discrete energy sums over the Matsubara frequencies and the full information about spectral functions is attained without analytic continuation in energy variables.

Thus, this is a formalism to be regarded as a natural extension of QFT (at $T=0°K$) to the case of $T\neq 0°K$. According to [2], we briefly describe here a field-theoretical formulation of gauge theory at $T\neq 0°K$ on the basis of the covariant operator formalism of gauge theory [3] and this "thermo field dynamics".

2. Thermo field dynamics

The point of thermo field dynamics [1] is to introduce fictitious "tilde" operators $\tilde{A}$ corresponding to each of the operators A describing the system and to perform a temperature-dependent Bogoliubov transformation mixing A's with $\tilde{A}^\dagger$'s, which realizes the state space at a finite temperature and the temperature-dependent "vacuum" $|0(\beta)\rangle$ giving statistical averages.

This is seen in the simplest example of a harmonic oscillator defined by

$$H = \varepsilon a^\dagger a,\ [a,a^\dagger] = 1. \tag{2.1}$$

By introducing tilde operators $\tilde{a}$, $\tilde{a}^\dagger[=(\tilde{a})^\dagger=\widetilde{(a^\dagger)}]$ as duplicates of $a,a^\dagger$ commuting with $a,a^\dagger$,

$$[\tilde{a},\tilde{a}^\dagger] = 1,\ [a,\tilde{a}] = [a^\dagger,\tilde{a}] = \ldots = 0, \tag{2.2}$$

the temperature-dependent "vacuum" $|0(\beta)\rangle$ is determined by the Bogoliubov transformation as follows:

$$a(\beta) \equiv a\cosh\theta(\beta)-\tilde{a}^\dagger\sinh\theta(\beta),\ \tilde{a}(\beta) \equiv \tilde{a}\cosh\theta(\beta)-a^\dagger\sinh\theta(\beta); \tag{2.3}$$

$$\cosh\theta(\beta) \equiv 1/(1-e^{-\beta\varepsilon})^{1/2},\ \sinh\theta(\beta) \equiv e^{-\beta\varepsilon/2}/(1-e^{-\beta\varepsilon})^{1/2}; \tag{2.4}$$

$$a(\beta)|0(\beta)\rangle = \tilde{a}(\beta)|0(\beta)\rangle = 0. \tag{2.5}$$

It can easily be checked by the aid of (2.3)∿(2.5) that statistical averages over the Gibbs ensemble are given by the "vacuum" expectation values:

$$\langle A\rangle \equiv \mathrm{Tr}(A\exp(-\beta H))/\mathrm{Tr}(\exp(-\beta H) = \langle 0(\beta)|A|0(\beta)\rangle. \tag{2.6}$$

The origin of the Bogoliubov transformation (2.3)∿(2.5) is traced to the following relations:

$$|0(\beta)\rangle = \sum_{n>0} e^{-\beta n\varepsilon/2}|n\rangle\otimes|n\rangle/(1-e^{-\beta\varepsilon})^{-1/2}$$
$$= \exp[\theta(\beta)(a^\dagger\tilde{a}^\dagger-\tilde{a}a)]|0\rangle \equiv \exp(-iG)|0\rangle, \tag{2.7}$$

$$a(\beta) = \exp(-iG)a\,\exp(iG),\ \tilde{a}(\beta) = \exp(-iG)a\,\exp(iG). \tag{2.8}$$

Although the unitary operator exp(-iG) has its proper meaning only for the system with finite degrees of freedom, these formulae will be useful heuristically also in the later discussion of gauge theory.

Now, the essence of (2.3)∿(2.5) can be summarized in a formula

$$\exp[\beta(H-\tilde{H})/2]M|0(\beta)\rangle = M^\dagger|0(\beta)\rangle, \tag{2.9}$$

which reproduces (2.5) in the case of $M=a,a^\dagger$ with the help of (2.1)∿(2.4) and $\tilde{H}=\varepsilon\tilde{a}^\dagger\tilde{a}$. By extending the definition of $\tilde{M}$ antilinearly, (2.9) holds for any polynomial M of a and $a^\dagger$. Taking account of the commutativity of tilde and non-tilde operators, we can derive from (2.9) the KMS condition [4] characterizing Gibbs states ,

$$\langle AB(t)\rangle = \langle B(t-i\beta)A\rangle. \tag{2.10}$$

The basis for (2.9) can be found [2] in connection with the algebraic formulation of statistical mechanics[5] due to Haag, Hugenholtz and Winnink [6] and Tomita-Takesaki theory [5]. The key concepts there are the modular conjugation operator J and the modular operator $\exp(-\beta\bar{H})$ defined by

$$J\exp(-\beta\bar{H}/2)M|0(\beta)\rangle = M^\dagger|0(\beta)\rangle \quad \text{for} \quad M\in\mathfrak{M}, \tag{2.11}$$

where $\mathfrak{M}$ is the algebra of operators desribing the system. J is an antiunitary operator satisfying

$$J^2 = 1,\ J|0(\beta)\rangle = |0(\beta)\rangle, \tag{2.12}$$

$$J\mathfrak{M}J = \mathfrak{M}'(\equiv \text{commutant of } \mathfrak{M}\ ), \tag{2.13}$$

and $\bar{H}$ satisfies

$$\bar{H}|0(\beta)\rangle = 0, \tag{2.14}$$

$$\exp(it\bar{H})\mathfrak{M}\exp(-it\bar{H}) = \mathfrak{M}, \tag{2.15}$$

$$J\bar{H}J = -\bar{H}. \tag{2.16}$$

Thus, by identifying $\tilde{M}$ with JMJ and $\bar{H}$ with $H-\tilde{H}$,

$$\tilde{M} = JMJ, \tag{2.17}$$

$$\bar{H} = H-\tilde{H} = H-JHJ, \tag{2.18}$$

(2.9) can be derived from (2.11) with the help of (2.16) and (2.12).

(2.17) and (2.13) explain the commutativity of tilde operators with nontilde ones. In the case of fermions, however, $\tilde{\psi}$ should anticommute with ψ in order to keep the Bogoliubov transformation meaningful, whence (2.17) for fermions is modified by the Klein transformation [2]:

$$\tilde{\psi} = iJ\psi J\exp(i\pi(F-JFJ)). \tag{2.19}$$

Here F is the fermion number operator. In contrast to (2.9) which requires modification to cover the cases with fermions, (2.11) remains unchanged, and hence, it should be taken as the key formula characterizing the temperature-dependent "vacuum" $|0(\beta)\rangle$ at $T=1/k_B\beta$ independently of a specific model (except for gauge theory discussed later in §4).

Now, corresponding to the "total" Hamiltonian (2.18) which is consistent with (2.16), the "total" Lagrangian $\bar{\mathcal{L}}$ of the "total" system consisting of tilde and non-tilde objects is given by

$$\bar{\mathcal{L}} = \mathcal{L} - \tilde{\mathcal{L}} = \mathcal{L}-J\mathcal{L}J. \tag{2.20}$$

Since the Gell-Mann-Low relation can be verified relative to the splitting of $\bar{\mathcal{L}}$ into the free and interaction parts, $\bar{\mathcal{L}} = \bar{\mathcal{L}}_0+\bar{\mathcal{L}}_I$, and since the Wick theorem holds (at the operator level in contrast to the Bloch-DeDominicis-Wick theorem in the imaginary time formulation), we can develop the Feynman diagram technique also here on the basis of the propagators such as

$$\langle 0(\beta)\, T\begin{bmatrix}\phi(x)\\ \tilde\phi^\dagger(x)\end{bmatrix}(\phi^\dagger(y),\tilde\phi(y))|\,0(\beta)\rangle$$

$$= i\int\frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}U_B(|\vec p|,\beta)\begin{pmatrix}1/(p^2-m^2+i0) & 0\\ 0 & -1/(p^2-m^2-i0)\end{pmatrix}U_B^\dagger(|\vec p|,\beta); \qquad (2.21)$$

$$U_B(|\vec p|,\beta) = \begin{bmatrix}\cosh\theta(\beta) & \sinh\theta(\beta)\\ \sinh\theta(\beta) & \cosh\theta(\beta)\end{bmatrix}, \qquad (2.22)$$

where $\theta(\beta)$ is given by (2.4) with ε replaced by $(\vec p^2+m^2)^{1/2}$.

3. Covariant operator formalism of gauge theory [3]

To apply thermo field dynamics discussed in §2 to the case of gauge fields, we recapitulate the points relevant here of the covariant operator formalism of gauge theory based upon the BRS invariance [3]. The Lagrangian density

$$\mathcal{L} = -F^a_{\mu\nu}F^{a,\mu\nu}/4+\mathcal{L}_{matter}(\varphi,\mathcal{D}_\mu\varphi)-A^a_\mu\partial^\mu B^a+\alpha B^aB^a/2-i\partial^\mu \bar c^a(D_\mu c)^a \qquad (3.1)$$

is invariant under the BRS transformation whose generator is given by

$$Q_B \equiv \int d^3x[B^a(D_0c)^a-\dot B^a c^a+(i/2)g\dot{\bar c}^a(c\times c)^a]; \qquad (3.2)$$

$$[iQ_B,A_\mu] = D_\mu c,\ [iQ_B,B] = 0,\ [iQ_B,\varphi] = igc^aT^a\varphi,$$

$$\{iQ_B,c\} = -(g/2)c\times c,\ \{iQ_B,\bar c\} = iB. \qquad (3.3)$$

Although the state space $\mathcal{V}$ necessarily contains unphysical particles with <u>negative norms</u>, they have been shown [3] to decouple from all the physical processes by quite a general norm-cancellation mechanism called <u>quartet mechanism</u>. This is based upon the <u>subsidiary condition</u> specifying the physical subspace $\mathcal{V}_{phys}\equiv\{|phys\rangle\}$ given by

$$Q_B|phys\rangle = 0, \qquad (3.4)$$

and the relation for the projection operator P onto the subspace $\mathcal{V}_{phys}$ of states containing physical particles alone,

$$P+\{Q_B,R\} = 1. \qquad (3.5)$$

The basic properties of Q_B leading to (3.5) are the following:

$$Q_B^2 = \{Q_B,Q_B\}/2 = 0, \qquad (3.6)$$

$$[iQ_c,Q_B] = Q_B, \qquad (3.7)$$

where Q_c is the Faddeev-Popov (FP) charge

$$[iQ_c,c] = c,\ [iQ_c,\bar c] = -\bar c,\ \text{otherwise}\ [iQ_c,\Phi] = 0. \qquad (3.8)$$

Since the <u>positive semi-definite</u> space $\mathcal{V}_{phys}$ contains <u>zero-norms</u>, the usual Hilbert space with positive definite metric is obtained by taking the quotient of $\mathcal{V}_{phys}$ with respect to its zero-norm subspace $\mathcal{V}_0 = Q_B\mathcal{V}$,

$$H_{phys} \equiv \mathcal{V}_{phys}/\mathcal{V}_0 = \{|\hat\Phi\rangle \equiv |\Phi\rangle+\mathcal{V}_0;\, |\Phi\rangle\in\mathcal{V}_{phys}\}, \qquad (3.9)$$

in which every physical process is described. The trace operation in the statistical average should also be taken in this Hilbert space H_{phys},

$$\langle A\rangle = Tr(e^{-\beta\hat H}\hat A)/Tr(e^{-\beta\hat H}) = \sum\langle\hat i|e^{-\beta\hat H}\hat A|\hat i\rangle/\sum\langle\hat i|e^{-\beta\hat H}|\hat i\rangle, \qquad (3.10)$$

where $\hat H$ and $\hat A$ are operators in H_{phys} obtained as quotient mappings from the Hamiltonian H and an observable A in $\mathcal{V}$ satisfying the defining relation for observables [7,3],

$$[Q_B,A]\mathcal{V}_{phys} = 0. \qquad (3.11)$$

For the observable A satisfying a stronger condition

$$[Q_B,A] = 0, \qquad (3.12)$$

(3.10) can be written [8] in terms of the trace in the total space $\mathcal{V}$ as

$$\langle A\rangle = Tr(A\exp(-\beta H+\pi Q_c))/Z(\beta);\ Z(\beta) = Tr(\exp(-\beta H+\pi Q_c)), \qquad (3.13)$$

by using (3.5) and the fact that Q_c vanishes in $\mathcal{H}_{phys}=P\mathcal{V}$. (3.13) applied (in disregard of (3.12)) to FP ghosts leads to the <u>periodic</u> boundary condition of their temperature Green's functions [9,8].

4. Gauge theory at finite temperature [2]

Following the discussion in §2, we first try to express the r.h.s. of (3.13) in a form of a "vacuum" expectation value

$$\mathrm{Tr}(A\exp(-\beta H+\pi Q_c))/Z(\beta) = \langle 0(\beta)|A|0(\beta)\rangle, \tag{4.1}$$

irrespective of the condition (3.12) to guarantee the equality in (3.13). In this case, the heuristic formula for $|0(\beta)\rangle$ corresponding to (2.7) is

$$|0(\beta)\rangle = \sum_{k,\ell} \exp(-\beta \bar{E}_k/2-iN_k\pi/2)\eta^{-1}_{k\ell}|k\rangle\otimes|\tilde{\ell}\rangle/Z(\beta)^{1/2} \tag{4.2}$$

with

$$H|k\rangle = E_k|k\rangle,\ iQ_c|k\rangle = N_k|k\rangle; \tag{4.3}$$

$$\eta_{k\ell} \equiv \langle k|\ell\rangle \propto \delta(N_k,-N_\ell);\ \langle\tilde{k}|\tilde{\ell}\rangle = \langle \ell|k\rangle = \eta_{\ell k} = \eta^*_{k\ell}. \tag{4.4}$$

Note that the trace in $\mathcal{V}$ with indefinite metric should be understood as

$$\mathrm{Tr}\,\mathcal{O} \equiv \sum_{k,\ell} (\eta^{-1})_{k\ell}\langle \ell|\mathcal{O}|k\rangle. \tag{4.5}$$

To generalize the relation (2.11) characterizing $|0(\beta)\rangle$ to the case of gauge theory with indefinite metric and to determine the propagators, we work out explicitly the Bogoliubov transformation leading to (4.1) and (4.2) for the free Abelian gauge theory in Feynman gauge (α=1):

$$\begin{aligned}|0(\beta)\rangle = \exp[\int d\mu(p)\theta(|\vec{p}|,\beta)\{-a^\dagger_\mu(p)\tilde{a}^{\mu\dagger}(p)+\tilde{a}_\mu(p)a^\mu(p)\\ -i(c^\dagger(p)\tilde{\bar{c}}^\dagger(p)+\tilde{c}(p)\bar{c}(p)+\bar{c}^\dagger(p)\tilde{c}^\dagger(p)+\tilde{\bar{c}}(p)c(p))\}]|0\rangle;\end{aligned} \tag{4.6}$$

$$\begin{aligned}a_\mu(p,\beta) &= a_\mu(p)\cosh\theta(|\vec{p}|,\beta)-\tilde{a}^\dagger_\mu(p)\sinh\theta(|\vec{p}|,\beta),\\ c(p,\beta) &= c(p)\cosh\theta(|\vec{p}|,\beta)-\tilde{c}^\dagger(p)\sinh\theta(|\vec{p}|,\beta),\\ \bar{c}(p,\beta) &= \bar{c}(p)\cosh\theta(|\vec{p}|,\beta)+\tilde{\bar{c}}^\dagger(p)\sinh\theta(|\vec{p}|,\beta),\end{aligned} \tag{4.7}$$

$$a_\mu(p,\beta)|0(\beta)\rangle = c(p,\beta)|0(\beta)\rangle = \bar{c}(p,\beta)|0(\beta)\rangle = 0, \tag{4.8}$$

with similar equations to (4.7), (4.8) for $\tilde{a}_\mu$, $\tilde{c}$, $\tilde{\bar{c}}$. According to (2.17) for bosons and (2.19) for fermions, tilde operators are defined by

$$\tilde{a}_\mu = Ja_\mu J,\ \tilde{c} = iJcJ\exp(-\pi\bar{Q}_c),\ \tilde{\bar{c}} = iJ\bar{c}J\exp(-\pi\bar{Q}_c), \tag{4.9}$$

with the antiunitary J satisfying (2.12), (2.13) and the "total" FP charge

$$\bar{Q}_c \equiv Q_c - JQ_cJ. \tag{4.10}$$

The relations (4.8) as well as the ones with tildes are unified into

$$J\exp[-(\beta\bar{H}-\pi\bar{Q}_c)/2]\mathcal{O}|0(\beta)\rangle = \mathcal{O}^\dagger|0(\beta)\rangle, \tag{4.11}$$

which is a generalization of (2.11). The propagators to be used in the Feynman diagrams are given in momentum space as

$$\begin{aligned}&\text{F.T.}\langle 0(\beta)|T\begin{pmatrix}A_\mu\\ \tilde{A}_\mu\end{pmatrix}(A_\nu,\tilde{A}_\nu)|0(\beta)\rangle = -iU_B(|\vec{p}|,\beta)g_{\mu\nu}\begin{bmatrix}1/(p^2+i0) & 0\\ 0 & -1/(p^2-i0)\end{bmatrix}U^\dagger_B(|\vec{p}|,\beta)\\ &\text{F.T.}\langle 0(\beta)|T\begin{pmatrix}c\\ \tilde{c}\end{pmatrix}(\bar{c},-\tilde{\bar{c}})|0(\beta)\rangle = -U_B(|\vec{p}|,\beta)\begin{bmatrix}1/(p^2+i0) & 0\\ 0 & -1/(p^2-i0)\end{bmatrix}U^\dagger_B(|\vec{p}|,\beta),\end{aligned} \tag{4.12}$$

where $U_B(|\vec{p}|,\beta)$ is given by (2.22) with m=0.

In (4.12), we have allowed the appearance of unphysical negative norms by applying (4.1) to <u>non</u>-observable quantities A_μ, c and $\bar{c}$ for the sake of developing the Feynman diagram method. Therefore, we have to treat again at finite temperature an <u>indefinite-metric</u> space $\mathcal{V}(\beta)$ obtained by applying A_μ, $\tilde{A}_\mu$, c, $\tilde{c}$, etc., to the "vacuum" $|0(\beta)\rangle$. This is taken care of by the "total" BRS charge defined by

$$\bar{Q}_B \equiv Q_B - \tilde{Q}_B \equiv Q_B - iJQ_BJ\exp(-\pi\bar{Q}_c) \tag{4.13}$$

which satisfies $\bar{Q}_B|0(\beta)\rangle=0$. Since $\bar{Q}_c$ defined by (4.10) and $\bar{Q}_B$ satisfy

$$\bar{Q}_B^2 = 0;\ [i\bar{Q}_c,\bar{Q}_B] = \bar{Q}_B, \tag{4.14}$$

similarly to (3.6) and (3.7), the <u>quartet mechanism</u> discussed in §3 works again here on the basis of the <u>subsidiary condition</u> for physical states

$$\overline{Q}_B|\text{phys}\rangle = 0;\ \mathcal{V}(\beta)_{\text{phys}} \equiv \{|\Phi\rangle\in\mathcal{V}(\beta);\overline{Q}_B|\Phi\rangle = 0\}, \tag{4.15}$$

ensuring the positive semi-definiteness of $\mathcal{V}(\beta)_{\text{phys}}$. Likewise, an observable $\mathcal{O}$ is defined similarly to (3.11) by the condition

$$[\overline{Q}_B,\mathcal{O}]\mathcal{V}(\beta)_{\text{phys}} = 0, \tag{4.16}$$

which allows us to transfer $\mathcal{O}$ into the physical Hilbert space $H(\beta)_{\text{phys}}\equiv\mathcal{V}(\beta)_{\text{phys}}/\mathcal{V}(\beta)_0$ by taking its quotient mapping $\hat{\mathcal{O}}$. Since $\overline{H}$, $\overline{H}-\pi\overline{Q}_c/\beta$, $\overline{Q}_c$ and J also satisfy (4.16), the relation (4.11) for the observable $\mathcal{O}$ satisfying (4.16) can be transferred into $H(\beta)_{\text{phys}}$,

$$\hat{J}\exp(-[\beta\hat{\overline{H}}-\pi\hat{\overline{Q}}_c]/2)\hat{\mathcal{O}}|0(\beta)\rangle = J\exp(-\beta\hat{\overline{H}}/2)\hat{\mathcal{O}}|0(\beta)\rangle = \hat{\mathcal{O}}^{\dagger}|0(\beta)\rangle, \tag{4.17}$$

where we have used $\hat{Q}_c=0$ valid in $H(\beta)_{\text{phys}}$. The relation (2.11) for the standard cases with positive metric is thus recovered in $H(\beta)_{\text{phys}}$.

At the end, we add a few comments. Firstly, as for the renormalization of the Feynman diagrams in this formalism, it has been proved [10] that all the UV divergences at $T\neq 0^{\circ}K$ are removed by the counterterms set up at $T=0^{\circ}K$. Secondly, the Lorentz (boost) invariance of relativistic QFT is shown to be broken spontaneously at $T\neq 0^{\circ}K$ without any Goldstone bosons but with continuous zero-energy spectrum due to particle pairs having opposite energy-momenta [11]. Thirdly, in view of the general condition imposed on the well-defined charges, supersymmetry turns out to be unable to be implemented at $T\neq 0^{\circ}K$ [11]. Finally, the application of this formalism to the curved space-time [12] and to quantum gravity will be of interest.

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