PATH INTEGRATION AT THE CROSSROAD OF STOCHASTIC AND DIFFERENTIAL CALCULUS

Cécile DeWitt-Morette, Department of Astronomy and Center for Relativity,
The University of Texas, Austin, Texas 78712

Ever since Newton differential calculus has been a very successful language for the description of physical systems; stochastic calculus on the other hand has only recently become an instrument of thought but has already been used in challenging problems. For instance the traditional quantization scheme can be summarized in the following chart.

The description of a physical system starts with its configuration space or its phase space	its dynamics being given by a lagrangian or an hamiltonian

The classical lagrangian or the classical hamiltonian is then used to quantize the system

Lagrangian → Hamiltonian ---→ Hamiltonian operator

Lagrangian ---→ Path integration ←--- Hamiltonian

where the dotted arrows indicate that the corresponding construction is not unique.

The stochastic scheme on the other hand proceeds as follows.

The description of a physical system starts with a fibre bundle on its configuration space or on its phase space	its dynamics being given by a stochastic differential equation

Then there is a unique prescription for the following constructions.

stoch. diff. eqn. → Path integral → Hamiltonian operator

Path integral ↓ Its WKB approximation (Lagrangian)

Here classical physics is obtained as the WKB limit of quantum physics.

Before presenting some applications of the stochastic scheme, I recall briefly the key concepts of stochastic calculus. Differential calculus is based on the assumption that $dx(t)$ is of order dt; but if $dx(t)$ is for instance of order $(dt)^{1/2}$, as in brownian motion, then $\lim dx(t)/dt$ is undefined and, at best, one can only speak of the probability that a particle which is at x_o at time t_o will be at $x_o + dx(t)$ at time $t_o + dt$. Thus stochastic calculus begins with a probability space $(\Omega,\mathcal{F},w)$ where Ω is a set of points $\omega \in \Omega$, the events are subsets of Ω which make a σ algebra $\mathcal{F}$, and random variables are measurable functions

$\phi: (\Omega,\mathcal{F}) \to (\mathbb{R}^n$, Borel σ-algebra) henceforth abbreviated to $\phi: \Omega \to \mathbb{R}^n$.

Random or stochastic processes are sets of random variables indexed, for instance by time T, $\{x(t)\}$ where

$$x: T \times \Omega \to \mathbb{R}^n \quad , \quad x(t): \Omega \to \mathbb{R}^n \quad .$$

w is a measure on Ω such that $w(\Omega) = 1$. For quantum physics the concept of measure has to be generalized to the concept of prodistribution - technically a projective family of tempered distributions on a projective system of finite dimensional spaces [1]; a prodistribution is defined by its Fourier transforms $\mathcal{F}w$. The condition $w(\Omega) = 1$ becomes $\mathcal{F}w(0) = 1$. Here I shall simply refer to measures, promeasures, prodistributions as "integrators".

A stochastic differential equation defines a stochastic process x in terms of a known process z, e.g. let $z: T \times \Omega \to \mathbb{R}$ be brownian with $z(t_o) = 0$ and let

$$x(t,\omega) = x_o(\omega) + \int_{t_o}^{t} X(x(t,\omega))dz(t,\omega) \quad \text{where X is given.}$$

It is usually abbreviated to $dx = X\,dz$, $x(t_o) = x_o$.

Stochastic integrals $\int X(x(t))dz(t)$ when $dz(t)$ is not of order dt have been given meaning by Itô and by Stratonovich.

Very few stochastic differential equations can be solved explicitly, but expectation values of functions of stochastic processes

$$\mathbb{E}\phi(x(t)) := \int_\Omega dw(\omega)\phi(x(t,\omega)) := \langle\phi(x(t))\rangle$$

are functions of t and the initial point $x_o = x(t_o)$ whose properties can be determined from the stochastic differential equation satisfied by x(t) without solving it.

The prototype of such a situation is the Feynman-Kac formula: given the stochastic differential equation

$$\begin{cases} dx^\alpha(t) = X^\alpha_i(x(t))dz^i(t) + A^\alpha(x(t))dt & x(t_o) = x_o \\ dv(t) = V(x(t))v(t)dt & v(t_o) = 1 \end{cases} \qquad (1)$$

Then $\Psi(t,x_o) := \int_\Omega dw(\omega)\ v(t,\omega)\phi(x(t,\omega))$ is a solution of the diffusion equation

$$\frac{\partial\Psi}{\partial t} = \sum_i X_i^\alpha(x_o)X_i^\beta(x_o)\,\frac{\partial^2\Psi}{\partial x_o^\alpha \partial x_o^\beta} + A^\alpha(x_o)\,\frac{\partial\Psi}{\partial x_o^\alpha} + V(x_o)\Psi := \mathfrak{H}\Psi$$

$$\Psi(t_o,x_o) = \phi(x_o) \quad . \tag{2}$$

To obtain a path integral solution of the Schrödinger equation for $H = i\hbar\mathfrak{H}$ with $\mathfrak{H} = \frac{i}{2}\frac{\hbar}{m}\Delta + A^\alpha \frac{\partial}{\partial x_o} + \frac{1}{i\hbar}V$ one must modify the above scheme and work with complex gaussian (technically prodistributions) over the space of paths vanishing at t rather than t_o. Note that the Schrödinger equation obtained from the stochastic process (1) where V is replaced by $V/i\hbar$, is not the equation of a particle in an electromagnetic potential; indeed there should be an A^2 term. One can, of course, modify the stochastic equation (1) to bring out the desired terms but then one is faced with having to choose the order of the factors in the hamiltonian operator. On the other hand if one sets up the problem on the appropriate fibre bundle one has an obvious choice for the stochastic differential equation. In the case of a particle in an electromagnetic field the appropriate fibre bundle is the U(1) bundle over the configuration space. Given a fibre bundle the "obvious" choice of stochastic differential equation is described in the two following examples where (i) the fibre bundle is a principal fibre bundle (ii) the fibre bundle is an associated vector bundle [3], [7].

i) A particle of mass m in a riemannian space M, dimension n, metric g, with the riemannian connection.

Let O(M) be the orthonormal frame bundle, $p \in O(M)$ is a pair (x,u) where $x \in M$ and u is a frame at x

$$u: \mathbb{R}^n \to T_xM$$

Given a path $x: T \to M$, $x(t_o) = x_o$, and a frame u_o at x_o, the connection defines a path $u: T \to O(M)$, $u(t_o) = u_o$ and a map

$$X: O(M) \times \mathbb{R}^n \to T\,O(M)$$

such that

$$\frac{du(t)}{dt} = X(u(t))(u(t))^{-1}\,\frac{dx(t)}{dt} \quad .$$

Recall that $dx(t)/dt \in T_{x(t)}M$ and $(u(t))^{-1}dx(t)/dt \in \mathbb{R}^n$. If the path x is not differentiable, one can define a stochastic frame by the corresponding stochastic differential equation (in the Stratonovich sense because Itô calculus is not tensorial)

$$du(t) = \mu X(u(t))dz(t) \quad ,$$

where $\mu = (\hbar/m)^{1/2}$ and z is brownian, the brownian path z being multipled by μ, so that μz has the dimension of length. A stochastic path on M is the projection of a path on the bundle: $x(t) = \pi(p(t))$. Given an arbitrary (good) function $\phi: M \to \mathbb{R}$, then $\int_\Omega dw(\omega)\phi(\pi(p(t,\omega))$ is the path integral representation of the solution $\Psi(t,x_o)$ of the diffusion equation

$$\begin{cases} \dfrac{\partial\Psi}{\partial t} = \dfrac{1}{2}\mu^2 \, \Delta \, \Psi \\ \Psi(t_o,x_o) = \phi(x_o) \end{cases}$$

where Δ is the Laplace-Beltrami operator on M defined by the metric g on M. It is an easy matter to write the stochastic differential equation which gives a diffusion equation with a vector and a scalar potential. The corresponding path integral has been computed as an expansion in powers of μ for the terms of order μ^{-2}, μ^{-1}, μ^0 in [1] and more recently for the terms of order μ^1 by a recursion method [2] which in principle can give higher order terms in μ.

ii) A particle in a gauge field in flat space.

Let $\tau^t_{t_o}\ \phi(x(t))$ be the parallel transport from t to t_o of $\phi(x(t))$ along the stochastic path $x(t) = \mu z(t)$ for $\mu = (\hbar(m))^{1/2}$ and z brownian. Then $\int_\Omega dw(\omega)\tau^t_{t_o}\phi(x(t,\omega))$ is the path integral solution $\Psi(t,x_o)$ of the diffusion equation

$$\begin{cases} \dfrac{\partial\Psi}{\partial t} = \dfrac{1}{2}\mu^2 \, \Delta \, \Psi \\ \Psi(t_o,x_o) = \phi(x_o) \end{cases}$$

where Δ is the laplacian constructed from the covariant derivatives defined by the gauge connection.

Both examples can be summarized by saying that a stochastic connection yields a diffusion equation with covariant derivatives. Starting from a stochastic process on a fibre bundle has the following advantages:

i. It is a unifying scheme which applies to a wide class of apparently different problems [3].

ii. It gives simple answers to such problems as parallel transport along a brownian path, short time propagator on riemannian manifolds, canonical relationship between lagrangian function and hamiltonian operator.

iii. It is cast in a framework which guarantees gauge invariance.

The Feynmn-Kac formula and its generalization to stochastic processes on fibre bundles is but a small example of the bartering which goes on at the crossroad of stochastic and differential calculus [4]. Stochastic calculus is also used in quantum field theory [5], [6].

[1] C. DeWitt-Morette, A. Maheshwari and B. Nelson, "Path Integration in Non-Relativistic Quantum Mechanics," Physics Reports 50 (1982), 255-372.
[2] K. D. Elworthy and A. Truman, "The Diffusion Equation and Classical Mechanics: An elementary formula". To appear in Stochastic Processes in Quantum Theory and Statistical Physics:Recent Progress and Applications, S. A. Albeverio, M. Sirugue and M. Sirugue-Collin, eds., (Springer Verlag Lecture Notes in Physics).
[3] C. DeWitt-Morette, K. D. Elworthy, B. L. Nelson and G. S. Sammelmann, "A stochastic scheme for constructing solutions of the Schrödinger equation," Ann. Inst. Henri Poincaré 32 (1980), 327-341.
[4] C. DeWitt-Morette, "Path Integration Quantization" (An expanded version of this talk with a more complete bibliography, presented at the III Marcel Grossmann conference, Shanghai, August 1982).
[5] B. Simon, The $P(\phi)_2$ Euclidean (Quantum) Field Theory, Princeton University Press (1974).
[6] J. Glimm and A. Jaffe, Quantum Physics: A Functional Integral Point of View, Springer Verlag, New York (1981).
[7] K. D. Elworthy, Stochastic Differential Equations on Manifolds, Cambridge University Press (1982).

Credit

Support in part by NSF grant No. PHY81-07381.