Manifestly Covariant Canonical Formalism of Quantum Gravity
—— A Brief Survey ——

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General relativity and quantum field theory are two brilliantly successful fundamental theories of physics. It is, therefore, of very fundamental importance to unify both theories in a consistent and beautiful manner, because if it were impossible to do so then either general relativity or quantum field theory might have to be abandoned in order to achieve the ultimate unified theory. The purpose of this talk is to claim that a quantum field-theoretical formalism of gravity has been formulated in quite a satisfactory way.

Unfortunately, however, there seems to be no consensus about what theory of quantum gravity is to be called satisfactory. Relativists usually expect that quantum gravity should inherit the geometrical concept of general relativity. On the contrary, according to most particle physicists, quantum gravity is nothing more than a quantum field theory of massless spin-two particles, and though the Einstein gravity is unitary and Lorentz invariant, it is not satisfactory because it is not renormalizable. Their primary concern in quantum gravity is thus <u>not</u> the connection between the spacetime structure and gravity at the quantum level <u>but</u> how to remove the ultraviolet divergence in perturbation theory.

I completely disagree to such a standpoint. It is totally nonsense to discuss the divergence problem of quantum gravity in perturbation theory, because Feynman integrals become meaningless when the contributions from the energies greater than the Planck mass are significant. In other words, since the gravitational constant κ is a fundamental constant at the same level as the light velocity c and the Planck constant $\hbar$, the perturbation expansion in quantum gravity is mathematically inadequate just as the expansion in powers of $1/c$ or $\hbar$ is. Accordingly, quantum gravity must be formulated in a <u>non-perturbative</u> way. Of course, it is a very important and extremely difficult problem to invent a divergence-free non-perturbative approximation method, but I emphasize that it is a problem at a stage <u>different</u> from constructing a satisfactory formalism itself. I believe that the invention of an appropriate approximation method is not a prerequisite for the judgement that the formalism is satisfactory.

I make some comments on the path-integral formalism because one might assert that it already provides a satisfactory formalism for quantum gravity. The path-integral formalism is well-formulated for scalar field theories and it yields correct results in perturbation theory. I must emphasize, however, that in gauge theories and in quantum gravity, the non-perturbative approach based on the path-integral formalism has no justification for its validity, because the proof of unitarity has been made only in the perturbative way. Since the non-perturbative functional measure of a path-integral is quite ambiguous, its definition should be made so as to guarantee unitarity, but there is no idea for finding such a definition. I believe that the path-integral formalism may be at best a convenient calculational technique, but it cannot be worth being called a fundamental formalism. Indeed, given an operator formalism, it may be possible to derive the corresponding path-integral, but the converse is generally impossible. This is because the path-integral formalism does not contain complete information; for example, it has no information concerning the hermiticity assignment for field operators in the indefinite-metric case.

In my opinion, a satisfactory operator formalism should be based on the following four principles:

1. Lagrangian and canonical formalism.
2. Manifest covariance.
3. Indefinite-metric Hilbert space with subsidiary conditions.
4. Asymptotic completeness.

I believe that mathematical rigor should not be regarded as a basic principle. The positivity of Hilbert-space metric is mathematically very convenient, but it is rather a source of various pathological features of quantum field theory. As is well known, it is inevitable to use indefinite metric in manifestly covariant gauge theories. Though a consistent positive-metric formalism of quantum electrodynamics is possible in the Coulomb gauge, I conjecture that no consistent non-perturbative formalism is possible in the positive-metric Hilbert space for non-abelian gauge theories nor for quantum gravity.

The celebrated operator formalism of a gauge theory is the Gupta-Bleuler formalism[1] for quantum electrodynamics. It is a satisfactory formalism from my point of view. But I note that it is more reasonable to introduce an auxiliary scalar field $B(x)$ into the fundamental Lagrangian. I believe it natural to have a gauge-fixing condition as an independent field equation. Then $B(x)$ should be regarded as one of fundamental fields. I wish to call such a formalism, in general, "B-field formalism." The Landau gauge can be properly formulated only in the B-field formalism.

It took longer than a quarter of a century to achieve the correct extension of the Gupta-Bleuler formalism to non-abelian gauge theories. This too much long delay has, unfortunately, brought people's blind confidence in the path-integral formalism. The manifestly covariant canonical formalism of non-abelian gauge theories was formulated quite successfully by Kugo and Ojima.[2,3] The basic ingradients of the Kugo-Ojima formalism are as follows:

1. B-field formalism.
2. Correct hermiticity assignment of the Faddeev-Popov (FP) ghosts.
3. Becchi-Rouet-Stora (BRS) invariance.[4]

Then it is crucial to note that there exists the BRS charge Q_B and that it is nilpotent ($Q_B{}^2=0$) and hermitian ($Q_B^\dagger=Q_B$). The physical states are defined by a subsidiary condition

$$Q_B|\text{phys}\rangle = 0.$$

Then one can quite generally prove the unitarity of the physical S-matrix under the postulate of asymptotic completeness.[3,5] This fact is extremely important because it is the only non-perturbative proof of unitarity in non-abelian gauge theories.

Now, I proceed to the manifestly covariant canonical formalism of quantum gravity.[6-20] As in the Kugo-Ojima formalism, it is natural to introduce the B-field $b_\rho(x)$ and a pair of the hermitian FP ghosts $c^\sigma(x)$ and $\bar{c}_\tau(x)$ and set up the fundamental Lagrangian in such a way that the action integral is BRS invariant. But it must be remarked that there is an important difference from the case of the Yang-Mills theory: The invariance under the general coordinate transformation in general relativity is a spacetime symmetry. Hence the corresponding BRS invariance is also a spacetime symmetry. I defined the "intrinsic" BRS transformation $\underline{\delta}$, as a fermionic derivation satisfying $\underline{\delta}^2=0$, by[6]

$$\underline{\delta}(\Phi^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell}) = \kappa\Big(\sum_{i=1}^{k}\partial_\lambda c^{\mu_i}\cdot\Phi^{\mu_1\cdots\lambda\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} - \sum_{j=1}^{\ell}\partial_{\nu_j}c^\lambda\cdot\Phi^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\lambda\cdots\nu_\ell}\Big)$$

for any classical tensor $\Phi^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell}$ and by (The signs of $\underline{\delta}$ and $\bar{c}_\rho$ are changed for convenience.)

$$\underline{\delta}(x^\mu) = \kappa c^\mu, \quad \text{whence} \quad \underline{\delta}(c^\mu) = 0,$$

$$\underline{\delta}(\bar{c}_\nu) = ib_\nu, \quad \text{whence} \quad \underline{\delta}(b_\nu) = 0,$$

$$[\underline{\delta},\partial_\nu] = -\kappa\partial_\nu c^\lambda\cdot\partial_\lambda .$$

Here $\underline{\delta}(x^\mu) = \kappa c^\mu$ is owing to the basic rule of constructing the BRS transform, namely, the rule that the infinitesimal transformation function is replaced by one of FP ghosts. The vanishing of $\underline{\delta}(c^\mu)$ represents the abelian nature of the translation which is the global version of the general coordinate transformation. The more conventional BRS transformation,[21-25] which I denote here by $\underline{\delta}_*$, is given by

$$\underline{\delta}_*(\varphi) = \underline{\delta}(\varphi) - \kappa c^\lambda\partial_\lambda\varphi$$

for any field $\varphi(\neq b_\nu,\bar{c}_\nu)$. The second term is the "orbital" part of the BRS transformation.

The gauge-fixing term $\mathcal{L}_{GF}$ is chosen so as to be a scalar density under the general linear transformation. I believe that the requirement of general linear invariance for gauge fixing is quite natural, because the introduction of the Minkowski metric $\eta_{\mu\nu}$ into the fundamental Lagrangian is too abrupt and destroys the spirit of general relativity unnecessarily violently. The simplest, most natural expression for $\mathcal{L}_{GF}$ is given by

$$\mathcal{L}_{GF} = -\kappa^{-1}\tilde{g}^{\mu\nu}\partial_\mu b_\nu,$$

where $\tilde{g}^{\mu\nu} \equiv \sqrt{-g}\, g^{\mu\nu}$ with $g \equiv \det g_{\mu\nu}$. It is noteworthy that general linear invariance can be realized only in the framework of the B-field formalism.

The FP-ghost term $\mathcal{L}_{FP}$ is determined by the requirement that

$$\sqrt{-g}^{\,-1}(\mathcal{L}_{GF} + \mathcal{L}_{FP}) = \underline{\delta}(i\kappa^{-1}g^{\mu\nu}\partial_\mu \bar{c}_\nu);$$

then

$$\mathcal{L}_{FP} = -i\tilde{g}^{\mu\nu}\partial_\mu \bar{c}_\rho \cdot \partial_\nu c^\rho.$$

The total Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_E + \mathcal{L}_{GF} + \mathcal{L}_{FP} + \mathcal{L}_M,$$

where $\mathcal{L}_E$ is the Einstein Lagrangian,

$$\mathcal{L}_E = (2\kappa)^{-1}\sqrt{-g}\, R,$$

and $\mathcal{L}_M$ denotes the matter-field Lagrangian.

It is very important to note that $\mathcal{L}_{FP}$ contains simple derivatives only but no covariant derivative. Its form is quite similar to the FP-ghost term of quantum electrodynamics. Though it is widely believed that quantum gravity is similar to the Yang-Mills theory, I emphasize that quantum gravity is much more similar to quantum electrodynamics. This feature is, of course, the consequence of the abelian nature of the translation group. The reason why people could not be aware of this very simple observation is that they did not make clear separation between the intrinsic part and the orbital one.

Since the operator formalism which follows from the above Lagrangian $\mathcal{L}$ is of outstanding beauty, I wish to claim that it is the "correct" formalism of quantum gravity. One might say that since the choice of $\mathcal{L}_{GF} + \mathcal{L}_{FP}$ is rather arbitrary, the fundamental thing is the classical Lagrangian. But I disagree to this opinion. Since quantum theory is to be more fundamental than the classical theory, the quantum Lagrangian must be more fundamental than than the classical one. Hence I believe that the expression for $\mathcal{L}_{GF} + \mathcal{L}_{FP}$ should be uniquely determined in the correct theory. This standpoint is very crucial also in considering quantum field theory in a background curved spacetime. Since it is usually constructed on the basis of classical general relativity, Duff[26] has criticized it by pointing out that the starting Lagrangian is quite ambiguous owing to the freedom of field redefinition. His difficulty is totally resolved if one starts with the full quantum-gravity theory having the uniquely specified gauge fixing.

Now, the field equations which follow from $\mathcal{L}$ are as follows:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}R - E_{\mu\nu} + \frac{1}{2} g_{\mu\nu}E = -\kappa T_{\mu\nu} \qquad \text{(quantum Einstein equation)},$$

with

$$E_{\mu\nu} \equiv \partial_\mu b_\nu + i\kappa\partial_\mu \bar{c}_\rho \cdot \partial_\nu c^\rho + (\mu \leftrightarrow \nu),$$

$$\partial_\mu \tilde{g}^{\mu\nu} = 0 \qquad \text{(de Donder condition)},$$

$$\partial_\mu (\tilde{g}^{\mu\nu}\partial_\nu c^\rho) = 0,$$

$$\partial_\mu (\tilde{g}^{\mu\nu}\partial_\nu \bar{c}_\rho) = 0.$$

Taking covariant derivative of the quantum Einstein equation, I obtain a remarkably simple equation,[6]

$$\partial_\mu (\tilde{g}^{\mu\nu}\partial_\nu b_\rho) = 0.$$

The four equations other than the quantum Einstein equation can be put together into

$$\partial_\mu (\tilde{g}^{\mu\nu}\partial_\nu X) = 0,$$

where $X = (x^\lambda/\kappa, b_\rho, c^\sigma, \bar{c}_\tau)$. This remark will become very important later.

Canonical quantization can be carried out consistently without employing Dirac's method of quantization[27] (which has serious difficulty in the quantum version of constraints[28]). The second derivatives of $g_{\mu\nu}$ in $\mathcal{L}_E$ and the derivative of b_ρ in $\mathcal{L}_{GF}$ are eliminated by integrating by parts. Canonical fields are $g_{\mu\nu}$, c^σ, $\bar{c}_\tau$, and matter fields; b_ρ is <u>not</u> regarded as a canonical field. Despite this difference from the standard canonical formalism, one can prove the equivalence between field equations and Heisenberg equations.[7]

It is quite remarkable that all equal-time (anti-)commutation relations between any two fields and between any field and the first time-derivative of any field can be calculated explicitly <u>in closed form</u>.[6] For example, I have

$$[g_{\mu\nu}(x), b_\rho(y)]_0 = -i\kappa(\tilde{g}^{00})^{-1}(\delta^0_\mu g_{\rho\nu} + \delta^0_\nu g_{\mu\rho})\delta^3(x - y),$$

$$[g_{\mu\nu}(x), \dot{g}_{\rho\lambda}(y)]_0 = -2i\kappa(\tilde{g}^{00})^{-1}[g_{\mu\nu}g_{\rho\lambda} - g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho} + (g^{00})^{-1}(\delta^0_\mu\delta^0_\rho g_{\nu\lambda} + \delta^0_\mu\delta^0_\lambda g_{\nu\rho} + (\mu \leftrightarrow \nu))]\delta^3(x - y),$$

where a subscript 0 of a commutator indicates to set $x^0 = y^0$.

From the BRS Noether current, I can calculate the expression for the gravitational BRS charge Q_b. By using the quantum Einstein equation and dropping total divergence, I find[6]

$$Q_b = \int d^3x\, \tilde{g}^{0\lambda}(b_\rho \partial_\lambda c^\rho - \partial_\lambda b_\rho \cdot c^\rho),$$

which is nilpotent ($Q_b{}^2=0$) and satisfies

$$i[Q_b, \varphi]_\mp = \underline{\delta}(\varphi) - \kappa c^\lambda \partial_\lambda \varphi$$

for any field $\varphi(x)$. The subsidiary condition is set up by

$$Q_b|\text{phys}\rangle = 0.$$

Then the unitarity of the physical S-matrix can be proved without recourse to perturbation theory.

The presence of the orbital term in $[Q_b, \varphi]_\mp$ is a very important characteristic of quantum gravity. As its consequence, any <u>local</u> operator $\Phi(x)$ has a <u>non-vanishing</u> (anti-)commutator with Q_b unless $\Phi(x)$ itself is a BRS transform of another operator. Hence, in contrast with the gauge-theory case, $\Phi(x)|0\rangle$ is <u>not</u> a physical state for any non-trivial local operator $\Phi(x)$. Accordingly, Lehmann's spectral function of any non-vanishing two point function can acquire the contribution from negative-norm intermediate states without contradicting unitarity. Thus Lehmann's theorem[29] breaks down in quantum gravity, that is, the exact two-point function may have a milder (perhaps oscillatory) high-energy asymptotic behavior than that of the corresponding free Feynman propagator.[30] This fact supports the old expection that quantum gravity may provide a natural ultraviolet cutoff, <u>without employing any kind of approximation</u>.

The above remark also resolves the Goto-Imamura difficulty[31] for the current-current commutator

$$\langle 0|[j_0(x), j_k(y)]_0|0\rangle.$$

Despite the fact that it must be zero if one relies upon the canonical anticommutation relations, one usually assumes the existence of the Schwinger term[32] in order to avoid the contradiction with the result of general framework. As is well known, however, the Schwinger term is the most pathological beast in quantum field theory. If quantum gravity is taken into account, one cannot prove the non-vanishing of the above commutator because of the presence of negative-norm intermediate states.[30] Thus the Goto-Imamura difficulty is resolved without introducing the Schwinger term.

It is straightforward to define the canonical energy-momentum tensor including gravity. It is interesting to note that the symmetric energy-momentum tensor cannot be defined so as to be a tensor density under the general linear transformation in contrast with the canonical one. By using the quantum Einstein equation and dropping total divergence, I find that the translation generator P_μ is given by a remarkably simple expression[8]

$$P_\mu = \kappa^{-1}\int d^3x\ \tilde{g}^{0\lambda}\partial_\lambda b_\mu$$

independently of the expression for $\mathcal{L}_M$. Of course, the well-defined energy-momentum operator depends on $\mathcal{L}_M$, but the above expression is sensible as a translation generator, because the volume integration should be carried out after commutator is taken.

Likewise, the generator of the general linear transformation is shown to be[8]

$$\hat{M}^\mu{}_\nu = \kappa^{-1}\int d^3x\ \tilde{g}^{0\lambda}[x^\mu\partial_\lambda b_\nu - \delta^\mu_\lambda b_\nu - i\kappa(\bar{c}_\nu\partial_\lambda c^\mu - \partial_\lambda\bar{c}_\nu\cdot c^\mu)].$$

It should be noted that $\hat{M}^\mu{}_\nu$ cannot be a well-defined operator. Indeed, if it were well-defined, one could exponentiate it, that is, one could consider finite general linear transformations. Then the equal-time (anti-)commutation relations imply that any two fields would (anti-)commute at all non-zero spacetime separations. Of course, however, $\hat{M}^\mu{}_\nu$ is sensible as a generator. For example, I have

$$i[\hat{M}^\mu{}_\nu, g_{\sigma\tau}] = \delta^\mu_\sigma g_{\nu\tau} + \delta^\mu_\tau g_{\sigma\nu} + x^\mu\partial_\nu g_{\sigma\tau}.$$

It is very important to note that general linear invariance is necessarily spontaneously broken. Since translational invariance should not be spontaneously broken, the vacuum expectation value of $g_{\sigma\tau}(x)$ must be a constant. Then I can set

$$\langle 0|g_{\sigma\tau}(x)|0\rangle = \eta_{\sigma\tau}$$

without loss of generality just as in the case of the Higgs model in which one can assume without loss of generality that the vacuum expectation value of the complex scalar field is a positive constant. From the above two formulae, I find

$$i\langle 0|[\hat{M}^{\mu}{}_{\nu}, g_{\sigma\tau}]|0\rangle = \delta^{\mu}_{\sigma}\eta_{\nu\tau} + \delta^{\mu}_{\tau}\eta_{\sigma\nu} \neq 0.$$

Thus $\hat{M}^{\mu}{}_{\nu}$ is spontaneously broken. But its antisymmetric part

$$\bar{M}_{\alpha\beta} \equiv \eta_{\alpha\mu}\hat{M}^{\mu}{}_{\beta} - \eta_{\beta\mu}\hat{M}^{\mu}{}_{\alpha}$$

is not broken. It is nothing but the Lorentz generator in the absence of spinor fields. The gravitational field is the Goldstone field of the broken ten components of $\hat{M}^{\mu}{}_{\nu}$.[33] Thus the physical graviton mass must vanish exactly.

Now, the most remarkable result of the manifestly covariant canonical formalism of quantum gravity is the existence of a sixteen-dimensional supersymmetry.[14,15] As mentioned already, the (super)current

$$\mathcal{P}^{\mu}(X) \equiv \tilde{g}^{\mu\nu}\partial_{\nu}X$$

is conserved, where $X = (x^{\lambda}/\kappa, b_{\rho}, c^{\sigma}, \bar{c}_{\tau})$, which is natural to be called the sixteen-dimensional supercoordinate. Let X and Y be two sixteen-dimensional supercoordinates. Then $\partial_{\mu}\mathcal{P}^{\mu}(X) = 0$ implies that

$$\mathcal{M}^{\mu}(X,Y) \equiv \sqrt{\epsilon(X,Y)}\tilde{g}^{\mu\nu}(X\partial_{\nu}Y - \partial_{\nu}X\cdot Y)$$

is also conserved, where $\epsilon(X,Y) = -1$ if both X and Y are FP ghosts and $\epsilon(X,Y) = +1$ otherwise; $\sqrt{+1} = +1$, $\sqrt{-1} = +i$. From those conservation laws, I obtain conserved (super)charges

$$P(X) \equiv \int d^3x\, \mathcal{P}^{0}(X),$$

$$M(X,Y) \equiv \int d^3x\, \mathcal{M}^{0}(X,Y).$$

Here X in $P(X)$ and X and Y in $M(X,Y)$ are not arguments but indices, each of which takes 16 values.

From the definition of $M(X,Y)$, it is evident that

$$M(Y,X) = -\epsilon(X,Y)M(X,Y),$$

whence one sees that $M(X,Y)$ has 128 independent components. Since, of course, $P(X)$ has 16 independent components, the theory possesses 144 independent symmetry generators altogether. The previously given generators P_{μ}, $\hat{M}^{\mu}{}_{\nu}$ and Q_b are expressible in terms of $P(X)$ and $M(X,Y)$.

By calculating the (anti-)commutators with field operators, I can determine the symmetry transformation laws corresponding to $P(X)$ and $M(X,Y)$, and verify the invariance of the action under them. It is very important to note that those transformation laws could not be discovered if quantization were made by the path-integral formalism. The Noether (super)currents of those symmetries can be confirmed to reproduce the original (super)currents $\mathcal{P}^{\mu}(X)$ and $\mathcal{M}^{\mu}(X,Y)$ apart from total divergence.[19]

The generators P(X) and M(X,Y) form a superalgebra quite similar to the Poincaré algebra. Hence I call it "sixteen-dimensional Poincaré-like superalgebra" (Some people[34,35] call it "choral symmetry" because it was proposed in the nineth paper of the series.). I define the sixteen-dimensional supermetric $\eta(X,Y)$ by

$$\eta(x^{\lambda}/\kappa, b_{\rho}) = \eta(b_{\rho}, x^{\lambda}/\kappa) = \eta(c^{\lambda}, \bar{c}_{\rho}) = -\eta(\bar{c}_{\rho}, c^{\lambda}) = \delta^{\lambda}_{\rho},$$

$$\eta(X,Y) = 0 \quad \text{otherwise.}$$

Then the (anti-)commutation relations between generators are as follows:[15]

$$[P(X),\ P(Y)]_{\mp} = 0,$$

$$[M(X,Y),\ P(U)]_{\mp} = \sqrt{-\epsilon(XY,U)}\,[\eta(Y,U)P(X) - \epsilon(X,Y)(X\leftrightarrow Y)],$$

$$[M(X,Y),\ M(U,V)]_{\mp} = \sqrt{-\epsilon(XY,UV)}\,\{[\eta(Y,U)M(X,V) - \epsilon(U,V)\eta(Y,V)M(X,U)] - \epsilon(X,Y)[X\leftrightarrow Y]\}$$

This superalgebra is a natural "super" version of the Poincaré algebra.

The symmetry implied by a generator is a spacetime symmetry if and only if it has the B-field b_{ρ} as an index. Thus the sixteen-dimensional Poincaré-like superalgebra contains both spacetime and internal symmetries in complete harmony. Furthermore, it is quite remarkable that it realizes, in some sense, the democracy between spacetime coordinates and quantum fields.

Some symmetries among the 144 generators are necessarily spontaneously broken. Unbroken ones are 10 Poincaré generators and 74 M(X,Y)'s involving no spacetime as an index. The Ward-Takahashi-type identities

$$\langle 0|[M(X,Y), \mathcal{O}]_{\mp}|0\rangle = 0$$

hold for those 74 generators, where $\mathcal{O}$ denotes an arbitrary T-product of field operators. The perturbation-theoretical validity of those Ward-Takahashi-type identities has been confirmed at one-loop level.[18]

Though the sixteen-dimensional Poincaré-like superalgebra includes no particle-physics symmetry other than the Poincaré algebra, in quantum gravi-electrodynamics it is possible to extend this superalgebra so as to include the electromagnetic U(1) symmetry.[17] In this way, therefore, there might be a possibility of unifying all physically relevant symmetries without contradicting the no-go theorem.[36,37]

Now, another very interesting feature of the manifestly covariant canonical formalism of quantum gravity is a revival of general covariance at the purely quantum level; more precisely, in this theory tensor analysis becomes relevant for certain commutation relations. Since the inevitable violation of general covariance in quantization has been quite regretable from the point of view of relativists, this revival of general covariance is quite noteworthy as the evidence showing that the theory is the rightful successor of Einstein's general relativity.

Since the B-field b_ρ is not a canonical field, the equal-time commutator between b_ρ and a canonical field may not necessarily vanish. It is found that the commutator between b_ρ and a local operator which is a tensor at the classical level has, in general, quite remarkable regularity. Let $\Phi^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell}(x)$ be a tensor generically. Then the general form of the equal-time commutator is[13]

$$[\Phi^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell}(x), b_\rho(y)]_0$$

$$= i\kappa(\tilde{g}^{00})^{-1}[\sum_{i=1}^{k}\delta_\rho^{\mu_i}\delta^0_{\mu_i{}'}\Phi^{\mu_1\cdots\mu_i{}'\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} - \sum_{j=1}^{\ell}\delta_\rho^{\nu_j{}'}\delta^0_{\nu_j}\Phi^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_j{}'\cdots\nu_\ell}]\delta^3(x-y).$$

This commutation relation is tensorlike in the sense that it is consistent with the rules of tensor analysis, that is, its form is preserved in raising or lowering tensor indices, in constructing tensor product of two tensors, and in contracting upper and lower indices. The validity of the above tensorlike commutation relation has been verified explicitly a large number of examples including the Ricci tensor $R_{\mu\nu}$.

Quite surprisingly, the tensorlike commutation relation can be extended into the four-dimensional form.[20] The four-dimensional commutation relation between b_ρ and a tensor consists of two parts: The main part is tensorlike, consistent with taking covariant derivative, and manifestly affine (i.e., translation and general linear) covariant, while the remaining part is of different character and has the same form as the four-dimensional commutation relation between an FP ghost and that tensor. Hence, in particular, one sees that the equal-time commutation relation between a tensor and b_ρ is tensorlike with no additional terms if and only if that tensor commutes with an FP ghost at the equal time.

In discussing the four-dimensional commutation relation, the quantum-gravity extension of the Pauli-Jordan invariant D-function has been introduced.[20] Since the metric tensor is now an operator, the new invariant D-function, which I denote by $\mathcal{D}(x,y)$, must be a bilocal operator. It is uniquely defined by the following four properties:

(1) $\mathcal{D}(x,y) = -\mathcal{D}(y,x)$.

(2) $\partial_\mu^x[\tilde{g}^{\mu\nu}(x)\partial_\nu^x\mathcal{D}(x,y)] = 0$.

(3) $\mathcal{D}(x,y)|_0 = 0.$

(4) $\partial_0^x \mathcal{D}(x,y)|_0 = -[\tilde{g}^{00}(x)]^{-1}\delta^3(x-y).$

[Here, even if the operator ordering is reversed in (2), the defined $\mathcal{D}(x,y)$ can be shown to be the same.] Then $\mathcal{D}(x,y)$ can be shown to be affine invariant in the sense that

$$i[P_\nu, \mathcal{D}(x,y)] = (\partial_\nu^x + \partial_\nu^y)\mathcal{D}(x,y),$$

$$i[\hat{M}^\mu{}_\nu, \mathcal{D}(x,y)] = (x^\mu\partial_\nu^x + y^\mu\partial_\nu^y)\mathcal{D}(x,y).$$

But, of course, $\mathcal{D}(x,y)$ is not invariant under finite general linear transformations. It has no c-number lightcone singularity, and therefore the short-distance expansion breaks down in quantum gravity. This fact is important in order for quantum gravity to play the role of a natural regulator.

In all the above, the gravitational field has been described by the metric tensor, but when there are Dirac fields, the fundamental gravitational field must be the vierbein (tetrad). Since the six additional degrees of freedom in the vierbein are of local Lorentz transformations, quantum theory can be constructed quite similarly to the Kugo-Ojima formalism for the Yang-Mills field.

It is very crucial that the local-Lorentz gauge-fixing term is chosen to be a scalar density under the general coordinate transformation. The right expression turns out to be[10]

$$\mathcal{L}_{LGF} = -\tilde{g}^{\mu\nu}\hat{\Gamma}_\mu{}^{ab}\partial_\nu s_{ab},$$

where $\hat{\Gamma}_\mu{}^{ab}$ denotes the spin connection and s_{ab} is a new antisymmetric scalar B-field. Since the spin connection contains first derivatives of the vierbein, s_{ab} cannot be regarded as a Lagrange-multiplier field. Owing to this form of $\mathcal{L}_{LGF}$, all components of both the vierbein and the B-field s_{ab} describe dynamical degrees of freedom.

The local-Lorentz FP-ghost term $\mathcal{L}_{LFP}$ is added in such a way that $\mathcal{L}_{LGF} + \mathcal{L}_{LFP}$ becomes a local-Lorentz BRS transform of some quantity. Then the manifestly covariant canonical formalism of quantum gravity can be extended quite beautifully to the vierbein case. Canonical quantization can be carried out consistently without using Dirac's method, and all equal-time (anti-)commutation relations are found explicitly in closed form.[11] Almost all results established in the metric-tensor case, such as the field equations and the equal-time commutation relations for $g_{\mu\nu}$, b_ρ, c^σ, and $\bar{c}_\tau$, various expressions for generators, the sixteen-dimensional Poincaré-like superalgebra, the tensorlike commutation relations, etc., remain intact.[10,11]

An important modification is necessary, however, for the spontaneous breakdown of general linear invariance. In the vierbein case, even the antisymmetric part, $\bar{M}_{\alpha\beta}$, of $\hat{M}^{\mu}{}_{\nu}$ is spontaneously broken. The unbroken one is given by[12]

$$M_{\alpha\beta} \equiv \bar{M}_{\alpha\beta} + \eta_{\alpha a}\eta_{\beta b}M_L{}^{ab},$$

where $M_L{}^{ab}$ is the generator of the global version of the local-Lorentz transformation. The true Lorentz generator $M_{\alpha\beta}$ is thus characterized at the level of the representation of field operators,[19] just as the electromagnetic charge is in the Weinberg-Salam model. This fact is conceptually very important: Lorentz invariance should not be regarded as a first principle determining the fundamental Lagrangian; Lorentz invariance is an S-matrix symmetry rather than a fundamental symmetry. I therefore conjecture that the usual supersymmetry having a spinor charge is not on the right way toward the ultimate unified theory.

Finally, I summarize the main achievements of the manifestly covariant canonical formalism of quantum gravity.

1. The theory is a beautiful and transparent canonical formalism of quantum gravity. Equal-time commutation relations such as $[g_{\mu\nu}, \dot{g}_{\lambda\rho}]$ are explicitly found in closed form.

2. Unitarity is proved in the Heisenberg picture. Since the perturbation series of quantum gravity is unrenormalizable, it is very important to construct the formalism without using perturbation theory.

3. There is a possibility that the ultraviolet divergence difficulty of quantum field theory may be ultimately resolved by taking account of quantum gravity: The theory achieves the evasion of Lehmann's theorem without violating unitarity.

4. The theory is manifestly covariant as in the Gupta-Bleuler formalism of quantum electrodynamics.

5. Though general covariance is broken by gauge fixing, which is necessary for quantization, general linear invariance still remains unbroken at the operator level. It is spontaneously broken up to Lorentz invariance, and the corresponding Goldstone field is nothing but $g_{\mu\nu}$.

6. The theory is invariant under a huge superalgebra, called "sixteen-dimensional Poincaré-like superalgebra", consisting of 144 symmetry generators. It contains both space-time and internal symmetries in complete harmony without contradicting the no-go theorem.

7. The theory has a very interesting property, called "tensor-like commutation relation". General covariance is revived in this way purely at the operator level.

8. All the above establishments are beautifully extended to the case in which vierbein is the fundamental field.

9. The Lorentz invariance of particle physics is characterized by spontaneous breakdown, whence it cannot be a first principle.

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