"The Gauge Invariant Effective Action for Quantum Gravity and Its Semi-Quantitative Approximation"

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Introduction

The configuration space of quantum gravity is PR(M), the set of all pseudo-Riemannian metrics on spacetime M. The gauge group of quantum gravity is $\mathrm{Diff}^*(M)$. A gauge transformation ξ acts on PR(M):

$$\xi : PR(M) \to PR(M) \qquad \xi \in \mathrm{Diff}^*(M)$$
$$\varphi \to \xi(\varphi) \qquad \varphi \in PR(M) \tag{1}$$

Gauge transformations divide PR(M) into orbits. The orbits can be shown to comprise an infinite dimensional manifold $PR(M)/\mathrm{Diff}^*(M)$. This manifold is the space of physically distinct fields.

Infinitesimal gauge transformations take the form[1]

$$\varphi \to \varphi + \delta\varphi \qquad \delta\varphi^i = Q^i{}_\alpha[\varphi]\,\delta\xi^\alpha \tag{2}$$

The $Q^i{}_\alpha$ are components of a set of vector fields on PR(M), the <u>Killing flows</u>. The Lie brackets of the Killing flows define the structure constants of $\mathrm{Diff}^*(M)$

$$[\underset{\sim}{Q}_\alpha, \underset{\sim}{Q}_\beta] = \underset{\sim}{Q}_\gamma c^\gamma{}_{\alpha\beta}. \tag{3}$$

PR(M) may be endowed with a gauge invariant metric $\underset{\sim}{\gamma}$. Gauge invariance is expressed by

$$\pounds_{\underset{\sim}{Q}_\alpha} \underset{\sim}{\gamma} = 0 \tag{4}$$

Any metric that satisfies this equation projects to a metric on $PR(M)/\mathrm{Diff}^*(M)$. In the case of quantum gravity Eq.4 has a unique one parameter family of local solutions,

[1] The dynamical variables (which in gravity theory are the metric components $g_{\mu\nu}$) are denoted by φ^i. The index i is to be understood as a combined discrete-continuous label. The implicit summation convention involves integrals as well as sums.

$$\gamma^{\mu\nu\sigma'\tau'} = g^{\frac{1}{2}}(g^{\mu\sigma}g^{\nu\tau}+g^{\mu\tau}g^{\nu\sigma}+\lambda g^{\mu\nu}g^{\sigma\tau})\delta(x,x')$$

$$\lambda \neq -\frac{1}{2} \qquad g = -\det(g_{\mu\nu}) \tag{5}$$

The dynamics of the gravitational field is described by the classical action S, which is a real valued scalar function on PR(M) [2]

$$S[\varphi] = \mu_P{}^2\int g^{\frac{1}{2}}R d^4x \qquad 16\pi G = \mu_P{}^{-2} \tag{6}$$

The classical action is gauge invariant:

$$\underset{\sim}{Q}_\alpha S = 0 \tag{7}$$

With $g_{\mu\nu}$ chosen for the basic dynamical variables φ^i the action of the gauge group on PR(M) is linear. Linearity may be expressed by

$$Q^i{}_{\alpha,jk} \equiv 0 \tag{8}$$

where the comma denotes functional differentiation. By repeatedly functionally differentiating Eq.7 and making use of Eq.8 one obtains the infinite sequence of equations

$$S_{,i}Q^i{}_\alpha \equiv 0$$

$$S_{,ij}Q^j{}_\alpha \equiv -S_{,j}Q^j{}_{\alpha,i}$$

$$S_{,ijk}Q^k{}_\alpha \equiv -S_{,kj}Q^k{}_{\alpha,i}-S_{,ik}Q^k{}_{\alpha,j}$$

$$S_{,ijk\ell}Q^\ell{}_\alpha \equiv -S_{,\ell jk}Q^\ell{}_{\alpha,i}-S_{,i\ell k}Q^\ell{}_{\alpha,j}-S_{,ij\ell}Q^\ell{}_{\alpha,k}, \text{ etc.} \tag{9}$$

These are the bare Ward-Takahashi identities of the theory.

The classical field equations are

$$S_{,i}[\varphi] = 0 \tag{10}$$

Given a solution φ of these equations one is often interested in a solution $\varphi + \delta\varphi$ which differs infinitesimally from φ . $\delta\varphi$ satisfies the equation of small disturbances

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2 We use units for which $\hbar = c = 1$ and a spacetime signature $-+++$.

$$0 = S_{,i}[\varphi+\delta\varphi] = S_{,i}[\varphi]+S_{,ij}[\varphi]\delta\varphi^j = S_{,ij}[\varphi]\delta\varphi^j \qquad (11)$$

The second functional derivative $S_{,ij}[\varphi]$ appearing in this equation is effectively a linear differential operator. Because of the gauge invariance of the theory this operator is singular. Equation 11 has a well defined solution for a given set of boundary conditions only if one imposes a supplementary condition

$$P^{\alpha}{}_{i}[\varphi]\delta\varphi^j = 0 \qquad (12)$$

When condition 12 is satisfied $\delta\varphi$ satisfies

$$F_{ij}[\varphi]\delta\varphi^j = 0 \qquad (13)$$

where

$$F_{ij} \stackrel{\text{def}}{=} S_{,ij} + \eta_{\alpha\beta}\, P^{\alpha}{}_{i} P^{\beta}{}_{j} \qquad (14)$$

The functions $P^{\alpha}{}_{i}$ appearing in Equation 12 are often chosen so that small disturbances are γ-orthogonal to gauge variations:

$$P^{\alpha}{}_{i} = \eta^{-1\alpha\beta}\, Q^j{}_{\beta}\, \gamma_{ji} \longrightarrow Q^j{}_{\beta}\, \gamma_{ji}\delta\varphi^i = 0 \qquad (15)$$

It is convenient to impose the following gauge covariance condition on the continuous matrix $\eta_{\alpha\beta}$:

$$\eta_{\alpha\beta,i}Q^i{}_{\gamma} \equiv -\eta_{\delta\beta}c^{\delta}{}_{\gamma\alpha} - \eta_{\alpha\delta}c^{\delta}{}_{\gamma\beta} \qquad (16)$$

The effective action

Suppose spacetime is such that we can introduce coherent "in" and "out" states, $|in,vac\rangle$, $|out,vac\rangle$. These states are sometimes known as relative vacua, i.e., they are vacua relative to a given background. The effective action Γ is defined by

$$\langle out,vac|in,vac\rangle = e^{i\Gamma[\varphi]} \tag{17}$$

Γ is a complex valued scalar field on the space CR(M) of complex metrics on spacetime M. The field φ appearing on the right hand side of Eq.17 is arbitrary. It does not need to be a classical background.

In what follows it will be convenient to define

$$\begin{aligned} \mathfrak{F}^{\alpha}{}_{\beta}[\varphi] &= P^{\alpha}{}_{i}[\varphi]Q^{i}{}_{\beta}[\varphi] \\ \mathfrak{F}^{\alpha}{}_{\gamma}[\varphi]\mathfrak{G}^{\gamma}{}_{\beta}[\varphi] &= -\delta^{\alpha}{}_{\beta} \\ V^{\alpha}{}_{\beta i}[\varphi] &= P^{\alpha}{}_{j}[\varphi]Q^{j}{}_{\alpha,i} \end{aligned} \tag{18}$$

and to extend the domain of all functionals to CR(M).

$\mathfrak{G}^{\alpha}{}_{\beta}$ is known as the <u>ghost propagator</u> and $V^{\alpha}{}_{\beta i}$ is known as the <u>ghost vertex</u>. The vacuum-to-vacuum amplitude (Eq.17) may be expressed by the following functional integral

$$\begin{aligned} e^{i\Gamma[\varphi]} &= \text{const.} \times \int e^{iS[\varphi+\phi]}d\phi \\ &= \text{const.} \times (\det\eta[\varphi])^{\frac{1}{2}}(\det\mathfrak{G}[\varphi])^{-1} \\ &\quad \times \int e^{i(S[\varphi+\phi]+\frac{1}{2}\eta_{\alpha\beta}[\varphi]P^{\alpha}{}_{i}[\varphi]P^{\beta}{}_{j}[\varphi]\phi^{i}\phi^{j})} \\ &\quad \times \det(1-\mathfrak{G}[\varphi]V[\varphi]\phi)^{-1}\, d\phi \end{aligned} \tag{19}$$

The <u>value</u> (although not in the explicit functional form) of the second integral is independent of the choice of $P^{\alpha}{}_{i}$ and $\eta_{\alpha\beta}$ provided a regularization is adopted that yields

$$c^{\beta}{}_{\alpha\beta} = 0 \quad , \qquad Q^{i}{}_{\alpha,i} = 0 \tag{20}$$

If the P's and η's are chosen covariantly then the effective action is gauge invariant ($\underset{\sim}{Q}_{\alpha}\Gamma \equiv 0$) and satisfies simple Ward-Takahashi identities:

$$\Gamma_{,i}Q^{i}{}_{\alpha} \equiv 0$$

$$\Gamma_{,ij}Q^{j}{}_{\alpha} \equiv -\Gamma_{,j}Q^{j}{}_{\alpha,i}$$

$$\Gamma_{,ijk}Q^{k}{}_{\alpha} \equiv -\Gamma_{,kj}Q^{k}{}_{\alpha,i} - \Gamma_{,ik}Q^{k}{}_{\alpha,i}$$

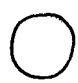
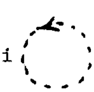
$$\Gamma_{,ijk\ell}Q^{\ell}{}_{\alpha} \equiv -\Gamma_{,\ell jk}Q^{\ell}{}_{\alpha,i} - \Gamma_{,i\ell k}Q^{\ell}{}_{\alpha,j} - \Gamma_{,ij\ell}Q^{\ell}{}_{\alpha,k} \qquad (21)$$

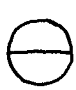
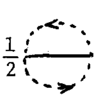
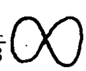
The equations

$$\Gamma_{,i}[\varphi] = 0 \qquad (22)$$

are called the <u>effective field equations</u>. The solution of these non-local equations satisfying the given boundary conditions is called the effective field. If the background is chosen to be the effective field the 1-particle reducible graphs may be omitted from the <u>loop expansion</u> of Γ:

$$\Gamma[\varphi] = S[\varphi] + \Sigma[\varphi]$$

$\Sigma[\varphi] = -\frac{i}{2}\ \ell n \det\eta[\varphi] - \frac{i}{2}$  $+\ i$ 

$-\frac{1}{12}$  $+\ \frac{1}{2}$  $-\ \frac{1}{8}$  $+ \cdots$ (23)

In these graphs solid lines denote the Feynman propagator for $F_{ij}[\varphi]$, written $G^{ij}[\varphi]$. A dotted line represents $\mathfrak{G}^{\alpha}{}_{\beta}[\varphi]$. A vertex at which a solid line meets two dotted lines represents the vertex function $V^{\alpha}{}_{\beta i}[\varphi]$. Vertices at which three or more solid lines come together represent functional derivatives of the classical action (the $S_{,ijk}$, $S_{,ijk\ell}$, etc.). The solid circle denotes $\ln \det G[\varphi]$. The dotted circle denotes $\ln \det \mathfrak{G}[\varphi]$.

The effective action has the following important properties:

1. The functional form of Γ depends on the P's and η's but the effective field and the value of Γ do <u>not</u>.

2. The tree functions built out of Γ yield the exact scattering amplitudes.

3. The effective field is an operator average:

$$\varphi^i = \frac{\langle \text{out,vac} | \underset{\sim}{\varphi}^i | \text{in,vac} \rangle}{\langle \text{out,vac} | \text{in,vac} \rangle} \tag{24}$$

4. If $\mathrm{Im}\Gamma \approx 0$ then the "in" and "out" states are nearly identical and φ^i becomes (approximately) an expectation value.

5. Γ , <u>not</u> S, governs the dynamics of quantized spacetime (e.g., near the Big Bang or near classical singularities).

6. In Yang-Mills theory the use of Γ simplifies the renormalization program. (See Abbot 1981, CERN preprints TH. 2973 and 3113,) and (Hart 1981, Ph.D. dissertation, University of Texas).

The general structure of Γ

In this section I shall attempt to adduce some plausibility arguments for the general structure of Γ . Begin by considering the functional $\Sigma[\varphi]$ which represents the difference between Γ and the classical action, i.e. the radiative corrections to the classical action (Eq.23). The functional Σ, like S and Γ , is gauge invariant. In quantum gravity this is expressed by:

$$(\delta\Sigma/\delta g_{\mu\nu})_{;\nu} \equiv 0 \qquad (25)$$

Let us assume that the Minkowski metric $\eta_{\mu\nu}$ is a stable solution of the effective field equation just as it is of the classical field equation $\delta S/\delta g_{\mu\nu} = 0$. That is, let us assume that Poincaré group,which is relevant for asymetrically flat spacetimes, is not dynamically broken.

$$(\delta\Sigma/\delta g_{\mu\nu})_{\underset{\sim}{g} = \underset{\sim}{\eta}} = 0 \qquad (26)$$

whence in virtue of 25,

$$\left[\left(\frac{\delta^2\Sigma}{\delta g_{\mu\nu}\,\delta g_{\sigma'\tau'}}\right)_{;\nu}\right]_{\underset{\sim}{g} = \underset{\sim}{\eta}} = 0 \qquad (27)$$

Denote by $\Sigma^{\mu\nu\sigma\tau}(p)$ the Fourier transform of $\left(\frac{\delta^2\Sigma}{\delta g_{\mu\nu}\delta g_{\sigma'\tau'}}\right)_{\underset{\sim}{g} = \underset{\sim}{\eta}}$, with the δ function expressing momentum conservation removed. Equation 25 is equivalent to

$$\Sigma^{\mu\nu\sigma\tau}(p)p_{\nu} = 0 \qquad (28)$$

of which the general solution, respecting Lorentz invariance and the index symmetries of $\Sigma^{\mu\nu\sigma\tau}$, is:

$$\begin{aligned}
&\Sigma^{\mu\nu\sigma\tau}(p)\\
&= \tfrac{1}{4}[(\eta^{\mu\sigma}\eta^{\nu\tau}+\eta^{\mu\tau}\eta^{\nu\sigma}-\tfrac{2}{3}\eta^{\mu\nu}\eta^{\sigma\tau})p^4\\
&-(\eta^{\mu\sigma}p^{\nu}p^{\tau}+\eta^{\nu\tau}p^{\mu}p^{\sigma}+\eta^{\mu\tau}p^{\nu}p^{\sigma}+\eta^{\nu\sigma}p^{\mu}p^{\tau}-\tfrac{2}{3}\eta^{\mu\nu}p^{\sigma}p^{\tau}-\tfrac{2}{3}\eta^{\sigma\tau}p^{\mu}p^{\nu})p^2\\
&+\tfrac{4}{3}p^{\mu}p^{\nu}p^{\sigma}p^{\tau}]\Sigma_1(p^2)\\
&-\tfrac{1}{3}[\eta^{\mu\nu}\eta^{\sigma\tau}p^4-(\eta^{\mu\nu}p^{\sigma}p^{\tau}+\eta^{\sigma\tau}p^{\mu}p^{\nu})p^2+p^{\mu}p^{\nu}p^{\sigma}p^{\tau}]\Sigma_2(p^2)
\end{aligned} \qquad (29)$$

If λ in Equation 5 is chosen to be -1, then the Fourier transform of $(F_{ij})_{g\,=\,\eta}$ is:

$$\frac{1}{4}\mu_P^2(\eta^{\mu\sigma}\eta^{\nu\tau}+\eta^{\mu\tau}\eta^{\nu\sigma}-\eta^{\mu\nu}\eta^{\sigma\tau})p^2+\Sigma^{\mu\nu\sigma\tau}(p) \tag{30}$$

The full graviton propagator is the 10 x 10 matrix inverse of this:

$$\begin{aligned}\Gamma_{\mu\nu\sigma\tau}(p) = \mu_P^{-2}[&\eta_{\mu\sigma}\eta_{\nu\tau}+\eta_{\mu\tau}\eta_{\nu\sigma}-\frac{2}{3}\eta_{\mu\nu}\eta_{\sigma\tau} \\ &+\mu_P^{-2}(\eta_{\mu\sigma}p_\nu p_\tau+\eta_{\nu\tau}p_\mu p_\sigma+\eta_{\mu\tau}p_\nu p_\sigma+\eta_{\nu\sigma}p_\mu p_\tau)\Sigma_1(p^2)] \\ &\times[p^2+\mu_P^{-2}p^4\Sigma_1(p^2)]^{-1} \\ &-\frac{1}{3}\mu_P^{-2}\eta_{\mu\nu}\eta_{\sigma\tau}[p^2+\mu_P^{-2}p^4\Sigma_2(p^2)]^{-1} \\ &-\frac{2}{3}\mu_P^{-4}[(\eta_{\mu\nu}p_\sigma p_\tau+\eta_{\nu\tau}p_\mu p_\sigma)p^2+2p_\mu p_\nu p_\sigma p_\tau] \\ &\times[\Sigma_1(p^2)-\Sigma_2(p^2)][p^2+\mu_P^{-2}p^4\Sigma_1(p^2)]^{-1} \\ &\times[p^2+\mu_P^{-2}p^4\Sigma_2(p^2)]^{-1}\end{aligned} \tag{31}$$

As can be seen from the form of this expression the particle spectrum is determined by the zeros of the functions $[p^2+\mu_P^{-2}p^4\Sigma_1(p^2)]$ and $[p^2+\mu_P^{-2}p^4\Sigma_2(p^2)]$. It is not difficult to show that if Σ is expanded as a functional power series in $\varphi_{\mu\nu}=g_{\mu\nu}-\eta_{\mu\nu}$ then the term of lowest order is quadratic in $\varphi_{\mu\nu}$ and is uniquely determined by Eq.29 to have the form

$$\Sigma^{(2)} = \int d^4x\int d^4x'\left[\frac{1}{2}\tilde{\Sigma}_1((x-x')^2)C_{\mu\nu\sigma\tau}(x)C^{\mu\nu\sigma\tau}(x') -\frac{1}{6}\tilde{\Sigma}_2((x-x')^2)R(x)R(x')\right] \tag{32}$$

where $C_{\mu\nu\sigma\tau}$ is the linearized Weyl tensor, R is the linearized curvature scalar and $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are the Fourier transforms of Σ_1 and Σ_2 respectively.

In the one-loop approximation <u>without</u> subtraction, the dominant singularities of both $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are proportional to $i/(x-x')^4$. This singularity structure, which renders expression (32) logarithmically divergent, arises from products of pairs of Green's functions $i/(x-x')^2$, together with loop factors -i, in typical self-energy graphs

How does it get modified in the exact theory?

A partial answer to this question is known [3,4] in the case of ladder graphs in which the free ends at the top of each ladder are joined together to make a single line, leaving only the two free ends at the bottom. The dominant high-energy contribution to the infinite sum of all such graphs can be expressed as the solution of the following simple integral equation

$$X(p) = \frac{1}{p^2-i0} - \frac{1}{(2\pi)^4\mu_P^{\,2}}\int \frac{X(k)}{(p-k)^2-i0}\, d^4k \qquad (33)$$

Since the integral in this equation is a convolution integral the equation is easily solved by taking the Fourier transform:

$$\tilde{X}(x) = G(x)[1-i\mu_P^{-2}\tilde{X}(x)] \qquad (34)$$

where G(x) is the standard scalar propagator,

$$G(x) = \frac{1}{(2\pi)^4}\int \frac{e^{ip\cdot x}}{p^2-i0}\, d^4p = \frac{i}{(2\pi)^2}\,\frac{1}{x^2+i0} \qquad (35)$$

yielding

$$\tilde{X}(x) = \frac{G(x)}{1+i\mu_P^{-2}G(x)} = \frac{i}{(2\pi)^2}\,\frac{1}{x^2-\lambda_P^{\,2}+i0} \qquad (36)$$

with

$$\lambda_P = \frac{1}{2\pi\mu_P} \qquad (37)$$

The line at the top of the ladder graph contributes a factor $1/(x-x')^2$ as always, but the rungs when summed to all orders, contribute expression 36 as a factor. The singularity of the rung factor lies on a hyperboloid at a distance λ_P outside the Minkowski light cone and implies noncausal propagation relative to Minkowski space-time. This is neither surprising nor alarming. When the metric itself undergoes quantum fluctuations "real" space-time is Minkowskian only in an average sense.

These results suggest that $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ may be well approximated by choosing each to be proportional to

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[3] B. S. DeWitt, Phys. Rev. Lett. 13, 114 (1964).

[4] I. B. Khriplovich, Yad. Fiz. 2, 950 (1965) [Sov. J. Nucl. Phys. 3, 415 (1966)].

$$iG(x-x')\tilde{X}(x-x') = \frac{i}{(2\pi)^4}\frac{1}{(x-x')^2[(x-x')^2-\lambda_p{}^2]}$$

$$= \frac{i}{(2\pi)^4\lambda_p{}^2}\left[\frac{1}{(x-x')^2-\lambda_p{}^2+i0} - \frac{1}{(x-x')^2+i0}\right]$$

$$= \frac{i}{(2\pi)^2}\mu_p{}^2\int_0^1\frac{d\xi}{[(x-x')^2-\xi\lambda_p{}^2+i0]^2} \qquad (38)$$

The final integral gives concrete expression to the old idea that quantum gravity smears the light cone. A more complete theory, which sums other graphs besides ladder graphs, would presumably insert a smearing function $w(\xi)$ in the integrand.

If $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ have the form 38 then their Fourier transforms are given by

$$\Sigma_{1,2}(p^2) \simeq A_{1,2}\left[-\frac{\pi i}{2}\frac{H_1^{(2)}((\lambda_p{}^2p^2-i0)^{\frac{1}{2}})}{(\lambda_p{}^2p^2-i0)^{\frac{1}{2}}} - \frac{1}{\lambda_p{}^2p^2-i0}\right] \qquad (39)$$

This function is complex for space-like momenta and real for time-like momenta. In both cases it tends to zero for $|p^2|>>\mu^2$. With this approximation the functions $p^2+\mu_p{}^{-2}p^4\Sigma_{1,2}(p^2)$ have no zeros other than $p^2=0$ on the real p^2 axis, provided $a_{1,2}$ avoid values lying between approximately .024 and .054 as well as an infinity of isolated points clustering about the origin between -.011 and .009. If $a_{1,2}$ lies between .024 and .054 then the graviton propagator has timelike ghosts. If $a_{1,2}$ = one of the discrete values then there are tachyon ghosts. The functions $p^2 + \mu_p^{-2}p^4\Sigma_{1,2}(p^2)$ have an infinity of zeros in the lower half p^2 plane, i.e., in the upper half $(p^0)^2$ plane. In the p^0 plane these zeros are in the first and third quadrants. Let $E=\omega+i\gamma\,(\omega>0,\gamma>0)$ be one of the first-quadrant zeros. The corresponding "instability" modes have time dependence $e^{\pm iEt}$. The mode that propagates positive frequencies into the future is

$$e^{-iEt} = e^{-i\omega t + \gamma t} \qquad (40)$$

The mode that propagates negative frequencies into the past is

$$e^{iEt} = e^{i\omega t - \gamma t} \qquad (41)$$

Both modes are eliminated by the "in" and "out" boundary conditions. Therefore they do not lead to real instabilities. However, they make Wick rotation impossible. This means that if the above approximation has any validity whatever, Euclideanization is not permitted in quantum gravity.

Equations 32 and 38 admit of immediate generalization to an approximation for Σ, and hence for Γ, that is invariant under the full diffeomorphism group:

$$\Gamma \approx \mu_P{}^2 \int g^{\frac{1}{2}} R d^4x$$
$$+\mu_P{}^2 \int d^4x \int d^4x' g^{\frac{1}{2}} g'^{\frac{1}{2}} \left[\frac{i}{\sigma(x,x')-\frac{1}{2}\lambda_P{}^2+i0} - \frac{i}{\sigma(x,x')+i0}\right]$$
$$\times\left[\tfrac{1}{4}A_1 g^{\mu\alpha'} g^{\nu\beta'} g^{\sigma\gamma'} g^{\tau\delta'} C_{\mu\nu\sigma\tau} C_{\alpha'\beta'\gamma'\delta'} - \frac{1}{12}A_2 RR'\right] \qquad (42)$$

Here g is $-\det(g_{\mu\nu})$, $g^{\mu\alpha'}$ is the parallel displacement bivector,[5] $\sigma(x,x')$ is half the square of the geodetic distance between x and x',[5] and $C_{\mu\nu\sigma\tau}$ and R are the Weyl tensor and curvature scalar of the full nonlinear theory. A_1 and A_2 are numerical coefficients whose precise values depend on the numbers and kinds of matter fields included, but whose magnitudes are not vastly different from unity. The $i0$ in the "propagators" specifies how the poles are to be skirted in the double integral, and the other factors i remind us that both Γ and the effective field, which is an "in-out" average, are generally complex valued.

Although expression (42) has been derived by arguments starting from flat space-time, I propose that it be taken seriously even under conditions of strong curvature ($R_{\mu\nu\sigma\tau} > \mu_P{}^2$) and with topologies other than $\mathbb{R}^4$. Efforts are currently underway at the University of Texas to test it on Friedmann-Robertson-Walker universes to see whether, under generic realistic conditions, it will suppress the initial curvature singularity. Among the properties of Friedmann-Robertson-Walker models that simplify this investigation is conformal flatness. The Weyl tensor disappears from expression (42) taking with it the parallel displacement bivectors, leaving $\sigma(x,x')$ as the only difficult geometrical quantity to compute and A_2 as the only adjustable constant.

Before describing these efforts, I wish to make a few comments on the reasonableness of expression (42) as an approximation to the true effective action. Expression (42), based as it is on a quadratic approximation to Σ that is determined solely by the graviton propagator, cannot be expected to yield accurate vertex functions (third functional derivatives and higher). Nevertheless it is well known that in regions of momentum space where $\Sigma_1(p^2)$ and $\Sigma_2(p^2)$ are slowly varying, e.g., in

[5] B. S. DeWitt, Dynamical Theory of Groups and Fields (Gordon and Breach, New York, 1965), Chap. 17.

the ultrahigh-energy region $|p^2| >> \mu p^2$ (see comments following Eq.(39)) the vertex functions are fully determined by the graviton propagator in virtue of the gauge-invariance condition (25). Therefore, if $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are well approximated by (38) then expression (42) has the correct structure as $x' \rightarrow x$ and will yield qualitatively correct dynamical behavior. More accurate vertex functions at lower energies could in principle be obtained by adding to expression (42) higher multiple integrals in which the curvature appears cubically, quartically, etc., along with factors involving $g^{\mu\alpha'}$ and $\sigma(x,x')$.

Numerical Work

The effort to solve the effective field equations based on the effective action (42) is being carried out by Richard Rohwer at the University of Texas. In the case of Friedmann-Robertson-Walker universes the line element may be written in the form

$$ds^2 = -\alpha^2(t)dt^2 + a^2(t)d\underset{\sim}{r}^2 \tag{43}$$

Here we are specializing to the case in which the spatial sections t = constant are flat. This is because we are primarily interested in the neighborhood of the Big Bang and in our actual universe the curvature in time at this epoch is much more important than the curvature in space. With this line element Eq.42 takes the form

$$\begin{aligned}\Gamma \approx\ & 6V\mu_P{}^2 \int_{-\infty}^{\infty} \alpha^{-1} a\ \dot{a}^2 dt \\ & -12\pi V\mu_P{}^2 A_2 \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \int_0^{\infty} s^2 ds\, \alpha^{-2} \alpha'^{-2} (\alpha a^2 \ddot{a} + \alpha a \dot{a}^2 - a^2 \dot{a}\dot{\alpha}) \\ & \times (\alpha' a'^2 \ddot{a}' + \alpha' a' \dot{a}'^2 - a'^2 \dot{a}' \dot{\alpha}') \left[\frac{i}{\sigma(t,t',s) - \frac{1}{2}\lambda_P{}^2 + i0} - \frac{i}{\sigma(t,t',s) + i0} \right] \\ & + 6V\mu_P{}^4 \int_{\infty}^{\infty} \alpha a^{-1} dt\end{aligned} \tag{44}$$

where V is the volume of space and the last term expresses the effect of the radiation which is assumed to fill the universe, a choice of scale being made so that the energy density is equal to $6\mu_P{}^4$ when a = 1.

There are two effective field equations for this effective action

$$\frac{\delta\Gamma}{\delta\alpha} = 0$$

$$\frac{\delta\Gamma}{\delta a} = 0 \tag{45}$$

Because of gauge invariance these two equations are not independent but satisfy the identity

$$\alpha \frac{d}{dt} \frac{\delta\Gamma}{\delta\alpha} \equiv \dot{a} \frac{\delta\Gamma}{\delta a} \tag{46}$$

It evidently suffices to work with $\frac{\delta\Gamma}{\delta\alpha} = 0$. Explicitly one finds

$$0 = -(6V\mu_P{}^2)^{-1}(\delta\Gamma/\delta\alpha)_{\alpha=1}$$

$$= a\dot{a}^2 - \mu_P{}^2 a^{-1}$$

$$+4\pi A_2\int_{-\infty}^{\infty} dt'\int_0^{\infty} s^2 ds \left\{ a\dot{a}^2\left[\frac{i}{\sigma(t,t',s)-\frac{1}{2}\lambda_P{}^2+i0} - \frac{i}{\sigma(t,t',s)+i0}\right]\right.$$

$$\left. -a^2\dot{a}\sigma_{,t}(t,t',s)\left[\frac{i}{[\sigma(t,t',s)-\frac{1}{2}\lambda_P{}^2+i0]^2} - \frac{i}{[\sigma(t,t',s)+i0]^2}\right]\right\}$$

$$\times(a'^2a'+a'\dot{a}'^2)$$

$$+4\pi A_2\int_{-\infty}^{\infty} dt'\int_{-\infty}^{\infty} dt''\int_0^{\infty} s^2 ds(a'^2\ddot{a}'+a'\dot{a}'^2)(a''^2a''+a''\dot{a}''^2)$$

$$\times\left\{-a^{-2}(t)[\sigma_{,s}(t',t'',s)]^2-2\sigma(t',t'',s)\right\}^{\frac{1}{2}}$$

$$\times\left[\frac{i}{\sigma(t',t'',s)-\frac{1}{2}\lambda_P{}^2+i0} - \frac{i}{\sigma(t',t'',s)+i0}\right] \quad (47)$$

The idea of the computation is the following. Begin with a Big Crunch followed by a Big Bang, having the form $a \sim |t|^{\frac{1}{2}}$ (which is a solution of the classical field equation) but with the Crunch at $t = 0$ smoothed out (by hand) over a region of the order of the Planck time. Substitute this value of a into the integrals appearing in Eq.47 and obtain new values for a by putting the terms involving no integrals on the left hand side of the equation. Then iterate this procedure, hoping that the sequence will converge. Unfortunately we have been unable as yet to get the program to this stage because we have been encountering unforseen difficulties in evaluating the bilinear σ . We began by attempting to compute it from the Hamilton-Jacobi equation,

$$2\sigma = g^{\mu\nu}\sigma_{,\mu}\sigma_{,\nu} = -(\sigma_{,t})^2 + a^{-2}(\sigma_{,s})^2 \quad (48)$$

but discovered subsequently that every reasonable way for converting this equation to a set of difference equations leads to unconditional instabilities. Another possibility is to solve directly the geodesic equations. However a new problem arises, namly that of caustics, which indeed occur for these metrics. We are now thinking in terms of computing the scalar propagator in the given metric and inverting it to obtain an approximation to σ . Note that the presence of the factors i and i0 in Eq.47 causes a to become complex. This in turn causes σ to become complex. Complex functions can be handled on the computer in a straightforward way but it is important to call attention to the added complication.