GENERALIZATIONS OF GRAVITATIONAL THEORY BASED ON GROUP COVARIANCE

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The justification for the validity of a symmetry group should be established as firmly as possible in the physical conditions and the mode of its breaking in modifications of the latter.

I proceed here in the opposite direction, letting myself be guided by the mathematical beauty of a symmetry group, I develop a formalism to derive the physics based on it. The physical theory is thus dictated by mathematics (not at all however by mathematicians).

One can expect from such a procedure in spite of its mathematical rigour, a physical fairy tale of determined harmony in which things may happen that are strange to physical reality. Fairy tales have however often a significant content of truth and I think it worth while to understand the models as well as possible before dismissing it as unrelated to reality.

In the beginning there was the group, a semisimple Lie group G with a semisimple Lie subgroup H, such that G/H, the factor space is homeomorphic to the unperturbed manifold of space-time. The group manifold G has a natural projection $\pi: G \to G/H$ so that $P(G, G/H, \pi, H)$ form a principal fibre bundle with typical fibre H.[1-4] Locally G is homeomorphic to $G/H \times H$. for most cases considered this is globally true; P is trivial.

G has a natural metric the Cartan-Killing metric γ given in a local orthogonal frame in terms of the structure constants by:

$$(1) \qquad \gamma_{RS} = C_{R\ V}^{\ U} C_{S\ U}^{\ V}$$

it's projection $g = \pi\gamma$ is the space-time metric of the universe.[5-9] G may for example be chosen as SO(4.1) or SO(3.2), the De Sitter groups and H as SO(3.1), the Lorentz group which yield the De Sitter or anti De Sitter universe as factor spaces.

It is postulated in general that physical quantities in the space-time manifold G/H are obtained as the projection of geometric quantities on G.

The orbits of one dimensional subgroups of G, which are the geodesics of the space with the metric γ, have as their projection on G/H all the geodesics of this space with metric g, but besides this other lines that have the form of trajectories of charged particles in electromagnetic fields. The projection from the higher dimensional space yields in general more than only the corresponding geometrical quantities on the base space - more things may occur in the fairy tale than what can be brought into accord with our (limited) experience - but these features could be the effect of inner degrees of freedom in a classical theory which should be brought in accord with reality by imposing quantum conditions. Especially the spiral motion analog to that of a charged particle in a magnetic field for the De Sitter groups, which unify momentum and angular momentum, may be the classical analog of spin motin.

The metric of the group manifold can only describe a space without inhomogenous mass distribution. This metric for a r-parameter semisimple group fulfills the relation:

$$R_{UV} - \frac{1}{2}\gamma_{UV} R + \frac{r-2}{8}\gamma_{UV} = 0 \qquad (2)$$

This can be interpreted as the homogenous Einstein equations in r dimensions with a cosmological member. (The radius of the De Sitter universe is of order unity so that here the cosmological term is of conventional magnitude)

The generalisation to the case of inhomogenous local mass distributions is made by introducing a source term to eq.(2) which generalizes the metric γ in such a way that the global topology of the manifolds and their character of a principal fibre bundle with group and typical fibre H is not altered. We have arrived at peculiar versions of multidimensional Kaluza-Klein theories with non-Abelian, in general non compact gauge groups H. If one believes in the fundamental character of exact invariance groups, one should analyze this mathematically consistent scheme with all suitable semisimple groups for its physical content.

There is no doubt that this scheme is closer adjusted to the invariance group than that of any other theory, even if the latter is formulated on the group manifold.

The effect of the metric in r dimensions with (r-k) Killing vectors, which must exist to meet the requirements of a principal fibre bundle, is equivalent to the effect of the metric projected on the k-dimensional base manifold together with (r-k) Yang-Mills fields. The latter are obtained from the curvature two form Ω of a connection on the principal fibre bundle P. (The generalisation) from that on the group manifold) This connection can be given by a Lie algebra valued one form ω:

(3) $$\Omega = d\omega + [\omega, \omega]$$

Horizontal vectors B are defined by $\omega(B) = 0$.

Ω has only horizontal components; it can be expressed in a local natural coordinate system where $P = B \times H$ by (r-k) Yang Mills fields:

(3a) $$F^M_{ik} = B^M_{k,i} - B^M_{i,k} + C^M_{PQ} B^P_i B^Q_k \qquad (M, P, Q \cdots k+1 \cdots r)$$

with (r-k) Yang-Mills potentials B^M_i and the structure constants of H.

The Lagrangian for eq.(2) has then the following k-dimensional form:

(4) $$\mathfrak{L} = \sqrt{\gamma}(R^{(r)} + \lambda) = \sqrt{g}(R^{(k)} + \tfrac{1}{4}F^{ikM}F_{ikM} + \lambda)$$

it does not depend on the coordinates of H. (γ metric on P, g metric on B) We require that the torsion two form vanishes. Horizontal and vertical vector spaces are perpendicular.

The projection of geodesics on P on B fulfills in our coordinates:

(5) $$\ddot{x}^i + \left\{{i \atop jk}\right\} \dot{x}^j \dot{x}^k = F^{iM}_k \dot{x}^k C_M$$

where the generalized "charges" C_M are given by the vertical components of an initial tangent vector of the geodesic.

The solution of the homogenous field equations (2) which constitutes the group manifold G has a nonflat metric g as well as a nonvanishing curvature form Ω due to the cosmological member. There exist thus "cosmological fields" F^M even in this case which can give rise to the nongeodesic motion. The Maurer-Cartan equations on the group manifold:

(6) $$dA^R + C^R_{ST}A^S A^T = 0$$

(R,S,T = 1$\cdots$r) for the left invariant forms A^R give:

(6a) $$F^M = dA^M + C^M_{PQ}A^P A^Q = - C^M_{EF}A^E A^F$$

(E, F summed over 1$\cdots$k, P, Q over k+1$\cdots$r) for the cosmological fields. Although these fields do not vanish, the total of their energy densities on the De Sitter universes vanishes. The energy of any one field F^M can thus be either positive or negative definite; this results from the non-compactness of the gauge group H. The cosmological fields are not felt except by a particle with suitable charge (≡ initial condition) C^M. The latter must be chosen so that no energy can be drawn from the fields. This property remains true along the world line. It is a generalisation of the restriction in relativity that a world line must be time like. In the De Sitter case it restricts the motion in a suitable frame to the analog of a spiral motion of a charge in a magnetic field.

The fairy tale could be well brought in accord with reality by letting the charges vanish or making them unobservably small; but this is not in the spirit of the present considerations. We have to see whether not some deep truth is in the fairy tale that properly applied may even give a better insight into reality. The classical equations of motion already in the De Sitter case are much more complicated. The Hamilton-Jacobi equation in a way has an oscillatory wave like character. The nature of the quantum of action in this context has first to be better understood to see whether the quantized equations of motion do not describe also a spin motion in the De Sitter case.

A heuristic considerations which contributed to this development was the following: The De Sitter groups for a large radius of the universe may be regarded as a unification of momentum and angular momentun in a similar way as the group of rotations on a sphere unifies rotation and translations of $\mathbb{E}^2$.

A system of reference which rotates should thus be on an equal footing with a uniformly translated system. Making a passive transformation to such a system we recognize that an observer there really sees the spiral motion described in our fairy tale - not just for one particle, but even for all the macroscopic bodies. From experience we know that he must pay for this by experiencing inertial forces - a result that can not adequately be derived from the general theory of relativity. The present formalism provides in addition to the metric g still the fields F^M which should give a stronger account of Mach's principle if they are interpreted as a spin-spin interaction in the De Sitter cases; it is strongly felt that spin and orbital angular momentum cannot be fully separated. To seek to account for this in the present theory one would have to describe the motion of orbiting bodies in detail including the gravitational fields generated by the fields causing the binding forces.

The only example presented here was that of the De Sitter group. The formalism is applicable to higher dimensional groups G. The conformal group SO(4.2) is 15-dimensional and has a 10-dimensional subgroup H. The metric base space B is thus five dimensional and can be interpreted to unify gravity and electromagnetism in a Kaluza-Klein theory of higher order by using the projective formalism of Veblen and Jordan.[10,11]

The present formalism is not extended to supersymmetry but a presentation by D. Ebner has shown that higher dimensional groups can take account of the antisymmetry of Fermions. Limiting oneself here to this case may provide a desirable criterium to restrict the growing number of possible theories.

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