Background Field Method of Gauge Theory and the Renormalization Problem

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§1. Introduction

The background field method was proposed by DeWitt[(1)] and has been discussed later in many papers.[(2)] This method has an interesting feature that we can quantize gauge theories without losing gauge invariance. As a result, in the renormalization procedure gauge invariant quantities need to be considered. This is a very important advantage in discussing the renormalization problem of non-Abelian gauge theories (especially of gravitation).

In this paper we will discuss a systematic renormalization procedure by using the background field method and by generalizing the counter-term formula of 't Hooft[(3)] to two loop processes.

§2. Background Field Method

In the background field method the generating functional of the S-metrix is given by

$$W(\tilde{\phi}) = \int D\phi e^{iS(\tilde{\phi}+\phi)} = \int D\phi \exp i\{S(\tilde{\phi}) + S_{,i}\phi_i + \frac{1}{2!} S_{,ij}\phi_i\phi_j + \cdots\}, \tag{2.1}$$

where $S_{,i}$, $S_{,ij}$ $\cdots$ denote functional derivatives of the action $S(\tilde{\phi})$ with respect to $\tilde{\phi}_i(x)$, $\tilde{\phi}_j(y)$, $\cdots$, and $S_{,i}\phi_i$, $S_{,ij}\phi_i\phi_j$ are abbreviations of

$$S_{,i}\phi_i = \int d^4x \frac{\delta S(\tilde{\phi})}{\delta\tilde{\phi}_i(x)} \phi_i(x) \quad , \quad S_{,ij}\phi_i\phi_j = \int d^4x d^4y \frac{\delta^2 S(\tilde{\phi})}{\delta\tilde{\phi}_i(x)\delta\tilde{\phi}_j(y)} \phi_i(x)\phi_j(y). \tag{2.2}$$

In (2.1) the background field $\tilde{\phi}$ is defined as a solution of the field equation $S_{,i}(\tilde{\phi}) = 0$.

If we consider a system which includes gauge fields in this formalism we can introduce two kinds of gauge transformations. By taking the Yang-Mills fields as an example we define such transformations as follows:

c-type gauge transformations

$$\begin{aligned} \tilde{\phi}_\mu^{a\prime} &= \tilde{\phi}_\mu{}^a + gf^{abc}\omega^b\tilde{\phi}_\mu{}^c - \partial_\mu\omega^a , \\ \phi_\mu'^a &= \phi_\mu{}^a + gf^{abc}\omega^b\phi_\mu{}^c , \end{aligned} \tag{2.3}$$

q-type gauge transformations

$$\tilde{\phi}'^{a}_{\mu} = \tilde{\phi}^{a}_{\mu} ,$$

$$\phi'^{a}_{\mu} = \phi^{a}_{\mu} + gf^{abc}\omega^{b}(\phi^{c}_{\mu} + \tilde{\phi}^{c}_{\mu}) - \partial_{\mu}\omega^{a} . \tag{2.4}$$

Under the both transformations (2.3) and (2.4) the total field $A = (\tilde{\phi}+\phi)$ has ordinary gauge transformation properties

$$A'^{a}_{\mu} = A^{a}_{\mu} + gf^{abc}\omega^{b}A^{c}_{\mu} - \partial_{\mu}\omega^{a} . \tag{2.5}$$

The important point is the fact that only $\tilde{\phi}$ field (ϕ field) has the transformation properties of gauge field under c-type (q-type) gauge transformations.

As concerning gauge fixing we have to fix the q-type gauge only in (2.1). Then we can choose a gauge fixing condition (the background gauge) such as

$$D_{\mu}\phi^{a}_{\mu} \equiv (\partial_{\mu}\phi^{a}_{\mu} + gf^{abc}\tilde{\phi}^{b}_{\mu}\phi^{c}_{\mu}) = 0 . \tag{2.6}$$

Since (2.6) transforms covariantly under (2.3) it is evident that $W(\tilde{\phi})$ is invariant under (2.3) even if the gauge fixing has been performed. This important fact simplifies the discussion of the renormalization problem of the gauge theories in the point that the counter Lagrangian can be expressed in term of gauge invariant combinations such as $(F^{a}_{\mu\nu})^2$.

§3. Counter-term Formula

Without loss of generality we can expand the action $S(\tilde{\phi}+\phi)$ around $\tilde{\phi}$ as

$$\begin{aligned} S(\tilde{\phi}+\phi) - S(\tilde{\phi}) - S_{,i}\phi_i \\ = \int d^4x\{\tfrac{1}{2}\, \partial_{\mu}\phi_i W^{ij}\partial_{\mu}\phi_j + \phi_i N^{\mu}_{ij}\partial_{\mu}\phi_j + \tfrac{1}{2}\,\phi_i M^{ij}\phi_j + \Xi^{ijk}_{\mu\nu}\phi_i\partial_{\mu}\phi_j\partial_{\nu}\phi_k \\ + \Omega^{ijk}_{\mu}\phi_i\phi_j\partial_{\mu}\phi_k + \Lambda^{ijk}\phi_i\phi_j\phi_k + \Gamma^{ijk\ell}_{\mu\nu}\phi_i\phi_j\partial_{\mu}\phi_k\partial_{\nu}\phi_{\ell} \\ + \Sigma^{ijk\ell}_{\mu}\phi_i\phi_j\phi_k\partial_{\mu}\phi_{\ell} + \Theta^{ijk\ell}\phi_i\phi_j\phi_k\phi_{\ell} + \cdots\} , \end{aligned} \tag{3.1}$$

where coefficients W, N_{μ}, M, Ξ, Ω_{μ}, Λ, $\Gamma_{\mu\nu}$, Σ_{μ} and Θ are functions of the $\tilde{\phi}$. In (3.1) we assumed that $S(\tilde{\phi}+\phi)$ has derivative couplings up to second order. This is satisfied both for Einstein gravity and for the Yang-Mills type gauge theories.

In this paper we will restrict ourselves to the case $\Xi = \Gamma = \Sigma = 0$, $W^{ij} = -\delta^{ij}$. By introducing a kind of covariant derivatives $\nabla_{\mu}\phi_i = \partial_{\mu}\phi_i + N^{\mu}_{ij}\phi_j$ under some transformations which will be mentioned later (3.1) can be rewritten by

$$S(\tilde{\phi}+\phi) - S(\tilde{\phi}) - S_{,i}\phi_i$$

$$= \int d^4x\{-\frac{1}{2}\nabla_\mu\phi_i\nabla_\mu\phi_i + \frac{1}{2}\phi_i X_{ij}\phi_j + \Omega_{ijk}\phi_i\phi_j\nabla_\mu\phi_k + \tilde{\Lambda}_{ijk}\phi_i\phi_j\phi_k + \Theta_{ijk\ell}\phi_i\phi_j\phi_k\phi_\ell\}, \quad (3.2)$$

where $X_{ij} = M_{ij} - (N^\mu N^\mu)_{ij}$,

$$\tilde{\Lambda}_{ijk} = \Lambda_{ijk} + \frac{1}{3}(N^\mu_{i\ell}\Omega^\mu_{jk\ell} + N^\mu_{j\ell}\Omega^\mu_{ik\ell} + N^\mu_{k\ell}\Omega^\mu_{ij\ell}) . \quad (3.3)$$

The action (3.1) is invariant under the following transformations

$$\phi'_i = \phi_i + \lambda_{ij}\phi_j , \quad N'^\mu_{ij} = N^\mu_{ij} + \lambda_{ik}N^\mu_{kj} - N^\mu_{ik}\lambda_{kj} - \partial_\mu\lambda_{ij} ,$$

$$M'_{ij} = M_{ij} + \lambda_{ik}M_{kj} - M_{ik}\lambda_{kj} - N^\mu_{ik}\partial_\mu\lambda_{kj} - \partial_\mu\lambda_{ik}N^\mu_{kj} ,$$

$$\Omega'^\mu_{ijk} = \Omega^\mu_{ijk} + \lambda_{i\ell}\Omega^\mu_{\ell jk} + \lambda_{j\ell}\Omega^\mu_{i\ell k} + \lambda_{k\ell}\Omega^\mu_{ij\ell} ,$$

$$\Lambda'_{ijk} = \lambda_{i\ell}\Lambda_{\ell jk} + \frac{1}{3}\partial_\mu\lambda_{i\ell}\Omega^\mu_{jk\ell} + \mathrm{cyclic}(i,j,k) ,$$

$$\Theta'_{ijk\ell} = \lambda_{im}\Theta_{mjk\ell} + \mathrm{cyclic}(i,j,k,\ell) . \quad (3.4)$$

By considering dimensions of the coefficient functions and their transformation properties under (3.4) which include c-type gauge transformations as a special case, one-loop counter-terms for (3.2) are given by(3)

$$\Delta L^{\text{one-loop}}(\tilde{\phi}) = -\frac{1}{8\pi^2\varepsilon}(\frac{1}{4}X_{ij}X_{ji} + \frac{1}{24}Y^{\mu\nu}_{ij}Y^{\mu\nu}_{ji}) , \quad (3.5)$$

where $Y^{\mu\nu}_{ij} = \partial_\mu N^\nu_{ij} - \partial_\nu N^\mu_{ij} + N^\mu_{ik}N^\nu_{kj} - N^\nu_{ik}N^\mu_{kj}$ and $\varepsilon = 4 - n$. Similarly we can obtain two-loop counter-terms which were given explicitly in our paper(4).

When we apply these counter-term formula to the pure Yang-Mills type theory as an example we find

$$\Delta L^{\text{one-loop}} + \Delta L^{\text{two-loop}} = -\{\frac{11}{96\varepsilon}\frac{g^2C_2}{\pi^2} + \frac{17}{3\cdot 29\varepsilon}\frac{(g^2C_2)^2}{\pi^4}\}(F_{\mu\nu}{}^a)^2 , \quad (3.6)$$

from which the renormalization group function $\beta(g)$ can be found to be

$$\beta = -\frac{11}{3\cdot 2^4}\frac{g^3C_2}{\pi^2} - \frac{17}{3\cdot 2^7}\frac{g^5C_2^2}{\pi^4} + \cdots . \quad (3.7)$$

In the calculation of (3.6) we used the background field gauge condition (2.6)

$$L_{\text{gauge}}(\alpha) = -\frac{1}{2\alpha}(D_\mu\phi_\mu{}^a)^2 , \quad (3.8)$$

where the gauge parameter α is fixed to be $\alpha = 1$. In order to cancel all the subdivergence it is necessary to be renormalized such that

$$\alpha = 1 - \frac{g^2}{4\pi^2 \varepsilon} C_2 + \cdots . \tag{3.9}$$

§4. Conclusions and Discussions

We obtained the counter-term formula up to two-loop by using the background field method. Although we did not discuss the cancellation mechanism of subdivergences in this formalism, it was examined in detail in our previous paper(4). The generalization of this formula to the gravitational theory is in progress now.

References

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