Algebraic Construction of Static Axially Symmetric Self-Dual Fields*

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In this short Note we will present a short introduction to the recent work by us[1),2)] on the problem of solving self-dual gauge field equations for an arbitrary compact semisimple gauge group G. The discussion is mostly focused on the simple case of the static and axially symmetric configuration, in which the effective space-time dimension is reduced to two and we can apply some of the solution generation techniques of two dimensional soliton theories.[3)] The central result is an algebraic method for the construction of solutions starting from one particular solution and solutions of the associated linear scattering problem. Here we see an interesting interplay of soliton theoretic techniques and group theoretic concepts. A main application is the construction of axially symmetric monopole solutions in gauge theories with an arbitrary group G and a single Higgs field in the adjoint representation.

Ansatz The ansatz for the static axially symmetric gauge potential is

$$A_z,\ A_\rho \in \mathcal{K}, \quad A_t,\ A_\phi \in \mathcal{P}\ , \tag{1}$$

in which $\mathcal{K}$ is some maximal subalgebra of $\mathcal{G}$ and $\mathcal{P}$ is its complementary subspace; $\mathcal{G} = \mathcal{K} + \mathcal{P}$,

$$[\mathcal{K},\mathcal{K}] \leq i\mathcal{K},\ [\mathcal{K},\mathcal{P}] \leq i\mathcal{P},\ [\mathcal{P},\mathcal{P}] \leq i\mathcal{K}\ . \tag{2}$$

The coefficient functions depend only on z and ρ.

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Generalized Self-Duality Equations with static axial symmetry

$$\partial_y A_{\bar{y}} - \partial_{\bar{y}} A_y - i[A_y, A_{\bar{y}}] + i[\phi_y, \phi_{\bar{y}}] = 0 \ , \qquad (3.a)$$

$$\partial_y \phi_{\bar{y}} - i[A_y, \phi_{\bar{y}}] + \frac{1}{2}(M_y \phi_{\bar{y}} + M_{\bar{y}} \phi_y) = 0 \ , \qquad (3.b)$$

$$\partial_{\bar{y}} \phi_y - i[A_{\bar{y}}, \phi_y] + \frac{1}{2}(M_y \phi_{\bar{y}} + M_{\bar{y}} \phi_y) = 0 \ , \qquad (3.c)$$

in which we used the coordinates $y \equiv \rho + iz$, $\bar{y} \equiv \rho - iz$ and gauge potentials $A_y = \frac{1}{2}(A_\rho - iA_z)$, $\phi_y = -\frac{1}{2}(A_\phi + iA_t)$, etc. In order to enlarge the symmetry of the system we have introduced the additional variables M_y and $M_{\bar{y}}$ which obey

$$\partial_y M_{\bar{y}} = \partial_{\bar{y}} M_y = -M_y M_{\bar{y}} \ . \qquad (4)$$

For the choice $M_y = M_{\bar{y}} = 1/2\rho$, eq. (3) reduces to the ordinary static, axially symmetric self-duality equation.

K-invariance The equation (3) as well as the ansatz (1) are invariant under the following gauge transformation,

$$A_\nu \to A'_\nu = \Omega A_\nu \Omega^{-1} - i\partial_\nu \Omega \Omega^{-1}, \quad \phi_\nu \to \phi'_\nu = \Omega \phi_\nu \Omega^{-1} \ , \qquad (5)$$

in which $\nu = y, \bar{y}$ and $\Omega = \Omega(y,\bar{y}) \varepsilon K$. This is a subgroup K of the original gauge group G which preserves the structure of the ansatz (1).

Pure gauge and Triangularity Two of the three equations (3) imply that the following combination of gauge potentials

$$a_\nu = A_\nu + i\phi_\nu, \quad \nu = y, \bar{y} \ , \qquad (6)$$

is pure gauge, i.e.,

$$a_\nu = -i(\partial_\nu g) g^{-1}, \quad g \varepsilon G^*, \quad \mathcal{G}^* = \mathcal{K} + i\mathcal{P} \ . \qquad (7)$$

A theorem of group theory (Iwasawa decomposition[4]) states that G^* factorizes into two parts,

$$G^* = KT \ , \qquad (8)$$

in which the Lie algebra of T consists of abelian and nilpotent parts only. By an appropriate K-transformation one can always make a_ν as

$$a_\nu = -i(\partial_\nu \tau)\tau^{-1} , \quad \tau \varepsilon T . \tag{9}$$

This gauge will be called "triangular".

Σ-invariance From a solution of (3) in the triangular gauge another solution in the triangular gauge is obtained in terms of the following discrete transformation

$$\left.\begin{aligned} A_y \to \tilde{A}_y &= RA_yR^{-1}, \ \phi_y \to \tilde{\phi}_y = S\phi_y S^{-1} + M_y\eta, \ M_y \to M_y , \\ A_{\bar{y}} \to \tilde{A}_y &= SA_{\bar{y}}S^{-1}, \ \phi_{\bar{y}} \to \tilde{\phi}_y = R\phi_{\bar{y}}R^{-1} + M_{\bar{y}}\eta, \ M_{\bar{y}} \to M_{\bar{y}} , \end{aligned}\right\} \tag{10}$$

in which R, S and η are constant matrices. Their explicit forms are found by considering the isomorphisms of the extended Dynkin diagrams of each algebra $\mathcal{G}$, which also characterizes the Kac-Moody algebra[4].

Γ-invariance The generalized self-dual equations (3) are invariant under

$$A'_\nu = A_\nu, \ \phi'_y = \gamma^{1/2}\phi_y, \ \phi_{\bar{y}}' = \gamma^{-1/2}\phi_{\bar{y}}, \ M'_y = \gamma M_y, \ M_{\bar{y}}' = \gamma^{-1}M_{\bar{y}}, \tag{11}$$

in which a scalar function γ should satisfy the following completely integrable Riccati equation

$$d\gamma = (\gamma-1)(\gamma M_y dy + M_y d\bar{y}) . \tag{12}$$

The Γ, Σ and K are three important symmetry transformations in terms of which new solutions are generated successively.

Associated Linear Problem By combining two facts that for the solution of (3), a_ν (6) is pure gauge and that the Γ-transformation maps a solution to another, we get the following linear problem,

$$\partial_y \mathcal{R} = i(A_y + i\gamma^{1/2}\phi_y)\mathcal{R}, \ \partial_{\bar{y}}\mathcal{R} = i(A_{\bar{y}} + i\gamma^{-1/2}\phi_{\bar{y}})\mathcal{R}. \tag{13}$$

The integrability condition is (3) and (4).

Triangularity Restoration. If a Γ-transformation is applied to a triangular solution, the result is not triangular any more. But the triangularity can be restored by a K-transformation in terms of Ω, which can be obtained from a solution $\mathcal{R}$ of the linear problem (13) in terms of the Iwasawa decomposition,

$$\mathcal{R} = \Omega^{-1}\tau, \quad \mathcal{R} \in G^{*}, \quad \Omega \in K, \quad \tau \in T \ . \tag{14}$$

Solution Generation By combining the Σ, Γ and K-transformations we can generate a host of new solutions from a given one $\overset{\circ}{A} \equiv (\overset{\circ}{A}_{\nu}, \overset{\circ}{\Phi}_{\nu}, \overset{\circ}{M}_{\nu})$ of eq. (3). A simple scheme is, for example,

$$\underset{(\text{Tr.})}{\overset{\circ}{A}} \rightarrow \underset{(\text{Tr.})}{\Sigma\overset{\circ}{A}} \rightarrow \underset{(\text{Non Tr.})}{\Gamma\Sigma\overset{\circ}{A}} \rightarrow \underset{(\text{Tr.})}{K\Gamma\Sigma\overset{\circ}{A}} \equiv \overset{1}{A} \rightarrow \underset{(\text{Tr.})}{\Sigma\overset{1}{A}} \rightarrow \cdots \ , \tag{15}$$

in which Tr. stands for triangular. This procedure can be repeated an arbitrary number of times. New parameters are introduced through constants of integration for γ and $\mathcal{R}$ at each stage.

Algebraic Construction The main point of our work is that we can construct a hierarchy of solutions $\overset{1}{A}, \overset{2}{A}, \cdots$, algebraically. The only analytical work needed is to solve the associated linear problem (12) and (13) for the initial solution A. We denote the solutions as

$$\overset{\circ}{A} \ ; \ (\overset{\circ}{\mathcal{R}}_1, \overset{\circ}{\gamma}_1) \ , \ \cdots \ , \ (\overset{\circ}{\mathcal{R}}_m, \overset{\circ}{\gamma}_m) \ , \ \cdots \ , \tag{16}$$

in which the suffix indicates a particular choice of constants of integration. The γ-functions and the triangularity restoring Ω functions at each stage can be constructed from (16) algebraically and step by step. For an explicit construction, a close relationship between the Σ-transformation and the associated linear problem must be revealed. We refer to our papers[1),2)] for further details.

Summary and Comments Applied to the simplest case, i.e., G = SU(2), the above procedure reproduces all the solution generation techniques of Neugebauer[5)] for the Ernst equation[6)] in general relativity, which describes the stationary axially symmetric vacuum gravitational field.

The algebraic solution generation method makes use of some specific combinations of elements of an infinite dimensional group of symmetries for eq. (3). This group is a natural generalization of Geroch-Kinnersley-Chitre[7] group for the Ernst equation. We believe that further investigation of the infinite dimensional group is very important for deeper understanding of gauge theories.

References

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